Interval Observer for Discrete Periodic Time-varying Descriptor Systems ⋆

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Abstract: This paper designs an interval observer for discrete linear periodic time-varying descriptor (LPTVD) systems. First, the discrete LPTVD system is transformed into a linear time-invariant (LTI) descriptor system by using a stacked form of periodic systems. Then, in order to reduce complexity, we further transform the equivalent LTI descriptor system into a minimal-order form. Finally, by using the obtained minimal-order implementation of the discrete LPTVD system, an interval observer is designed to estimate the system state interval at each step. At the end, an example is used to illustrate the effectiveness of the proposed results.

Keywords: Interval observer, system equivalent transformation, discrete periodic time-varying descriptor systems, robust state estimation, bounded uncertainties.

1. INTRODUCTION

State estimation is an important topic in control theory and applications, which plays a key role in controller design, fault diagnosis and so on. In order to handle robust state estimation problems under effects of uncertainties such as disturbances, parametric uncertainties and measurement noises, two well-known types of observers can be found in the literature, i.e., interval observers and set-valued observers. Both observers are based on a classical assumption that uncertainties are bounded and their bounds are known, which both have their own advantages and disadvantages. Generally, the interval observers employ the lower and upper bounds of uncertainties to design two point-wise observers to estimate the lower and upper bounds of states, respectively Efimov et al. (2013a,b); Gouzé et al. (2000); Mazenc and Bernard (2011), while in a different way, the set-valued observers obtain robust state estimations by propagating the sets of uncertainties through a system model to generate state estimation sets of actual system states Xu et al. (2014, 2019a,b).

This paper focuses on the design problem of interval observers. Actually, the design methods of interval observers are widely investigated for different types of systems, which are generally based on system positivity. Particularly, the key to design interval observers is to find a coordinate transformation such that the dynamics of state estimation error is transformed into a cooperative one and then interval observers are designed. In the literature, the design methods of interval observers were proposed for linear time-invariant (LTI) systems (Mazenc and Bernard (2010, 2011) for continuous systems and Efimov et al. (2013b); Wang et al. (2018) for discrete systems). In Raïssi et al. (2012), the interval observers were designed for a class of nonlinear system using partial exact linearizations. In Krebs et al. (2016); Wang et al. (2015), the design problems of interval observers for linear parameter varying (LPV) systems were investigated. Thabet et al. (2014) proposed a method to design the interval observer for continuous linear time-varying (LTV) systems. In Efimov et al. (2013a), the design method of interval observers for discrete LTV systems was presented. However, the method in Efimov et al. (2013a) only gave a way to design a constant coordinate transformation to transform the discrete time-varying system matrix into a nonnegative matrix, which requires that the time-varying system matrix is bounded and that the bound is sufficiently thin. Moreover, under the notion of reducible discrete LTV systems, the method was extended to design an interval observer for periodic LTV systems. However, the reducible condition is a limit to general periodic LTV systems.

In real situations, many physical systems can be modeled as systems of differential and algebraic equations, which are called descriptor systems, singular systems or differential algebraic equations. However, many descriptor systems are nonlinear or time-varying. If using time-invariant linearizations to simplify descriptor systems, we sometimes result in incorrect approximations. Moreover, nonlinear systems, if correctly linearized along a trajectory, naturally result in linear time-varying descriptor systems. Within the knowledge scope of the authors, the design of interval observers for the discrete LPTVD systems is a relatively open issue. Thus, we are motivated to design interval observers for the discrete LPTVD systems for robust state estimation of such systems in this paper.

This paper transforms the discrete LPTVD system into a discrete LTI descriptor system by using a stacked form

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of periodic systems. Then, by using a series of system equivalence transformations, the discrete LTI descriptor system is transformed and decomposed into a minimal-order realization composed of a discrete LTI dynamic subsystem and a discrete static subsystem. Consequently, an interval observer is designed for the discrete LPTVD system based on the minimal-order implementation, which has relatively low complexity and can be widely used for robust state estimation of discrete LPTVD systems.

In the remaining part of this paper, Section II introduces preliminary knowledge. The main results on the design of interval observer for the LPTVD systems are presented in Section III. Section IV uses a numerical example to illustrate the proposed interval observer. The paper is finally concluded in Section V.

2. PRELIMINARIES AND SYSTEM MODEL

2.1 Preliminaries

The compatible zero matrix is denoted as O, the n-dimensional identity matrix is denoted as I_n and $blockdiag\{\}$ denotes the diagonal matrix full of block matrix elements. The vector inequalities are understood element-wise.

Definition 1. For a matrix A, we define $A^+_{i,j} = max(0, A_{i,j})$, where $A^+_{i,j}$ and $A_{i,j}$ are the i-th row and j-th column elements of A^+ and A, respectively. Moreover, corresponding to the matrix A^+, we further define $A^- = A^+ - A$.

Definition 2. A matrix A is called nonnegative if all its elements are nonnegative.

Lemma 1. Given a non-autonomous discrete-time system
$$x_{k+1} = Ax_k + b_k,$$
where A is nonnegative and $b_k \geq 0$. If the initial condition $x_0 \geq 0$ is given, one always has $x_k \geq 0$ for all $k > 0$

Lemma 2. For a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $x \in \mathbb{R}^n$ with $\underline{x} \leq x \leq \overline{x}$, one has
$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \tag{1}$$

Lemma 3. For $A \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, if there is a matrix $L \in \mathbb{R}^{n\times n}$ such that $A - LC$ and R have the same eigenvalues, then there is a matrix $S \in \mathbb{R}^{n\times n}$ such that $R = S(A - LC)S^{-1}$ provided that the pairs $(A-LC, e_1)$ and (R, e_2) are observable for some $e_1 \in \mathbb{R}^{1\times n}$ and $e_2 \in \mathbb{R}^{1\times n}$.

Lemma 3 is used to design a matrix S such that $R = S(A-LC)S^{-1}$ is nonnegative. The details on Lemma 3 could be found in Efimov et al. (2013b) and Raïssi et al. (2012).

2.2 System Model

This paper considers the discrete LPTVD system
$$G_{k+1}x_{k+1} = A_kx_k + B_ku_k + E_k\nu_k, \tag{2a}$$
$$y_k = C_kx_k + F_k\eta_k, \tag{2b}$$
where k denotes the k-th time instant, $G_{k+1} \in \mathbb{R}^{n\times n}$, $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times p}$, $E_k \in \mathbb{R}^{n\times r}$, $C_k \in \mathbb{R}^{q\times n}$ and $F_k \in \mathbb{R}^{q\times s}$ are time-varying periodic parametric matrices of period ω, $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^q$ denote the state and output vectors, $u_k \in \mathbb{R}^p$ represents the control input vector, $\nu_k \in \mathbb{R}^r$ represents the unknown input vector (including disturbances, modeling errors, etc.), and $\eta_k \in \mathbb{R}^s$ is the measurement noise vector. In this paper, the system (2) is called an ω-periodic descriptor system.

Assumption 1. w_k and η_k are bounded by
$$\mathcal{W} = \{\nu_k \in \mathbb{R}^r : \underline{\nu} \leq \nu_k \leq \overline{\nu}\},$$
$$\mathcal{V} = \{\eta_k \in \mathbb{R}^s : \underline{\eta} \leq \eta_k \leq \overline{\eta}\},$$
respectively, where $\underline{\nu}$, $\overline{\nu}$, $\underline{\eta}$ and $\overline{\eta}$ are constant vectors.

3. INTERVAL OBSERVER FOR LPTVD SYSTEMS

3.1 Equivalence of ω-stacked Form

The periodic system (2) can be equivalently rewritten into the following dynamics:
$$\mathcal{G}\mathcal{R}(\lambda)\mathbf{x}_k(h) = \mathcal{A}\mathbf{x}_k(h) + \mathcal{B}\mathbf{u}_k(h) + \mathcal{E}\nu_k(h), \tag{3a}$$
$$\mathbf{y}_k(h) = \mathcal{C}\mathbf{x}_k(h) + \mathcal{D}\mathbf{u}_k(h) + \mathcal{F}\eta_k(h), \tag{3b}$$
with
$$\mathcal{R}(\lambda) = \begin{bmatrix} O & I_{(\omega-1)n} \\ \lambda I & O \end{bmatrix}, \tag{4a}$$
$$\mathcal{G} = block\ diag\{G_{k+1},\ G_{k+2}, \cdots,\ G_{k+\omega}\}, \tag{4b}$$
$$\mathcal{A} = block\ diag\{A_k,\ A_{k+1}, \cdots,\ A_{k+\omega-1}\}, \tag{4c}$$
$$\mathcal{B} = block\ diag\{B_k,\ B_{k+1}, \cdots,\ B_{k+\omega-1}\}, \tag{4d}$$
$$\mathcal{C} = block\ diag\{C_k,\ C_{k+1}, \cdots,\ C_{k+\omega-1}\}, \tag{4e}$$
$$\mathcal{D} = block\ diag\{D_k,\ D_{k+1}, \cdots,\ D_{k+\omega-1}\}, \tag{4f}$$
$$\mathcal{E} = block\ diag\{E_k,\ E_{k+1}, \cdots,\ E_{k+\omega-1}\}, \tag{4g}$$
$$\mathcal{F} = block\ diag\{F_k,\ F_{k+1}, \cdots,\ F_{k+\omega-1}\} \tag{4h}$$
and
$$\mathbf{u}_k(h) = [u^T(k+h\omega)\ u^T(k+h\omega+1) \tag{5a}$$
$$\cdots u^T(k+h\omega+\omega-1)]^T, \tag{5b}$$
$$\mathbf{x}_k(h) = [x^T(k+h\omega)\ x^T(k+h\omega+1) \tag{5c}$$
$$\cdots x^T(k+h\omega+\omega-1)]^T, \tag{5d}$$
$$\mathbf{y}_k(h) = [y^T(k+h\omega)\ y^T(k+h\omega+1) \tag{5e}$$
$$\cdots y^T(k+h\omega+\omega-1)]^T, \tag{5f}$$
$$\nu_k(h) = [\nu^T(k+h\omega)\ \nu^T(k+h\omega+1) \tag{5g}$$
$$\cdots \nu^T(k+h\omega+\omega-1)]^T, \tag{5h}$$
$$\eta_k(h) = [\eta^T(k+h\omega)\ \eta^T(k+h\omega+1) \tag{5i}$$
$$\cdots \eta^T(k+h\omega+\omega-1)]^T, \tag{5j}$$
where h denotes the number of periods, and λ denotes the one-step forward time operator in the variable h or the ω-step forward time operator in the variable k. Since (2) is periodic, $\mathcal{G}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are constant, which implies that the system (2) is transformed into an equivalent linear time-invariant (LTI) descriptor system (3).

3.2 System Transformation

The method in Misra (1996) is used to transform the system (3) into a time-invariant system. Thus, we further rewrite the parametric matrix $\mathcal{G}$ in (3) into
$$\mathcal{G} = \begin{bmatrix} \mathcal{G}_{11} & O \\ O & \mathcal{G}_{22} \end{bmatrix}, \tag{6}$$
where $\mathcal{G}_{11} = block\ diag\{G_{k+1},\ G_{k+2}, \cdots,\ G_{k+\omega-1}\} \in \mathbb{R}^{(\omega-1)n\times(\omega-1)n}$ and $\mathcal{G}_{22} = G_{k+\omega}$. This implies that
$$\mathcal{G}\mathcal{R}(\lambda) = \begin{bmatrix} O & \mathcal{G}_{11} \\ \lambda\mathcal{G}_{22} & O \end{bmatrix}. \tag{7}$$

By observing the structure of $\mathcal{GR}(\lambda)$ in (7), we could transform $\mathcal{GR}(\lambda)$ into a block diagonal matrix by premultiplying (3a) with a permutation matrix

$$\mathcal{P} = \begin{bmatrix} O & \cdots & O & I_n \\ I_n & O & O & O \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & I_n & O \end{bmatrix} \tag{8}$$

such that

$$\begin{aligned} \mathcal{P}\mathcal{GR}(\lambda) =& block\ diag\{\lambda G_{k+\omega}, G_{k+1}, \cdots, G_{k+\omega-1}\}, \\ \mathcal{P}\mathcal{A} =& \begin{bmatrix} O & \cdots & O & A_{k+\omega-1} \\ A_k & O & O & O \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & A_{k+\omega-2} & O \end{bmatrix}, \\ \mathcal{P}\mathcal{B} =& \begin{bmatrix} O & \cdots & O & B_{k+\omega-1} \\ B_k & O & O & O \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & B_{k+\omega-2} & O \end{bmatrix}, \\ \mathcal{P}\mathcal{E} =& \begin{bmatrix} O & \cdots & O & E_{k+\omega-1} \\ E_k & O & O & O \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & E_{k+\omega-2} & O \end{bmatrix}. \end{aligned} \tag{9}$$

Thus, the system (3a) can be further transformed into a new descriptor system:

$$\lambda\bar{\mathcal{G}}\mathbf{x}_k(h) = \bar{\mathcal{A}}\mathbf{x}_k(h) + \bar{\mathcal{B}}\mathbf{u}_k(h) + \bar{\mathcal{E}}\nu_k(h), \tag{10}$$

where

$$\bar{\mathcal{G}} = \begin{bmatrix} G_{k+\omega} & \cdots & O & O \\ O & O & O & O \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & O & O \end{bmatrix}, \bar{\mathcal{B}} = \mathcal{P}\mathcal{B}, \bar{\mathcal{E}} = \mathcal{P}\mathcal{E}, \tag{11}$$

$$\bar{\mathcal{A}} = \begin{bmatrix} O & \cdots & O & A_{k+\omega-1} \\ A_k & -G_{k+1} & O & O \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & A_{k+\omega-2} & -G_{k+\omega-1} \end{bmatrix}. \tag{12}$$

3.3 Minimal-Order System Implementation

In order to reduce (10) to obtain a minimal-order implementation for less complexity, we recall Fact 1.

Fact 1. (Misra and Patel (1989)). For a regular matrix pencil $\lambda G - A$, i.e., $\det(\lambda G - A) \not\equiv 0$ where $G \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}$, it can always find two orthogonal matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{n\times n}$ such that $U^T G V = \left[\begin{smallmatrix} G_{11} & G_{12} \\ O & O \end{smallmatrix}\right]$ and $U^T A V = \left[\begin{smallmatrix} A_{11} & A_{12} \\ O & A_{22} \end{smallmatrix}\right]$, where $G_{11} \in \mathbb{R}^{l\times l}$, $G_{12} \in \mathbb{R}^{l\times n-l}$, $A_{11} \in \mathbb{R}^{l\times l}$, $A_{12} \in \mathbb{R}^{l\times n-l}$ and $A_{22} \in \mathbb{R}^{n-l\times n-l}$.

Note that since the matrix pencil $\lambda G - A$ is assumed to be regular, it is concluded that the submatrix A_{22} is a full rank matrix, i.e., $\text{rank}(A_{22}) = n - l$. Thus, under Fact 1, we can find two orthogonal matrices $\bar{\mathcal{U}} \in \mathbb{R}^{n\times n}$, $\bar{\mathcal{V}} \in \mathbb{R}^{n\times n}$ and a coordinate transformation

$$\mathring{\mathbf{x}}_k(h) = \bar{\mathcal{V}}^{-1}\mathbf{x}_k(h)$$

to transform the system (10) into the following form:

$$\lambda\mathring{\mathcal{G}}\mathring{\mathbf{x}}_k(h) = \mathring{\mathcal{A}}\mathring{\mathbf{x}}_k(h) + \mathring{\mathcal{B}}\mathbf{u}_k(h) + \mathring{\mathcal{E}}\nu_k(h), \tag{13a}$$

$$\mathbf{y}_k(h) = \mathring{\mathcal{C}}\mathring{\mathbf{x}}_k(h) + \mathcal{D}\mathbf{u}_k(h) + \mathcal{F}\eta_k(h), \tag{13b}$$

where

$$\mathring{\mathcal{G}} = \bar{\mathcal{U}}^T\bar{\mathcal{G}}\bar{\mathcal{V}} = \begin{bmatrix} \mathring{\mathcal{G}}_{11} & \mathring{\mathcal{G}}_{12} \\ O & O \end{bmatrix}, \mathring{\mathcal{B}} = \bar{\mathcal{U}}^T\bar{\mathcal{B}} = \begin{bmatrix} \mathring{\mathcal{B}}_1 \\ \mathring{\mathcal{B}}_2 \end{bmatrix}, \tag{14a}$$

$$\mathring{\mathcal{A}} = \bar{\mathcal{U}}^T\bar{\mathcal{A}}\bar{\mathcal{V}} = \begin{bmatrix} \mathring{\mathcal{A}}_{11} & \mathring{\mathcal{A}}_{12} \\ O & \mathring{\mathcal{A}}_{22} \end{bmatrix}, \mathring{\mathcal{C}} = \mathcal{C}\bar{\mathcal{V}} = \begin{bmatrix} \mathring{\mathcal{C}}_1 & \mathring{\mathcal{C}}_2 \end{bmatrix}, \tag{14b}$$

$$\mathring{\mathcal{E}} = \bar{\mathcal{U}}^T\bar{\mathcal{E}} = \begin{bmatrix} \mathring{\mathcal{E}}_1 \\ \mathring{\mathcal{E}}_2 \end{bmatrix} \tag{14c}$$

with $\mathring{\mathcal{G}}_{11} \in \mathbb{R}^{l\times l}$ and $\mathring{\mathcal{A}}_{11} \in \mathbb{R}^{l\times l}$.

Corresponding to the system structure of (13) and (14), we define $\mathring{\mathbf{x}}_k(h) = \left[\mathring{\mathbf{x}}^T_{1,k}(h)\ \mathring{\mathbf{x}}^T_{2,k}(h)\right]^T$, where $\mathring{\mathbf{x}}_{1,k}(h) \in \mathbb{R}^l$ and $\mathring{\mathbf{x}}_{2,k}(h) \in \mathbb{R}^{n\omega-l}$. Thus, the system (13) can be equivalently rewritten into

$$\begin{aligned} \lambda\mathring{\mathcal{G}}_{11}\mathring{\mathbf{x}}_{1,k}(h) =& \mathring{\mathcal{A}}_{11}\mathring{\mathbf{x}}_{1,k}(h) + \mathring{\mathcal{A}}_{12}\mathring{\mathbf{x}}_{2,k}(h) - \lambda\mathring{\mathcal{G}}_{12}\mathring{\mathbf{x}}_{2,k}(h) \\ &+ \mathring{\mathcal{B}}_1\mathbf{u}_k(h) + \mathring{\mathcal{E}}_1\nu_k(h), \end{aligned} \tag{15a}$$

$$\mathbf{0} = \mathring{\mathcal{A}}_{22}\mathring{\mathbf{x}}_{2,k}(h) + \mathring{\mathcal{B}}_2\mathbf{u}_k(h) + \mathring{\mathcal{E}}_2\nu_k(h), \tag{15b}$$

$$\begin{aligned} \mathbf{y}_k(h) =& \mathring{\mathcal{C}}_1\mathring{\mathbf{x}}_{1,k}(h) + \mathring{\mathcal{C}}_2\mathring{\mathbf{x}}_{2,k}(h) + \mathcal{D}\mathbf{u}_k(h) \\ &+ \mathcal{F}\eta_k(h). \end{aligned} \tag{15c}$$

Under Fact 1, $\mathring{\mathcal{A}}_{22}$ is a full rank matrix, (15b) can be further transformed into

$$\mathring{\mathbf{x}}_{2,k}(h) = -\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h) - \mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2\nu_k(h), \tag{16a}$$

$$\lambda\mathring{\mathbf{x}}_{2,k}(h) = -\lambda\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h) - \lambda\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2\nu_k(h). \tag{16b}$$

By substituting (16) into (15), an equivalent form of the system (15) is obtained as

$$\begin{aligned} \lambda\mathring{\mathcal{G}}_{11}\mathring{\mathbf{x}}_{1,k}(h) =& \mathring{\mathcal{A}}_{11}\mathring{\mathbf{x}}_{1,k}(h) + (\mathring{\mathcal{B}}_1 - \mathring{\mathcal{A}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2)\mathbf{u}_k(h) + (\mathring{\mathcal{E}}_1 \\ &- \mathring{\mathcal{A}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)\nu_k(h) + \lambda\mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h) \\ &+ \lambda\mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2\nu_k(h), \\ \mathbf{y}_k(h) =& \mathring{\mathcal{C}}_1\mathring{\mathbf{x}}_{1,k}(h) + (\mathcal{D} - \mathring{\mathcal{C}}_2\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2)\mathbf{u}_k(h) \\ &- \mathring{\mathcal{C}}_2\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2\nu_k(h) + \mathcal{F}\eta_k(h). \end{aligned} \tag{17}$$

It can be observed that there exists a delay operation on $\mathbf{u}_k(h)$ in (17). Thus, it is necessary to further transform the system (17) to avoid the delay operation on $\mathbf{u}_k(h)$. Consequently, a coordinate transformation is defined as

$$\mathring{\mathcal{G}}_{11}\mathring{\mathbf{x}}_{1,k}(h) = \mathring{\mathcal{G}}_{11}\check{\mathbf{x}}_{1,k}(h) + \mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h), \tag{18a}$$

$$\lambda\mathring{\mathcal{G}}_{11}\mathring{\mathbf{x}}_{1,k}(h) = \lambda\mathring{\mathcal{G}}_{11}\check{\mathbf{x}}_{1,k}(h) + \lambda\mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h). \tag{18b}$$

Without loss of generality, only the case that $\mathring{\mathcal{G}}_{11}$ is nonsingular is considered here. Thus, with the coordinate transformation (18), the system (17) is be transformed into

$$\begin{aligned} \lambda\mathring{\mathcal{G}}_{11}\check{\mathbf{x}}_{1,k}(h) =& \mathring{\mathcal{A}}_{11}\check{\mathbf{x}}_{1,k}(h) + (\mathring{\mathcal{B}}_1 + \mathring{\mathcal{A}}_{11}\mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2 \\ &- \mathring{\mathcal{A}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2)\mathbf{u}_k(h) + \mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2\nu_k(h+1) \\ &+ (\mathring{\mathcal{E}}_1 - \mathring{\mathcal{A}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)\nu_k(h), \\ \mathbf{y}_k(h) =& \mathring{\mathcal{C}}_1\check{\mathbf{x}}_{1,k}(h) + (\mathcal{D} - \mathring{\mathcal{C}}_2\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)\nu_k(h) + \mathcal{F}\eta_k(h) \\ &+ (\mathring{\mathcal{C}}_1\mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2 - \mathring{\mathcal{C}}_2\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2)\mathbf{u}_k(h). \end{aligned} \tag{19}$$

Note that as long as the descriptor system (10) is regular, it is always able to transform (10) into a minimal-order nonsingular descriptor system in the form (19) with $\mathring{\mathcal{G}}_{11}$ nonsingular. Thus, in the case that $\mathring{\mathcal{G}}_{11}$ in (14) is singular, we could further find orthogonal transformation matrices to transform (19) into a nonsingular descriptor system in the same form (19) by following the same procedure above (see Misra and Patel (1989)). Thus, all the following analysis will be done based on the nonsingular case of (19) and the system (19) is equivalently transformed into

$$\check{\mathbf{x}}_{1,k}(h+1) = \check{\mathcal{A}}\check{\mathbf{x}}_{1,k}(h) + \check{\mathcal{B}}\mathbf{u}_k(h) + \check{\mathcal{E}}\check{\nu}_k(h), \tag{20a}$$

$$\mathbf{y}_k(h) = \check{\mathcal{C}}\check{\mathbf{x}}_{1,k}(h) + \check{\mathcal{D}}\mathbf{u}_k(h) + \check{\mathcal{F}}\check{\eta}_k(h). \tag{20b}$$

where

$$\begin{aligned}
&\check{\mathcal{A}} = \mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{A}}_{11},\ \check{\mathcal{C}} = \mathring{\mathcal{C}}_1,\ \ \check{\mathcal{F}} = \left[\,(\mathcal{D}-\mathring{\mathcal{C}}_2\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)\ \mathcal{F}\,\right],\\
&\check{\mathcal{B}} = \mathring{\mathcal{G}}_{11}^{-1}(\mathring{\mathcal{B}}_1 - \mathring{\mathcal{A}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2 + \mathring{\mathcal{A}}_{11}\mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2),\\
&\check{\mathcal{D}} = (\mathring{\mathcal{C}}_1\mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2 - \mathring{\mathcal{C}}_2\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2),\\
&\check{\mathcal{E}} = \left[\,\mathring{\mathcal{G}}_{11}^{-1}(\mathring{\mathcal{E}}_1-\mathring{\mathcal{A}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)\ \mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2\,\right],\\
&\check{\nu}_k(h) = \left[\nu_k(h)^T\ \ \nu_k(h+1)^T\right]^T,\\
&\check{\eta}_k(h) = \left[\nu_k(h)^T\ \ \eta_k(h)^T\right]^T.
\end{aligned} \tag{21}$$

Under Assumption 1, it is known that $\nu_k(h)$, $\check{\nu}_k(h)$ and $\check{\eta}_k(h)$ are also bounded, which are denoted as

$$\underline{\nu_h} \le \nu_k(h) \le \overline{\nu_h},\ \underline{\check{\nu}} \le \check{\nu}_k(h) \le \overline{\check{\nu}},\ \underline{\check{\eta}} \le \check{\eta}_k(h) \le \overline{\check{\eta}}$$

for all $h \ge 0$ and $k \ge 0$ with

$$\begin{aligned}
&\underline{\nu_h} = [\underbrace{\,\underline{\nu}^T \cdots \underline{\nu}^T\,}_{\omega}]^T,\ \overline{\nu_h} = [\underbrace{\,\overline{\nu}^T \cdots \overline{\nu}^T\,}_{\omega}]^T,\\
&\underline{\check{\nu}} = [\underbrace{\,\underline{\nu_h}^T\ \underline{\nu_h}^T\,}_{2}]^T,\ \overline{\check{\nu}} = [\underbrace{\,\overline{\nu_h}^T\ \overline{\nu_h}^T\,}_{2}]^T,\\
&\underline{\check{\eta}} = [\underbrace{\,\underline{\nu}^T \cdots \underline{\nu}^T\ \underline{\eta}^T \cdots \underline{\eta}^T\,}_{\omega+\omega}]^T,\ \overline{\check{\eta}} = [\underbrace{\,\overline{\nu}^T \cdots \overline{\nu}^T\ \overline{\eta}^T \cdots \overline{\eta}^T\,}_{\omega+\omega}]^T.
\end{aligned}$$

3.4 Design of Interval Observer

In Sections 3.2 and 3.3, the discrete LPTVD system (2) is transformed into a discrete LTI system (20). This implies that the design problem of interval observer for (2) can be transformed into that of interval observer for (20).

Assumption 2. The pair ($\check{\mathcal{A}}$, $\check{\mathcal{C}}$) in (20) is detectable.

Under Assumption 2, we can design a stable observer to estimate the states of (20), which assures the existence of interval observers for the LPTVD system (2). Thus, under Definition 2, Lemmas 1 and 3, and the equivalent system transformations in Sections 3.2 and 3.3, an interval observer for the system (2) is designed in Theorem 1.

Theorem 1. For an invertible matrix $S \in \mathbb{R}^{l\times l}$ and a gain matrix $L \in \mathbb{R}^{l\times q\omega}$ such that $S(\check{\mathcal{A}}-L\check{\mathcal{C}})S^{-1}$ is nonnegative and $\check{\mathcal{A}}-L\check{\mathcal{C}}$ is a Schur matrix, an interval observer can be designed for the LPTVD system (2) as

$$\underline{\mathbf{x}}_k(h) \le \mathbf{x}_k(h) \le \overline{\mathbf{x}}_k(h)$$

with

$$\underline{\mathbf{x}}_k(h) = \bar{\mathcal{V}}^{+}\underline{\mathring{\mathbf{x}}}_k(h) - \bar{\mathcal{V}}^{-}\overline{\mathring{\mathbf{x}}}_k(h), \tag{24a}$$

$$\overline{\mathbf{x}}_k(h) = \bar{\mathcal{V}}^{+}\overline{\mathring{\mathbf{x}}}_k(h) - \bar{\mathcal{V}}^{-}\underline{\mathring{\mathbf{x}}}_k(h), \tag{24b}$$

where

$$\underline{\mathring{\mathbf{x}}}_k(h) = \left[\,\underline{\mathring{\mathbf{x}}}_{1,k}^T(h)\ \underline{\mathring{\mathbf{x}}}_{2,k}^T(h)\,\right]^T,\ \overline{\mathring{\mathbf{x}}}_k(h) = \left[\,\overline{\mathring{\mathbf{x}}}_{1,k}^T(h)\ \overline{\mathring{\mathbf{x}}}_{2,k}^T(h)\,\right]^T, \tag{25a}$$

$$\begin{aligned}
\underline{\mathring{\mathbf{x}}}_{1,k}(h) =& (\mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{11}S^{-1})^{+}\underline{\bar{\mathbf{x}}}_{1,k}(h) - (\mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{11}S^{-1})^{-}\times\\
&\overline{\bar{\mathbf{x}}}_{1,k}(h) + \mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h),
\end{aligned} \tag{25b}$$

$$\begin{aligned}
\overline{\mathring{\mathbf{x}}}_{1,k}(h) =& (\mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{11}S^{-1})^{+}\overline{\bar{\mathbf{x}}}_{1,k}(h) - (\mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{11}S^{-1})^{-}\times\\
&\underline{\bar{\mathbf{x}}}_{1,k}(h) + \mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h),
\end{aligned} \tag{25c}$$

$$\begin{aligned}
\underline{\bar{\mathbf{x}}}_{1,k}(h+1) =& S(\check{\mathcal{A}}-L\check{\mathcal{C}})S^{-1}\underline{\bar{\mathbf{x}}}_{1,k}(h) + S(\check{\mathcal{B}}-L\check{\mathcal{D}})\mathbf{u}_k(h)\\
&+ SL\mathbf{y}_k + (-SL\check{\mathcal{F}})^{+}\underline{\check{\eta}} - (-SL\check{\mathcal{F}})^{-1}\overline{\check{\eta}}\\
&+ (S\check{\mathcal{E}})^{+}\underline{\check{\nu}} - (S\check{\mathcal{E}})^{-}\overline{\check{\nu}},
\end{aligned} \tag{25d}$$

$$\begin{aligned}
\overline{\bar{\mathbf{x}}}_{1,k}(h+1) =& S(\check{\mathcal{A}}-L\check{\mathcal{C}})S^{-1}\overline{\bar{\mathbf{x}}}_{1,k}(h) + S(\check{\mathcal{B}}-L\check{\mathcal{D}})\mathbf{u}_k(h)\\
&+ SL\mathbf{y}_k + (-SL\check{\mathcal{F}})^{+}\overline{\check{\eta}} - (-SL\check{\mathcal{F}})^{-1}\underline{\check{\eta}}\\
&+ (S\check{\mathcal{E}})^{+}\overline{\check{\nu}} - (S\check{\mathcal{E}})^{-}\underline{\check{\nu}},
\end{aligned} \tag{25e}$$

$$\begin{aligned}
\underline{\mathring{\mathbf{x}}}_{2,k}(h) =& -\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h) + (-\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)^{+}\underline{\nu}_h\\
&- (-\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)^{-}\overline{\nu}_h,
\end{aligned} \tag{25f}$$

$$\begin{aligned}
\overline{\mathring{\mathbf{x}}}_{2,k}(h) =& -\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h) + (-\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)^{+}\overline{\nu}_h\\
&- (-\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)^{-}\underline{\nu}_h.
\end{aligned} \tag{25g}$$

Proof : (20) can be equivalently transformed into

$$\begin{aligned}
\check{\mathbf{x}}_{1,k}(h+1) =& \check{\mathcal{A}}\check{\mathbf{x}}_{1,k}(h) + \check{\mathcal{B}}\mathbf{u}_k(h) + L(\mathbf{y}_k - \check{\mathcal{C}}\check{\mathbf{x}}_{1,k}(h)\\
&- \check{\mathcal{D}}\mathbf{u}_k(h) - \check{\mathcal{F}}\check{\eta}_k(h)) + \check{\mathcal{E}}\check{\nu}_k(h),\\
=& (\check{\mathcal{A}}-L\check{\mathcal{C}})\check{\mathbf{x}}_{1,k}(h) + (\check{\mathcal{B}}-L\check{\mathcal{D}})\mathbf{u}_k(h) + L\mathbf{y}_k\\
&- L\check{\mathcal{F}}\check{\eta}_k(h) + \check{\mathcal{E}}\check{\nu}_k(h),
\end{aligned} \tag{26}$$

where L is designed to stabilize the dynamics (i.e., such that $\check{\mathcal{A}}-L\check{\mathcal{C}}$ is a Schur matrix). For the system (26), it is able to find a coordinate transformation matrix S with

$$\bar{\mathbf{x}}_{1,k}(h) = S\check{\mathbf{x}}_{1,k}(h)$$

such that the matrix $S(\check{\mathcal{A}}-L\check{\mathcal{C}})S^{-1}$ is nonnegative. Thus, with this coordinate change, (26) is transformed into

$$\begin{aligned}
\bar{\mathbf{x}}_{1,k}(h+1) =& S(\check{\mathcal{A}}-L\check{\mathcal{C}})S^{-1}\bar{\mathbf{x}}_{1,k}(h) + S(\check{\mathcal{B}}-L\check{\mathcal{D}})\mathbf{u}_k(h)\\
&+ SL\mathbf{y}_k - SL\check{\mathcal{F}}\check{\eta}_k(h) + S\check{\mathcal{E}}\check{\nu}_k(h).
\end{aligned} \tag{27}$$

Thus, an interval observer for $\bar{\mathbf{x}}_{1,k}(h)$ of (27) is designed as (25d) and (25e). We define estimation errors:

$$\tilde{\underline{\bar{\mathbf{x}}}}_{1,k}(h) = \bar{\mathbf{x}}_{1,k}(h) - \underline{\bar{\mathbf{x}}}_{1,k}(h),$$

$$\tilde{\overline{\bar{\mathbf{x}}}}_{1,k}(h) = \overline{\bar{\mathbf{x}}}_{1,k}(h) - \bar{\mathbf{x}}_{1,k}(h)$$

subject to

$$\tilde{\underline{\bar{\mathbf{x}}}}_{1,k}(h+1) = S(\check{\mathcal{A}}-L\check{\mathcal{C}})S^{-1}\tilde{\underline{\bar{\mathbf{x}}}}_{1,k}(h) + \underline{d}, \tag{29a}$$

$$\tilde{\overline{\bar{\mathbf{x}}}}_{1,k}(h+1) = S(\check{\mathcal{A}}-L\check{\mathcal{C}})S^{-1}\tilde{\overline{\bar{\mathbf{x}}}}_{1,k}(h) + \overline{d}, \tag{29b}$$

where

$$\begin{aligned}
\underline{d} =& -SL\check{\mathcal{F}}\check{\eta}_k(h) + S\check{\mathcal{E}}\check{\nu}_k(h) - (-SL\check{\mathcal{F}})^{+}\underline{\check{\eta}}\\
&+ (-SL\check{\mathcal{F}})^{-1}\overline{\check{\eta}} - (S\check{\mathcal{E}})^{+}\underline{\check{\nu}} + (S\check{\mathcal{E}})^{-}\overline{\check{\nu}},
\end{aligned} \tag{30a}$$

$$\begin{aligned}
\overline{d} =& (-SL\check{\mathcal{F}})^{+}\overline{\check{\eta}} - (-SL\check{\mathcal{F}})^{-1}\underline{\check{\eta}} + (S\check{\mathcal{E}})^{+}\overline{\check{\nu}} - (S\check{\mathcal{E}})^{-}\underline{\check{\nu}}\\
&+ SL\check{\mathcal{F}}\check{\eta}_k(h) - S\check{\mathcal{E}}\check{\nu}_k(h).
\end{aligned} \tag{30b}$$

Under Lemma 2, it is known that $\underline{d} \ge 0$ and $\overline{d} \ge 0$ always hold. Moreover, since the matrix $S(\check{\mathcal{A}}-L\check{\mathcal{C}})S^{-1}$ is designed to be nonnegative and Schur, as long as the given initial conditions $\tilde{\underline{\bar{\mathbf{x}}}}_{1,k}(0) \ge 0$ and $\tilde{\overline{\bar{\mathbf{x}}}}_{1,k}(0) \ge 0$ are satisfied, we always have $\tilde{\underline{\bar{\mathbf{x}}}}_{1,k}(h) \ge 0$ and $\tilde{\overline{\bar{\mathbf{x}}}}_{1,k}(h) \ge 0$ for all $h \ge 0$ and $k \ge 0$. This implies that we always have

$$\underline{\bar{\mathbf{x}}}_{1,k}(h) \le \bar{\mathbf{x}}_{1,k}(h) \le \overline{\bar{\mathbf{x}}}_{1,k}(h).$$

By considering (18), we could further know

$$\mathring{\mathbf{x}}_{1,k}(h) = \mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{11}S^{-1}\bar{\mathbf{x}}_{1,k}(h) + \mathring{\mathcal{G}}_{11}^{-1}\mathring{\mathcal{G}}_{12}\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h).$$

Thus, using the results in Lemma 2, we could obtain

$$\underline{\mathring{\mathbf{x}}}_{1,k}(h) \leq \mathring{\mathbf{x}}_{1,k}(h) \leq \overline{\mathring{\mathbf{x}}}_{1,k}(h)$$

as presented in (25b) and (25c). Similar to (24), we could also use Lemma 2 to obtain the interval of $\mathring{\mathbf{x}}_{2,k}(h)$ as

$$\underline{\mathring{\mathbf{x}}}_{2,k}(h) \leq \mathring{\mathbf{x}}_{2,k}(h) \leq \overline{\mathring{\mathbf{x}}}_{2,k}(h)$$

with

$$\underline{\mathring{\mathbf{x}}}_{2,k}(h) = -\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h) + (-\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)^{+}\underline{\nu}_h - (-\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)^{-}\overline{\nu}_h, \tag{32a}$$

$$\overline{\mathring{\mathbf{x}}}_{2,k}(h) = -\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{B}}_2\mathbf{u}_k(h) + (-\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)^{+}\overline{\nu}_h - (-\mathring{\mathcal{A}}_{22}^{-1}\mathring{\mathcal{E}}_2)^{-}\underline{\nu}_h. \tag{32b}$$

Based on (24) and (32), we could obtain the interval of $\mathring{\mathbf{x}}_k(h)$ presented in (25f) and (25g) as

$$\underline{\mathring{\mathbf{x}}}_k(h) \leq \mathring{\mathbf{x}}_k(h) \leq \overline{\mathring{\mathbf{x}}}_k(h).$$

Thus, by considering the transformation $\mathbf{x}_k(h) = \bar{\mathcal{V}}\mathring{\mathbf{x}}_k(h)$ and using Lemma 2, we obtain the interval in (24a) as

$$\underline{\mathbf{x}}_k(h) \leq \mathbf{x}_k(h) \leq \overline{\mathbf{x}}_k(h).$$

Thus, the proof is completed. □

In order to better present the proposed results, a systematic procedure is summarized in Algorithm 1.

Algorithm 1 Interval observer for LPTVD systems

Require: Parameter matrices G_{i+1}, A_i, B_i, E_i, C_i and F_i $(i = 0, 1, ..., \omega - 1)$, initial values x_0, $\underline{x}_0$, $\overline{x}_0$, $\underline{\nu}$, $\overline{\nu}$, $\underline{\eta}$ and $\overline{\eta}$, periodic iteration initial time $k_0 = 0$, input vector u, and simulation period from $k = 1$ to $k = T\omega$;

1: Equivalently rewrite the original LPTVD system (2) into the ω-stacked form as presented in (3);
2: Using $\mathcal{P}$ to transform the ω-stacked form (3) into a new equivalent descriptor system form (10);
3: Using Fact 1 to find an orthogonal transformation pair $(\bar{\mathcal{U}}, \bar{\mathcal{V}})$ to obtain a minimal-order form (19);
4: After a series of orthogonal transformations till $\mathcal{G}_{11}$ is non-singular, the system can be rewritten into (20);
5: Under Theorem 1, design a pair $(S,\ L)$ to realize the interval observer using the minimal-order form;
6: **for** $h = 0$ to $T - 2$ **do**
7:   **if** $h = 0$ **then**
8:     According to the given initial values, calculate $\mathbf{x}_k(h)$, $\underline{\mathbf{x}}_k(h)$ and $\overline{\mathbf{x}}_k(h)$ based on (2);
9:     Calculate $\overline{\bar{\mathbf{x}}}_{1,k}(h)$, $\underline{\bar{\mathbf{x}}}_{1,k}(h)$, $\overline{\mathring{\mathbf{x}}}_k(h)$ and $\underline{\mathring{\mathbf{x}}}_k(h)$;
10:   **else**
11:     Update $\overline{\bar{\mathbf{x}}}_{1,k}(h)$, $\underline{\bar{\mathbf{x}}}_{1,k}(h)$, $\overline{\mathring{\mathbf{x}}}_k(h)$ and $\underline{\mathring{\mathbf{x}}}_k(h)$ based on (25);
12:     Update $\mathbf{x}_k(h)$ and $\underline{\mathbf{x}}_k(h)$ based on (24);
13:   **end if**
14: **end for**
15: **return** state interval estimations $\overline{\mathbf{x}}_k$ and $\underline{\mathbf{x}}_k$.

Remark 1. In Algorithm 1, due to the existence of $\nu_k(h+1)$ in (20), when the simulation is set from $k = 1$ to $k = T\omega$ (T periods), we can only show the results of $T - 1$ periods (i.e., h can only take values from 0 to $T - 2$). By observing (2) and (3), we know that all the stacked signals $\mathbf{u}_k(h)$, $\mathbf{x}_k(h)$, $\mathbf{y}_k(h)$, $\nu_k(h)$ and $\eta_k(h)$ are dependent of a stacked initial time instant k for (3), which is denoted as k_0 in Algorithm 1 and is set as $k_0 = 0$ for brevity.

4. ILLUSTRATIVE EXAMPLE

To illustrate the effectiveness of the proposed method, a 3-periodic discrete LPTVD system is considered as

$$A_0 = \begin{bmatrix} 0.2209 & 0 \\ 0.6000 & 1.5956 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.3618 & 0 \\ 0.6000 & 0.3298 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.1737 & 0 \\ 0.6000 & 0.4746 \end{bmatrix},\ B_0 = \begin{bmatrix} 1 & 0.9 \\ 0.5 & 1.3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0.1 \\ -0.6 & -0.3 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.5 & 0 \end{bmatrix},\ G_{i+1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_i = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ E_i = F_i = I_2,\ (i = 0, 1, 2).$$

The system initial input needs to meet the initial value constraints since the matrix G_1 is singular. Thus, we give the suitable initial conditions, input signal, disturbance signal, noise signal and their bounds as

$$u = \begin{bmatrix} \sin(kT_s) + \cos(2kT_s) \\ 1 + 0.2\cos(kT_s) \end{bmatrix},\ \nu = \begin{bmatrix} 0.1\cos(5kT_s) \\ 0.1\sin(3kT_s) \end{bmatrix},$$

$$\eta = \begin{bmatrix} 0.12\cos(4kT_s) \\ 0.12\sin(6kT_s) \end{bmatrix},\ x_0 = \begin{bmatrix} -0.4516 \\ -1.2009 \end{bmatrix},$$

$$\overline{x}_0 = \begin{bmatrix} 0.1 \\ -0.8 \end{bmatrix},\ \underline{x}_0 = \begin{bmatrix} -1 \\ -1.8 \end{bmatrix},\ \overline{\nu} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},\ \underline{\nu} = \begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix},$$

$$\overline{\eta} = \begin{bmatrix} 0.12 \\ 0.12 \end{bmatrix},\ \underline{\eta} = \begin{bmatrix} -0.12 \\ -0.12 \end{bmatrix},$$

where $T_s = \pi/10$ is the sampling period and the simulation period is set from $k = 1$ to $k = 30$. Under Fact 1, an orthogonal transformation pair $(\bar{\mathcal{U}}, \bar{\mathcal{V}})$ is obtained:

$$\bar{\mathcal{U}} = I_6,\ \bar{\mathcal{V}} = \begin{bmatrix} 0.8549 & 0.0741 & -0.3739 & 0.3520 & 0 & 0 \\ -0.3215 & -0.0279 & 0.1406 & 0.9360 & 0 & 0 \\ 0.1888 & 0.0164 & 0.4350 & 0 & 0.0827 & -0.8763 \\ -0.3436 & -0.0298 & -0.7915 & 0 & -0.1505 & -0.4817 \\ 0.0683 & 0.0059 & 0.1574 & 0 & -0.9852 & 0 \\ -0.0864 & 0.9963 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Under the orthogonal transformation, the obtained parametric matrices $\mathring{\mathcal{G}}_{11}$ and $\mathring{\mathcal{A}}_{22}$ are nonsingular. Thus, consequently, the minimal-order form (20) can be obtained and the corresponding parametric matrices are presented as

$$\breve{\mathcal{A}} = 0.0139, \breve{\mathcal{B}} = \begin{bmatrix} 0.0735 & 0.0662 & 0.2032 & 0.0203 & 0.2339 & 0.4679 \end{bmatrix},$$

$$\breve{\mathcal{C}} = \begin{bmatrix} 0.8549 & -0.6429 & 0.1888 & -0.6871 & 0.0683 & -0.1728 \end{bmatrix}^T,$$

$$\breve{\mathcal{V}} = \begin{bmatrix} \breve{\mathcal{V}}_{1:6} & \breve{\mathcal{V}}_{7:12} \end{bmatrix},$$

$$\breve{\mathcal{D}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -0.6267 & -1.6295 & 0 & 0 & 0 & 0 \\ 1 & 0.9 & 0 & 0 & 0 & 0 \\ -3.6386 & -3.2747 & 3.6386 & 1.8193 & 0 & 0 \\ 0.3618 & 0.3256 & 1 & 0.1000 & 0 & 0 \\ -0.9148 & -0.8233 & -2.5284 & -0.2528 & -2.1070 & 0 \end{bmatrix},$$

$$\breve{\mathcal{V}}_{7:12} = \begin{bmatrix} 0.8781 & 0.2015 & 0.1775 & 1.0417 & 0 & 0.1820 \end{bmatrix},$$

$$\breve{\mathcal{V}}_{1:6} = \begin{bmatrix} 0.0613 & -0.0028 & 0.2007 & -0.0145 & 1.1697 & -0.0025 \end{bmatrix},$$

$$\breve{\mathcal{F}} = \begin{bmatrix} \breve{\mathcal{F}}_{1:6} & I_6 \end{bmatrix},$$

$$\breve{\mathcal{F}}_{1:6} = \begin{bmatrix} -0.7507 & -0.1722 & -0.1518 & -0.8906 & 0 & -0.1556 \\ 0.5646 & -1.1239 & 0.1141 & 0.6698 & 0 & 0.1170 \\ 0.8342 & -0.0380 & -0.0335 & -0.1967 & 0 & -0.0344 \\ -3.0352 & 0.1384 & 0.1220 & -5.3485 & 0 & 0.1251 \\ 0.3018 & -0.0138 & 0.9879 & -0.0712 & 0 & -0.0124 \\ -0.7631 & 0.0348 & -2.4978 & 0.1800 & 0 & -4.1826 \end{bmatrix}.$$

The pair $(S,\ L)$ as seen at Step 5 of Algorithm 1 for the design of the interval observer is obtained as

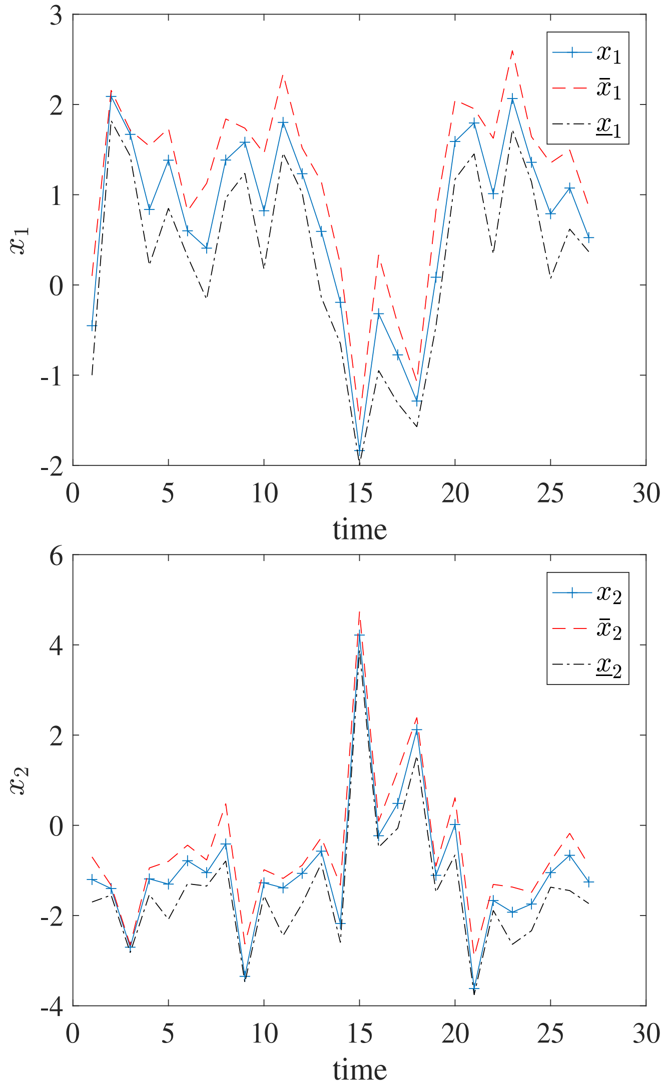

Fig. 1. Interval estimations of a 3-periodic LPTVD system.

$$S = I_6,\ L = [-0.0774\ 0\ -0.0774\ 0\ -0.0774\ 0]\,.$$

In this example, the state vector has two components x_1 and x_2 and their bounds are denoted as $\overline{x}_1$, $\underline{x}_1$, $\overline{x}_2$ and $\underline{x}_2$, respectively. Using the parameters presented above and the proposed design method in Section III, the state intervals estimated by the designed interval observer and real states are both shown in Figure 1, which illustrate the effectiveness of the proposed interval observer for state estimations of the LPTVD system. Note that, as explained in Remark 1, only 27 steps can be shown in Figure 1.

5. CONCLUSIONS

This paper designs an interval observer for the discrete LPTVD system. In this proposed method, the discrete LPTVD system is first transformed into a discrete LTI stacked form. Then by using system equivalence transformations, the discrete LTI stacked form is transformed into a minimal-order system implementation. Based on this minimal-order implementation, a new interval observer is finally designed for the discrete LPTVD system. In the future research, we will explore methods to design an interval observer for discrete linear time-varying systems.

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