A convex optimization based solution for the robotic manipulator control design problem subject to input saturation [*]

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Abstract: This paper proposes a state-feedback design procedure for robotic manipulator systems with saturating actuators, a solution which is based on convex optimization subject to constraints in the form of linear matrix inequalities. Our fundamental idea is to express the system dynamics in a novel differential-algebraic representation with state-derivative components. This approach allows us to provide a systematic control design framework with formal theoretical guarantees, such as the asymptotic stabilization of the manipulator attitude reference error within a prescribed exponential decay-rate. Our method is capable of dealing with the nonlinearities of a mechanical manipulator system, including the input saturation effect, without relying on any kind of linearization or approximation. A two-link planar robotic manipulator example is employed in order to illustrate the proposed approach.

Keywords: Robotic Manipulator Control, Linear Matrix Inequalities, Differential-Algebraic Representation, Input Saturation.

1. INTRODUCTION

Robotic manipulators have been widely studied in control engineering in areas ranging from manufacturing (Cherubini et al., 2016) to medical applications (Taylor et al., 2016) and search and rescue devices (Wu et al., 2017), among others. In practice, manipulator control design is a particularly challenging task due to the nonlinear multivariable nature of its model (Spong et al., 2006) and due to physical limitations of the actuators, such as input saturation. This observation motivates the development of robust and systematic control design methodologies able to address all of these intrinsic characteristics.

One can find a myriad of nonlinear control techniques for robotic manipulator systems. One example is Passivity-based control (Walsh and Forbes, 2015), where an extension of the Lyapunov function is presented by using the system energy. Sliding mode techniques were also proposed in Kali et al. (2015) and Pan Zhang (2019), which apply a discontinuous control law to stabilize the system. Based on the inverse dynamic method and the feedback linearization apporach, a model-based controller was designed in Farhadmanesh et al. (2015). Moreover, robust techniques such as $\mathcal{H}_\infty$ control (Costa et al., 2018) were devised, where the uncertainties of the manipulator

is considered for the trajectory tracking problem. The implementation of a robust predictive controller in real-time was also presented in Andres Lara-Molina et al. (2014) and a robust Neural Network controller has been proposed in order to deal with external disturbances (Rahmani and Belkheiri, 2016). Alternatively, one may find methods based on evolutionary algorithms (Pan Zhang, 2018), hybrid intelligent optimization (Wang et al., 2017) and fuzzy-logic control (Chen and Lai, 2018). Based on the aforementioned studies, an open challenge is to provide a control design method that does not require the full knowledge of the plant model and is able to rigorously address the nonlinearities of a manipulator dynamics, including the input saturation effect from the actuators.

This paper proposes a novel convex optimization based solution able to systematically synthesize a state feedback controller for robotic manipulator systems subject to saturating actuators. Our methodology provides formal closed-loop stability and performance guarantees considering all the nonlinearities involved in the equations of motion of a manipulator system and also the input torque saturation effect of the joint actuators (i.e., no linearization or approximation is performed). In order to achieve this objective, the system dynamics are cast in a new differential algebraic representation (DAR) with state-derivative components, an idea that we conceived so as to deal with rational inertia-related components of mechanical systems. The devised model representation allowed us to express the closed-loop stability and performance conditions by a

feasibility problem subject to constraints in the form of linear matrix inequalities (LMIs), which can be efficiently addressed by numerical solvers. In this paper, we focus the study in a two-link robotic manipulator system, even though the methodology to be presented here is naturally expansible for any robotic manipulator system with an arbitrary number of links.

This paper is structured as follows. Section 2 presents the mathematical modeling of the system dynamics; Section 3 introduces the main concepts for rewriting the model in the porposed DAR; Section 4 addresses the problem of systematically synthesizing a state-feedback controller subject to input saturation; Section 5 illustrates the main results with a numerical example and Section 6 concludes the paper.

Notation: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n \times m}$ is the set of $n \times m$ real matrices. I_n is the $n \times n$ identity matrix and diag$\{A,B\}$ denotes a block-diagonal matrix formed by A and B. For a real matrix M, M^T denotes its transpose, He$\{M\} = M + M^T$ and $M > 0$ means that M is symmetric and positive-definite. For a symmetric block matrix, the symbol $\star$ stands for the transpose of the blocks outside the main diagonal block. For a polytopic set $\mathcal{X}$, the notation $\mathcal{V}(\mathcal{X})$ means the set of all vertices of $\mathcal{X}$.

2. MATHEMATICAL MODELING

This section presents the mathematical model of a planar two-link manipulator and its equations of motion. Consider the two-link planar manipulator as shown in Fig. 1, with masses m_i and lengths ℓ_i, $i = 1, 2$. Based on the Newton-Euler modeling approach (Spong et al., 2006), it follows that the dynamics of this system is governed by

$$M(\theta)\dot{\omega} + v(\theta, \omega) = u, \qquad (1)$$

where $\theta \in \mathbb{R}^2$ is the vector of link angles, $\omega = \dot{\theta} \in \mathbb{R}^2$ is the vector of angular velocities and $u \in \mathbb{R}^2$ is the vector of torque control inputs applied to each link. Moreover, $v(\theta, \omega) \in \mathbb{R}^2$ is a vector related to centrifugal and Coriolis effects, which is given by

$$v(\theta, \omega) = \begin{bmatrix} -m_2 \ell_1 \ell_2 \sin(\theta_2)\omega_2^2 - 2m_2 \ell_1 \ell_2 \sin(\theta_2)\omega_1 \omega_2 \\ m_2 \ell_1 \ell_2 \sin(\theta_2)\omega_1^2, \end{bmatrix}, \quad (2)$$

and $M(\theta) \in \mathbb{R}^{2 \times 2}$ is the system inertia matrix, which is described as

$$M(\theta) = \begin{bmatrix} M_{11}(\theta) & \ell_2^2 m_2 + \ell_1 \ell_2 m_2 \cos(\theta_2) \\ \ell_2^2 m_2 + \ell_1 \ell_2 m_2 \cos(\theta_2) & \ell_2^2 m_2 \end{bmatrix}, \quad (3)$$

for $M_{11}(\theta) = \ell_2^2 m_2 + 2\ell_1 \ell_2 m_2 \cos(\theta_2) + \ell_1^2 (m_1 + m_2)$.

So as to represent the difference between the desired attitude angles θ_r and the actual manipulator angles θ, we consider an angle error vector θ_e defined as:

$$\theta_e = \theta - \theta_r, \qquad (4)$$

Assuming that $\dot{\theta}_r = 0$, it follows that $\dot{\theta}_e = \omega$.

3. DIFFERENTIAL ALGEBRAIC REPRESENTATION

As proposed by Trofino (2000), the original differential algebraic representation (DAR) consists in representing a rational nonlinear system by a differential equation com-
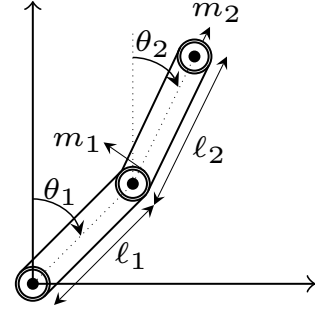

Fig. 1. Schematic of two-link planar robotic manipulator.

bined with an equality relation. The DAR representation was originally given as

$$\begin{cases} \dot{x} = A_1(x, \delta) \, x + A_2(x, \delta) \, \xi + B(x, \delta) \, u \\ 0 = \Omega_1(x, \delta) \, x + \Omega_2(x, \delta) \, \xi \end{cases} \qquad (5)$$

where $x \in \mathbb{R}^{n_x}$ is the system state vector, $u \in \mathbb{R}^{n_u}$ is the control input vector, $\delta \in \mathbb{R}^{n_\delta}$ is a vector of uncertain parameters and $\xi \in \mathbb{R}^{n_\xi}$ is an auxiliary vector that lumps the rational nonlinearities [1]. Terms $A_1(x, \delta) \in \mathbb{R}^{n \times m}$, $A_2(x, \delta) \in \mathbb{R}^{n_x \times n_x}$ e $B(x, \delta) \in \mathbb{R}^{n_x \times n_u}$, $\Omega_1(x, \delta) \in \mathbb{R}^{n_\xi \times n_x}$ and $\Omega_2(x, \delta) \in \mathbb{R}^{n_\xi \times n_\xi}$ are affine matrix functions in (x, δ). An important advantage of the system structure (5) is the possibility to solve stability analysis and control design problems by convex optimizations subject to linear matrix inequalities (LMIs) (Trofino and Dezuo, 2014).

In order to represent the robotic manipulator system equations shown earlier in a DAR format such as (5), it would be necessary to multiply both sides of (1) by the inverse of the inertia matrix $M(\theta)$, so as to isolate the angular acceleration vector component $\dot{\omega}$, a procedure that would significantly increase the complexity of the system dynamics. Towards avoiding this difficulty, we propose in this paper a new DAR structure with descriptor components, which will allow to directly address (1) without the complete inversion of $M(\theta)$. For instance, our DAR is expanded as

$$\begin{cases} A_0 \dot{x} = A_1(x, \delta) \, x + A_2(x, \delta) \, \xi + A_3(x, \delta) \, \dot{x} + B(x, \delta) \, u \\ 0 = \Omega_1(x, \delta) \, x + \Omega_2(x, \delta) \, \xi \end{cases} \qquad (6)$$

where the new matrices $A_0, A_3(x, \delta) \in \mathbb{R}^{n_x \times n_x}$ can be employed to easily deal with inertia term $M(\theta)$ term in (1).

The goal now is to express the system equations from (1) in the form of (6). In this sense, we consider the state variable

$$x = \begin{bmatrix} \theta_e \\ \omega \end{bmatrix} \in \mathbb{R}^4, \qquad (7)$$

which contains the angular velocities and angle errors of each link. In turn, the domain of interest related to these variables is specified by the polytope

$$\mathcal{X} = \left\{ x \in \mathbb{R}^4 : |\theta_{e_i}| \leq \overline{\theta}_{e_i}, |\omega_i| \leq \overline{\omega}_i, i = 1, 2 \right\}, \qquad (8)$$

where $\overline{\theta}_{e_i} > 0$ and $\overline{\omega}_i > 0$ are denoting the maximum admissible angular position error and velocity of the i-th

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[1] The term ξ is clearly a vector function of (x, δ), however this dependence will be omitted to simplify the notation in this paper.

manipulator link. In conjunction with these definitions, we consider a vector of time-varying parameters δ defined as

$$\delta = \begin{bmatrix} \sin(\theta_2) \\ \cos(\theta_2) \end{bmatrix} \in \mathbb{R}^2 \ , \qquad (9)$$

which is bounded inside the following polytope:

$$\Delta = \left\{ \delta \in \mathbb{R}^2 : |\delta_i| \leq 1, \ i = 1, 2 \right\} \ . \qquad (10)$$

At last, the DAR vector responsible for grouping the remaining rational nonlinearities is considered as

$$\xi = \begin{bmatrix} \delta_1 x_4 \\ \delta_1 x_3 \end{bmatrix} \in \mathbb{R}^2. \qquad (11)$$

Given these definitions for x, δ and ξ, the robotic manipulator system introduced in the previous section can be expressed in the descriptor DAR (6) with the following matrices:

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (m_1 + m_2)\ell_1^2 + m_2\ell_2^2 & \ell_2^2 m_2 \\ 0 & 0 & \ell_2^2 m_2 & \ell_2^2 m_2 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_2(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ m_2\ell_1\ell_2 x_4 + 2m_2\ell_1\ell_2 x_3 & 0 \\ 0 & -m_2\ell_1\ell_2 x_3 \end{bmatrix}, \qquad (12)$$

$$A_3(\delta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2m_2\ell_1\ell_2\delta_2 & -m_2\ell_1\ell_2\delta_2 \\ 0 & 0 & -m_2\ell_1\ell_2\delta_2 & 0 \end{bmatrix},$$

$$\Omega_1(\delta) = \begin{bmatrix} 0 & 0 & 0 & \delta_1 \\ 0 & 0 & \delta_1 & 0 \end{bmatrix}, \ \Omega_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By now using an augmented vector $\xi_a = [\xi^\mathsf{T} \ \dot{x}^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^6$ that combines ξ with $\dot{x}$, we can show that the descriptor DAR from (6) can be re-arranged to appear as a traditional DAR, such as the one in (5). In this case, we simply need to include an extra equality constraint related to state derivatives, i.e., $0 = A_1(\delta) x + A_2(x) \xi + (A_3(\delta) - A_0) \dot{x} + B u$. Moreover, since all of the descriptor nonlinearities were grouped into $A_3(\delta) \dot{x}$, the invariant descriptor component $A_0 \dot{x}$ can be cancelled out by inversion of A_0, which is clearly non-singular from (12). This process allows one to re-write (6) as

$$\begin{cases} \dot{x} = \mathbf{A_1}(\delta) x + \mathbf{A_2}(x, \delta) \xi_a + \mathbf{B} u \\ 0 = \mathbf{\Omega_1}(\delta) x + \mathbf{\Omega_2}(x, \delta) \xi_a + \mathbf{\Omega_3} u \end{cases} \qquad (13)$$

where the augmented matrices shown in here are constructed in the following manner:

$$\mathbf{A_1}(\delta) = A_0^{-1} A_1(\delta), \quad \mathbf{A_2}(x, \delta) = A_0^{-1} \begin{bmatrix} A_2(x) & A_3(\delta) \end{bmatrix},$$

$$\mathbf{\Omega_1}(\delta) = \begin{bmatrix} \Omega_1(\delta) \\ A_1(\delta) \end{bmatrix}, \quad \mathbf{\Omega_2}(x, \delta) = \begin{bmatrix} \Omega_2 & 0 \\ A_2(x) & A_3(\delta) - A_0 \end{bmatrix},$$

$$\mathbf{B} = A_0^{-1} B, \quad \mathbf{\Omega_3} = \begin{bmatrix} 0 \\ B \end{bmatrix}. \qquad (14)$$

4. CONTROL DESIGN

This section provides a systematic design approach able to design a feedback controller for the robotic manipulator system presented in Section 2, where the proposed DAR developed in Section 3 will be used to deal with the nonlinear dynamics. Beyond dealing with the nonlinearities in the equations of motion, we will complementary address the input saturation of the actuators, which is an inherent effect of every practical application. It is important to emphasize that this methodology will provide rigorous stability and performance guarantees, without resorting to any kind of linearization or approximation.

The robotic manipulator controller to be considered in here is a state-feedback law subject to saturation, i.e.

$$u = \text{sat}(Kx) \ , \qquad (15)$$

where $u \in \mathbb{R}^2$ denotes saturated control vector to be delivered to the plant, the state vector $x \in \mathbb{R}^4$ is as in (7) and the gain matrix $K \in \mathbb{R}^{2 \times 4}$ is a free design term to be synthesized. In turn, the nonlinear function $\text{sat}(\cdot)$ is defined as

$$\text{sat}(\mu_i) \triangleq \min \left\{ \max\{\mu_i, -\overline{u}_i\} , \overline{u}_i \right\} \ , \qquad (16)$$

where $\overline{u}_i$ represents the maximum input torque of the i-th robot link. In order to deal with this saturation effect in the control loop, we consider an approach similar to Tarbouriech et al. (2011), where the saturation function is re-expressed in terms of a dead-zone type nonlinearity:

$$\varphi(\mu) \triangleq \mu - \text{sat}(\mu). \qquad (17)$$

In this case, it follows that closed-loop system formed by (13) and (15) can be written as

$$\begin{cases} \dot{x} = (\mathbf{A_1}(\delta) + \mathbf{B}K) x + \mathbf{A_2}(x, \delta) \xi_a - \mathbf{B} \varphi(Kx) \\ 0 = (\mathbf{\Omega_1}(\delta) + \mathbf{\Omega_3}K) x + \mathbf{\Omega_2}(x, \delta) \xi_a - \mathbf{\Omega_3} \varphi(Kx) \end{cases}. \qquad (18)$$

Our primary control objective is to ensure the asymptotic stabilization of the trajectories $x(t)$ of system (18) with respect to the origin for a given set of admissible initial conditions $x(0) \in \mathcal{R}$, recalling that $\|x\| \to 0$ means that the manipulator configuration approaches and settles in a prescribed reference attitude defined by the reference angles. In this sense, we want to systematically synthesize a feedback matrix K such that a domain of attraction estimate $\mathcal{R}$ is made as large as possible. Beyond this primary goal, we also want to ensure that the trajectories $x(t)$ converge exponentially with a decay rate greater then a pre-determined value $\lambda > 0$. This is equivalent to guaranteeing that $\exists \beta > 0$:

$$\|x(t)\| \leq \beta \|x(0)\| \, e^{-\lambda t} \ \forall t \geq 0 \, , \ \forall x(0) \in \mathcal{R} \ . \qquad (19)$$

Towards addressing the established control objectives, we will utilize the following lemma.

Lemma 1. (Tarbouriech et al., 2011) Consider matrices K, $R : \mathbb{R}^{n_x} \to \mathbb{R}^{n_u}$ and a dead-zone nonlinearity defined as in (17). If $x \in \mathcal{S}$, where $\mathcal{S}$ is the polyhedral set

$$\mathcal{S} = \left\{ x \in \mathbb{R}^{n_u} : |(K_{[i]} - R_{[i]}) x| \leq \overline{u}_i, i = 1, \dots, n_u \right\}, \qquad (20)$$

then it verifies that

$$\varphi^\mathsf{T}(Kx) \, T \, (\varphi(Kx) - Rx) \leq 0 \qquad (21)$$

for any diagonal and positive-definite matrix $T \in \mathbb{R}^{n_u \times n_u}$.

Prior to showing our results, it is also convenient to re-express the domain of interest $\mathcal{X}$ from (8) in a standard mathematical form such as

$$\mathcal{X} = \left\{ x \in \mathbb{R}^4 : |\alpha_k x_k| \leq 1 \,, k = 1, 2, 3, 4 \right\}, \quad (22)$$

where the vectors $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}^{1 \times 4}$ are defined by

$$\alpha_1 = [\overline{\theta}_{e_1}^{-1} \ \ 0 \ \ 0 \ \ 0], \ \alpha_3 = [0 \ \ 0 \ \ \overline{\omega}_1^{-1} \ \ 0], \\ \alpha_2 = [0 \ \ \overline{\theta}_{e_2}^{-1} \ \ 0 \ \ 0], \ \alpha_4 = [0 \ \ 0 \ \ 0 \ \ \overline{\omega}_2^{-1}]. \quad (23)$$

Our main result from Theorem 2 provides a set of LMI conditions that allows one to synthesize the feedback matrix K of the robotic manipulator controller (15) by a convex optimization problem, which will be shown in the end of this section.

Theorem 2. Suppose that there exist a symmetric matrix $\hat{P} \in \mathbb{R}^{4 \times 4}$, a diagonal matrix $\hat{T} \in \mathbb{R}^{2 \times 2}$ and generic matrices $\hat{L} \in \mathbb{R}^{6 \times 6}$ and $\hat{K}, \hat{R} \in \mathbb{R}^{2 \times 4}$ such that:

$$\hat{P} > 0\,, \quad \hat{T} > 0\,, \quad (24)$$

$$\begin{bmatrix} 1 & \alpha_k \hat{P} \\ \star & \hat{P} \end{bmatrix} > 0 \ \forall\, k = 1, 2, 3, 4\,, \quad (25)$$

$$\begin{bmatrix} \overline{u}_i^2 & \hat{K}_{[i]} - \hat{R}_{[i]} \\ \star & \hat{P} \end{bmatrix} > 0 \ \forall\, i = 1, 2\,, \quad (26)$$

$$\text{He} \left\{ \begin{bmatrix} \mathbf{A_1}(\delta)\hat{P} + \lambda\hat{P} + \mathbf{B}\hat{K} & \mathbf{A_2}(x, \delta)\hat{L} & -\mathbf{B}\hat{T} \\ \mathbf{\Omega_1}(\delta)\hat{P} + \mathbf{\Omega_3}\hat{K} & \mathbf{\Omega_2}(x, \delta)\hat{L} & -\mathbf{\Omega_3}\hat{T} \\ \hat{R} & 0 & -\hat{T} \end{bmatrix} \right\} < 0\,, \quad (27)$$

$\forall\, (x, \delta) \in \mathcal{V}(\mathcal{X}) \times \mathcal{V}(\Delta)$. Then the trajectories $x(t)$ of the closed-loop system (18) with $K = \hat{K}\hat{P}^{-1}$ exponentially approach the origin with decay rate greater then λ for every initial condition $x(0)$ in

$$\mathcal{R} = \left\{ x \in \mathbb{R}^4 : x^\mathsf{T} P x \leq 1 \right\}, \quad P \triangleq \hat{P}^{-1}\,. \quad (28)$$

Proof. Consider the Lyapunov candidate function

$$V(x) = x^\mathsf{T} P x\,, \quad (29)$$

where $P \in \mathbb{R}^{4 \times 4}$ is a symmetric and positive-definite matrix. By differentiating this function along the trajectories of system (18), one obtains $\dot{V}(x, \delta) = \text{He}\{x^\mathsf{T}\Pi_1(x, \delta)\zeta\}$, where $\Pi_1(x, \delta)$ and ζ are defined as:

$$\Pi_1(x, \delta) \triangleq \left[P(\mathbf{A_1}(\delta) + \mathbf{B}\hat{K}) \ P\mathbf{A_2}(x, \delta) \ -P\mathbf{B} \right], \\ \zeta \triangleq \left[x^\mathsf{T} \ \xi_a^\mathsf{T} \ \varphi^\mathsf{T}(Kx) \right]^\mathsf{T}. \quad (30)$$

Observe that, for any matrix $L \in \mathbb{R}^{6 \times 6}$, the equality constraint in (18) implies that $0 = \text{He}\{\xi_a^\mathsf{T}\Pi_2(x, \delta)\zeta\}$ is true, where $\Pi_2(x, \delta)$ here denotes:

$$\Pi_2(x, \delta) \triangleq \left[L(\mathbf{\Omega_1}(\delta) + \mathbf{\Omega_3}\hat{K}) \ L\mathbf{\Omega_2}(x, \delta) \ -L\mathbf{\Omega_3} \right]. \quad (31)$$

Consider the Lemma 1 for $n_x = 4$, $n_u = 2$ and assume that $x \in \mathcal{S}$, where $\mathcal{S}$ is the polyhedral set introduced in (20). In this case, for any diagonal and positive-definite $T \in \mathbb{R}^{2 \times 2}$, relation (21) is verified, which is in turn equivalent to $\text{He}\{\varphi^\mathsf{T}(Kx)\Pi_3\zeta\} \geq 0$, where

$$\Pi_3 \triangleq \left[TR \ 0 \ -T \right]. \quad (32)$$

Now suppose the following holds $\forall\, (x, \delta) \in \mathcal{X} \times \Delta$:

$$\text{He}\{x^\mathsf{T}\Pi_1(x, \delta)\zeta + \xi_a^\mathsf{T}\Pi_2(x, \delta)\zeta + \dots \\ \dots + \varphi^\mathsf{T}(Kx)\Pi_3\zeta + x^\mathsf{T}\lambda P x\} < 0\,, \quad (33)$$

which can be developed into

$$\text{He} \left\{ \begin{bmatrix} P(\mathbf{A_1}(\delta) + \mathbf{B}\hat{K}) + \lambda P & P\mathbf{A_2}(x, \delta) & -P\mathbf{B} \\ L(\mathbf{\Omega_1}(\delta) + \mathbf{\Omega_3}\hat{K}) & L\mathbf{\Omega_2}(x, \delta) & -L\mathbf{\Omega_3} \\ TR & 0 & -T \end{bmatrix} \right\} < 0. \quad (34)$$

Since $\text{He}\{x^\mathsf{T}\Pi_1(x, \delta)\zeta\} = \dot{V}(x, \delta)$, $\text{He}\{\xi_a^\mathsf{T}\Pi_2(x, \delta)\zeta\} = 0$, $\text{He}\{\varphi^\mathsf{T}(Kx)\Pi_3\zeta\} \geq 0$ and $\lambda > 0$, it follows from expression (34) that the derivative of the Lyapunov function (29) is negative-definite inside the domain $(\mathcal{X} \cap \mathcal{S}) \times \Delta$, i.e.

$$\dot{V}(x, \delta) < -2\lambda V(x) < 0 \ \forall\, x \in (\mathcal{X} \cap \mathcal{S}) - \{0\}\,, \forall\, \delta \in \Delta\,. \quad (35)$$

Our idea is therefore to consider an invariant domain of attraction estimate $\mathcal{R}$ as a level set of the candidate Lyapunov function, for instance $\mathcal{R} = \{x \in \mathbb{R}^4 : x^\mathsf{T}Px \leq 1\}$, such that $\mathcal{R} \subset \mathcal{X}$ and $\mathcal{R} \subset \mathcal{S}$, so as to ensure validity of Lemma 1.

Based on the definition of $\mathcal{X}$ in (22), in order to ensure that $\mathcal{R} \subset \mathcal{X}$, it is necessary and sufficient to guarantee that $x^\mathsf{T}\alpha_k^\mathsf{T}\alpha_k x < x^\mathsf{T}Px \leq 1 \Leftrightarrow P - \alpha_k^\mathsf{T}\alpha_k > 0 \ \forall\, k = 1, 2, 3, 4$. Hence, from Schür's complement, the last relation can be equivalently stated as

$$\begin{bmatrix} 1 & \alpha_k \\ \star & P \end{bmatrix} > 0 \ \forall\, k = 1, 2, 3, 4\,. \quad (36)$$

Likewise, the inclusion condition $\mathcal{R} \subset \mathcal{S}$ holds if and only if $x^\mathsf{T}(K_{[i]} - R_{[i]})^\mathsf{T}(K_{[i]} - R_{[i]})x < x^\mathsf{T}Px^\mathsf{T} \leq 1 \Leftrightarrow P - (K_{[i]} - R_{[i]})^\mathsf{T}(K_{[i]} - R_{[i]}) > 0 \Leftrightarrow$

$$\begin{bmatrix} 1 & K_{[i]} - R_{[i]} \\ \star & P \end{bmatrix} > 0 \ \forall\, i = 1, 2\,. \quad (37)$$

Consequently, if conditions the conditions $P > 0$, $T > 0$, (36), (37) and (34) are satisfied, the trajectories $x(t)$ of the closed-loop system (18) asymptotically approach the origin for every initial condition $x(0) \in \mathcal{R}$. Moreover, from relation (35), it is also noticeable that

$$V(x(t)) < V(x(0))\,\mathrm{e}^{-2\lambda t} \ \forall\, x(0) \in \mathcal{R}\,. \quad (38)$$

Observe that $\gamma_{min}\|x\|^2 \leq V(x) \leq \gamma_{max}\|x\|^2$, where γ_{min} and γ_{max} denote the smallest and largest eigenvalue of P respectively. Thus, it also follows that the exponential performance criterion (19) holds with

$$\beta = \sqrt{\frac{\gamma_{max}}{\gamma_{min}}} > 0\,. \quad (39)$$

By then pre- and post-multiplying the matrix inequalities $P > 0$, $T > 0$, (36), (37) and (34) respectively by P^{-1}, T^{-1}, $\text{diag}\{1, P^{-1}\}$, $\text{diag}\{1, P^{-1}\}$, $\text{diag}\{P^{-1}, L^{-1}, T^{-1}\}$, and their transposes, one should finally obtain the conditions presented in (24), (25), (26) and (27) when considering the change of variables $\hat{P} \triangleq P^{-1}$, $\hat{K} \triangleq KP^{-1}$, $\hat{R} \triangleq RP^{-1}$, $\hat{L} \triangleq L^{\mathsf{T}-1}$ and $\hat{T} \triangleq T^{-1}$.

To conclude the proof, if the LMI (27) is satisfied just for (x, δ) at the vertices $\mathcal{V}(\mathcal{X}) \times \mathcal{V}(\Delta)$, by convexity they are also satisfied $\forall\, (x, \delta) \in \mathcal{X} \times \Delta$. $\square$

In order to maximize the size of the domain of attraction estimate $\mathcal{R}$, one should minimize the trace of $\hat{P}^{-1}$ subject to the LMIs from Theorem 2, which is clearly an optimization with a non-convex objective function. However, this optimization problem can be easily convexified by considering a new symmetric matrix $X \in \mathbb{R}^{4 \times 4}$ which

satisfies $X > \hat{P}^{-1}$, or equivalently by Schür's complement:

$$\begin{bmatrix} X & I \\ \star & \hat{P} \end{bmatrix} \succ 0 \ . \tag{40}$$

The optimal control design which maximizes the domain of attraction estimate $\mathcal{R}$ can therefore be synthesized by the following convex optimization:

$$\underset{X,\hat{P},\hat{T},\hat{L},\hat{K},\hat{R}}{\text{minimize}} \ \text{tr}(X) \ \text{s.t.} \ \left\{ (24),\,(25),\,(26),\,(27),\,(40) \right\} . \tag{41}$$

5. NUMERICAL EXAMPLE

This section presents a numerical simulation of a robotic manipulator control system in order to illustrate the contribution presented in this paper. The numerical results were obtained in the MATLAB R2012b software and its native LMILAB toolbox was employed to solve the proposed convex optimization problem with LMI constraints.

An ideal two-link manipulator is here considered with dynamics governed by equation (1), where the constructive parameters are $m_1 = 10\,\text{kg}$, $m_2 = 1\,\text{kg}$, $\ell_1 = 1\,\text{m}$, $\ell_2 = 1\,\text{m}$, $\overline{u}_1 = 10^3\,\text{Nm}$ and $\overline{u}_2 = 10^2\,\text{Nm}$. Towards employing the proposed control synthesis approach, the target domain of interest $\mathcal{X}$ was defined with $\overline{\theta}_{e_1} = 100°$, $\overline{\theta}_{e_2} = 100°$, $\overline{\omega}_1 = 3\,\text{rad/s}$ and $\overline{\omega}_2 = 10\,\text{rad/s}$. At last, the minimum exponential convergence rate criterion, as defined in (19), was set with $\lambda = 0.6$. Given these setup parameters, the proposed convex optimization problem (41) yielded the feedback matrix

$$K = \begin{bmatrix} -343.36 & -9.9162 & -530.36 & 35.671 \\ 108.34 & -110.01 & 185.77 & -139.12 \end{bmatrix}, \tag{42}$$

in which case the domain of attraction estimate (28) is defined with

$$P = \begin{bmatrix} 0.3770 & -0.0000 & 0.0788 & 0.0000 \\ -0.0000 & 0.3327 & -0.0001 & 0.0067 \\ 0.0788 & -0.0001 & 0.1276 & 0.0000 \\ 0.0000 & 0.0067 & 0.0000 & 0.0101 \end{bmatrix}, \tag{43}$$

where $\text{tr}(P) = 0.8474$.

Fig. 2 shows, in bold line, two contour slices of the ellipsoidal set $\mathcal{R}$ defined by the matrix P obtained in this numerical design example. One should recall that this region $\mathcal{R}$ represents the set of admissible initial states, for which our synthesized controller is guaranteed to asymptotically stabilize the system trajectories within the specified exponential convergence rate criterion. Also in Fig. 2, the dashed-dotted and dotted bars denote the control input saturation borderlines, where $u_1 = \overline{u}_1$ and $u_2 = \overline{u}_2$ respectively. This graphical analysis demonstrates that the proposed method is capable of providing stability and performance guarantees even for initial conditions that saturates both of the input signals.

Fig. 3 shows the time series produced by a numerical simulation of the closed-loop system with the controller designed by the proposed methodology. The target reference attitude angles were defined as $\theta_{r_1} = 120°$ and $\theta_{r_2} = -90°$ and the initial manipulator angles as $\theta_1(0) = 150°$ and $\theta_2(0) = -60°$, yielding the initial error $\theta_{e_i}(0) = x_i(0) = 30°$ for both joint angles $i = 1,2$. In turn, the initial joint angular velocities were considered as $\omega_1(0) = x_3(0) = 2\,\text{rad/s}$ and $\omega_2(0) = x_4(0) = -3\,\text{rad/s}$. This system initial
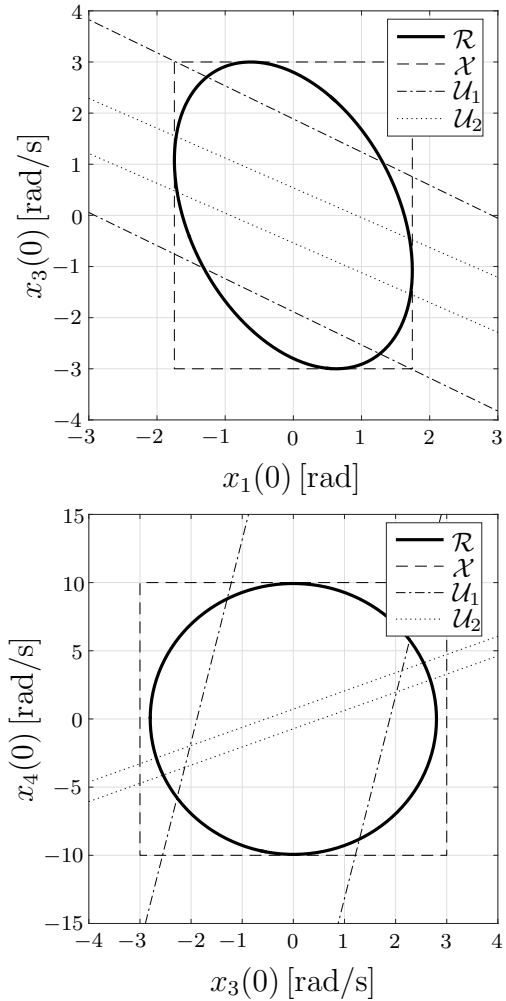

Fig. 2. Slices of the estimated domain of attraction $\mathcal{R}$ compared to the validity polytope $\mathcal{X}$ and the control input saturation regions $\mathcal{U}_i \triangleq \{x \in \mathbb{R}^4 : |K_{[i]}x| \leq \overline{u}_i\}$.

state is marginally close to the border of the domain of attraction estimate $\mathcal{R}$.

6. CONCLUSION AND PERSPECTIVES

This paper proposed a systematic state-feedback control design method for robotic manipulator systems with input saturation, where the feedback gain matrix is synthesized by means of a convex optimization problem subject to LMI constraints. Towards this final result, one important advancement was the development of a new DAR with descriptor components, which allowed to easily represent the dynamics of a manipulator system.

Since this paper was mainly focused on the model of a planar two-link robotic manipulator, future works might explore our methodologies for higher order mechanisms. Beyond that, an interesting and relevant future study proposal is the output regulation of robotic manipulators, where some developments from Castro et al. (2019) may be employed. In this sense, one could investigate dynamic trajectory tracking and dealing with exogenous disturbances forces due to load or gravity.
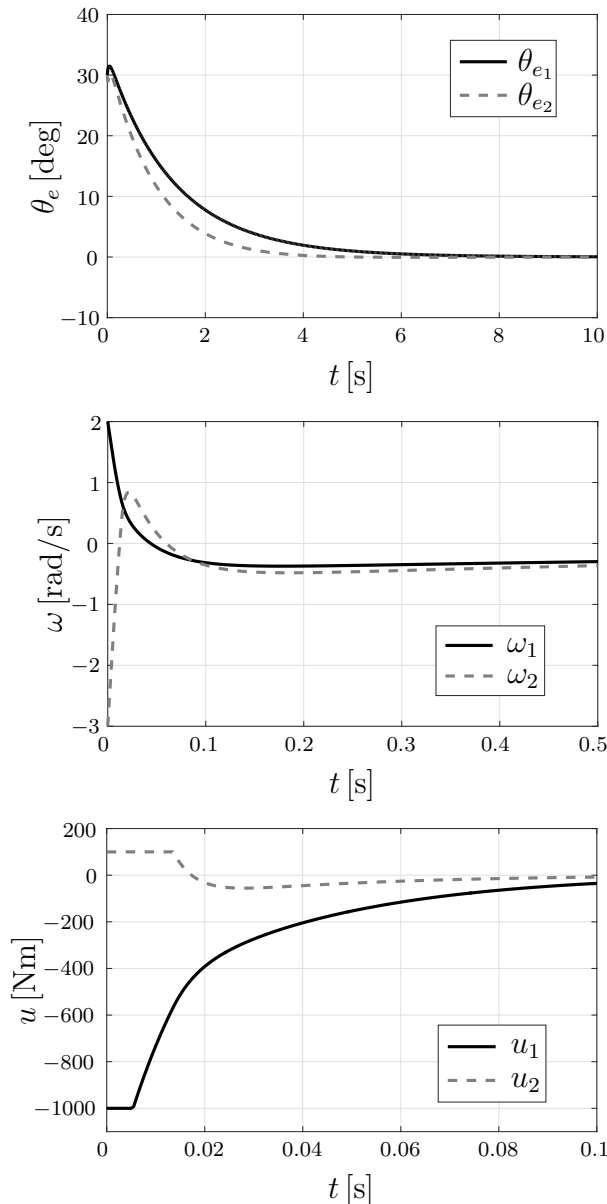
Fig. 3. Time series of the error angles $\theta_e(t)$, angular velocities $\omega(t)$ and control inputs $u(t)$.

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