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Measurement-Induced Boolean Dynamics from Closed Quantum Networks ⋆

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Abstract: In this paper, we study the induced probabilistic Boolean dynamics for dynamical quantum networks subject to sequential quantum measurements. In this part of the paper, we focus on closed networks of quits whose states evolve according to a Schrödinger equation. Sequential measurements may act on the entire network, or only on a subset of qubits. First of all, we show that this type of hybrid quantum dynamics induces probabilistic Boolean recursions as a Markov chain representing the measurement outcomes. Particularly, we establish an explicit and algebraic representation of the underlying recursive random mapping driving such induced Markov chains. Next, with local measurements, we establish a recursive way of computing such non-Markovian probability transitions.

Keywords: quantum networks, quantum measurements, bilinear systems, Boolean networks

1. INTRODUCTION

Quantum systems admit drastically different behaviors compared to classical systems in terms of state representations, evolutions, and measurements, based on which there holds the promise to develop fundamentally new computing and cryptography infrastructures for our society Nielsen and Chuang (2010). Quantum states are described by vectors in finite or infinite dimensional Hilbert spaces; isolated quantum systems exhibit closed dynamics described by Schrödinger equations; performing measurements over a quantum system yields random outcomes and creates back action to the system being measured. When interacting with environments, quantum systems admit more complex evolutions which are often approximated by various types of master equations. The study of the evolution and manipulation of quantum states has been one of the central problems in the fields of quantum science and engineering Altafini and Ticozzi (2012); Jurdjevic and Sussman (1972); Brockett (1972); Brockett and Khaneja (2000); Schirmer et al. (2001); Albertini and D'Alessandro (2003); Li and Khaneja (2009); Tsopelakos et al. (2019).

Qubits, the so-called quantum bits, are the simplest quantum states with a two-dimensional state space. Qubits naturally form networks in various forms of interactions: they can interact directly with each other by coupling Hamiltonians in a quantum composite system Altafini (2002); implicitly through coupling with local environments Shi et al. (2016); or through local quantum operations such as measurements and classical communications on the operation outcomes Perseguers et al. (2010). Qubit networks have become canonical models for quantum mechanical states and interactions between particles and fields under the notion of spin networks Kato and Yamamoto (2014), and for quantum information processing platforms in computing and communication Perseguers et al. (2010); Shi et al. (2017). The control of qubit networks has been studied in various forms Albertini and D'Alessandro (2002); Wang et al. (2012); Dirr and Helmke1 (2008); Shi et al. (2016); Li et al. (2017).

In this paper, we study dynamical qubit networks which evolve as a collective isolated quantum system but subject to sequential local or global measurements. Sequential measurements have been used be as a way of manipulation quantum states Pechen et al. (2006). Global measurements are represented by observables applied to all qubits in the network, and local measurements only apply to a subset of qubits and therefore the state information of the remaining

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qubits becomes hidden. We reveal that this type of hybrid quantum dynamics induces probabilistic Boolean recursions representing the measurement outcomes, defining a quantum-induced probabilistic Boolean network Kauffman (1969); Shmulevich et al. (2002). Under global measurements, the induced Boolean recursions define Markov chains for which we establish a purely algebraic representation of the underlying recursive random mapping. The representation is in the form of random linear systems embedded in a high dimensional real space. Under local measurements, the resulting probabilistic Boolean dynamics is no longer Markovian. The transition probability at any given time relies on the entire history of the sample path, for which we establish a recursive computation scheme.

A more complete version of the work with full proofs and more detailed examples can be found in Qi et al. (arXiv). The remainder of the extended abstract is organized as follows. Section 2 presents the qubit network model for the study. Section 3 focuses on the induced Boolean network dynamics from the measurements of the dynamical qubit network. Finally Section 4 concludes the paper with a few remarks.

2. THE QUANTUM NETWORK MODEL

In this section, we present the quantum networks model for our study. We consider a network of qubits subject to bilinear control, which aligns with the spin-network models in the literature. We also consider a sequential measurement process where global or local qubit measurements take place periodically.

2.1 Qubit Networks

Qubit is the simplest quantum system whose state space is a two-dimensional Hilbert space $\mathcal{H}$ ($:= \mathcal{H}_2$). Let n qubits indexed by $\mathrm{V} = \{1, \dots, n\}$ form a network with state space $\mathcal{H}^{\otimes n}$. The (pure) states of the qubit network are then in the space $\mathcal{Q}(2^n) := \{q \in \mathcal{H}_{2^n} : |q|^2 = 1\}$.

Let there be a projective measurement (or an observable) for a single qubit as

$$\mathsf{M} = \lambda_0 \mathsf{P}_0 + \lambda_1 \mathsf{P}_1,$$

where $\mathsf{P}_m = |v_m\rangle\langle v_m|$ is the projector onto the eigenspace generated by $|v_m\rangle$ with eigenvalue λ_m, $m \in \{0, 1\}$. For the n-qubit network, we can have either global or local measurements.

Definition 1. (i) We term $\mathsf{M}^{\otimes n} = \mathsf{M} \otimes \cdots \otimes \mathsf{M}$ as a global measurement over the n-qubit network.

(ii) Let $\mathrm{V}_* = \{i_1, \dots, i_k\} \subset \mathrm{V}$. Then

$$\mathsf{M}^{\mathrm{V}_*} = \mathrm{I} \otimes \cdots \otimes \mathrm{I} \otimes \overbrace{\mathsf{M}}^{i_1\text{-th}} \otimes \mathrm{I} \otimes \cdots \otimes \mathrm{I} \otimes \overbrace{\mathsf{M}}^{i_k\text{-th}} \otimes \mathrm{I} \otimes \cdots \otimes \mathrm{I}$$

is defined as a local measurement over V_*.

The global measurement $\mathsf{M}^{\otimes n}$ measures the individual qubit states of the entire network, which yields 2^n possible outcomes $[\lambda_{m_1}, \dots, \lambda_{m_n}], m_j \in \{0, 1\}, j = 1, \dots, n$. Upon measuring the state $|\varphi\rangle$, the probability of getting result $[\lambda_{m_1}, \dots, \lambda_{m_n}]$ is given by $p([\lambda_{m_1}, \dots, \lambda_{m_n}]) = \langle \varphi | \mathsf{P}_{m_1} \otimes \cdots \otimes \mathsf{P}_{m_n} | \varphi \rangle$. Given that the outcome $[\lambda_{m_1}, \dots, \lambda_{m_n}]$ occurred, the qubit network state immediately after the measurement is $|\varphi\rangle_{\mathrm{p}} = |v_{m_1}\rangle \otimes \cdots \otimes |v_{m_n}\rangle$. On the other hand, the local measurement $\mathsf{M}^{\mathrm{V}_*}$ measures the states of the qubits in the set V_* only, which yields 2^k possible outcomes $[\lambda_{m_{i_1}}, \dots, \lambda_{m_{i_k}}], i_j \in \{0, 1\}, j = 1, \dots, k$ corresponding to the qubits $\{i_1, \dots, i_k\}$. Upon measuring the state $|\varphi\rangle$, the probability of getting result $[\lambda_{m_{i_1}}, \dots, \lambda_{m_{i_k}}]$ is

$$p([\lambda_{m_{i_1}}, \dots, \lambda_{m_{i_k}}]) = \langle \varphi | \mathrm{I} \otimes \cdots \otimes \mathrm{I} \otimes \mathsf{P}_{m_{i_1}} \otimes \mathrm{I} \otimes \cdots \otimes \mathrm{I} \otimes \mathsf{P}_{m_{i_k}} \otimes \mathrm{I} \otimes \cdots \otimes \mathrm{I} | \varphi \rangle,$$

where $m_{i_j} \in \{0, 1\}, j = 1, \dots, k$. Since the local measurement reveals no information about the nodes in $\mathrm{V} \setminus \mathrm{V}_*$, we term the qubits in V_* as the measured qubits, and those in $\mathrm{V} \setminus \mathrm{V}_*$ as the dark qubits. For the ease of presentation and without loss of generality, we assume $\mathrm{V}_* = \{1, \dots, k\}$ throughout the remainder of the paper.

2.2 Hybrid Qubit Network Dynamics

Consider the continuous time horizon represented by $s \in [0, \infty)$. Let $\mathsf{A} = \imath \mathsf{H}_0$ and $\mathsf{B}_\ell = \imath \mathsf{H}_\ell, \ell = 1, \dots, p$ with H_0 and H_ℓ being Hermitian operators as Hamiltonians. Let $|q(s)\rangle$ denote the qubit network state at time s. The evolution of $|q(s)\rangle$ is defined by a Schrödinger equation with controlled Hamiltonians, and the network state be measured globally or locally from $s = 0$ periodically with a period T. To be precise, $|q(s)\rangle$ satisfies the following hybrid dynamical equations

$$|\dot{q}(s)\rangle = \left(\mathsf{A} + \sum_{\ell=1}^{p} u_\ell(s) \mathsf{B}_\ell \right) |q(s)\rangle, \quad s \in [tT, (t+1)T), \tag{1}$$

$$|q((t+1)T)\rangle = |q((t+1)T)^-\rangle_{\mathrm{p}}, \tag{2}$$

for $t = 0, 1, 2, \cdots$, where $|q((t+1)T)^-\rangle$ represents the quantum network state right before $(t+1)T$ along (1) starting from $|q(tT)\rangle$, and $|q((t+1)T)^-\rangle_{\mathrm{p}}$ is the post-measurement state of the network when a measurement is performed at time $s = (t+1)T$. For the ease of presentation, we define quantum states

$$|\psi(t)\rangle = |q((tT)^-)\rangle,$$
$$|\psi(t)\rangle_{\mathrm{p}} = |q(tT)\rangle$$

for the pre- and post-measurement network states at the $(t+1)$-th measurement.

In particular, the control signals $u_\ell(s)$, $\ell = 1, \dots, p$ will have feedforward or feedback forms.

Definition 2. (i) The control signals $u_\ell(s)$, $\ell = 1, \dots, p$ are feedforward if their values are determined deterministically at $t = 0^-$ for the entire time horizon $s \geq 0$.

(ii) The control signals $u_\ell(s)$, $\ell = 1, \dots, p$ are feedback if each $u_\ell(s)$ for $s \in [tT, (t+1)T)$ depends on the post-measurement state $|\psi(t')\rangle_{\mathrm{p}}$, $t' = 0, 1, \dots, t$.

3. BOOLEAN DYNAMICS FROM QUANTUM MEASUREMENTS

In this section, we focus our attention on the induced Boolean dynamics from the sequential measurements of the qubit networks. We impose the following assumption.

Assumption 3. The $u_\ell(s)$, $\ell = 1, \dots, p$ are feedforward signals. Consequently, there exist a sequence of deterministic U_t, $t = 0, 1, 2, \dots$ such that $|\psi(t+1)\rangle = \mathsf{U}_t |\psi(t)\rangle_{\mathrm{p}}$.

3.1 Induced Probabilistic Boolean Networks

Under the global measurement $\mathsf{M}^{\otimes n}$, we can use the Boolean variable $x_i(t) \in \{0,1\}$ to represent the measurement outcome at qubit i for step t, where $x_i(t) = 0$ corresponds to λ_0 and $x_i(t) = 1$ corresponds to λ_1. We can further define the n-dimensional random Boolean vector

$$\mathbf{x}(t) = [x_1(t), \cdots, x_n(t)] \in \{0,1\}^n$$

as the outcome of measuring $|\psi(t)\rangle$ under $\mathsf{M}^{\otimes n}$ at step t. The recursion of $|\psi(t)\rangle_{\mathrm{p}}$ generates the corresponding recursion of $\mathbf{x}(t)$ for $t = 0, 1, 2, \ldots$, resulting in an induced probabilistic Boolean network (PBN). Similarly, subject to local measurement, we can define $\mathbf{x}_{\mathrm{k}}(t) = [x_1(t), \cdots, x_k(t)] \in \{0,1\}^k$ as the outcome of measuring $|\psi(t)\rangle$ by $\mathsf{M}^{\mathsf{V}_*}$, where $x_i(t) \in \{0,1\}$ continues to represent the measurement outcome at qubit i.

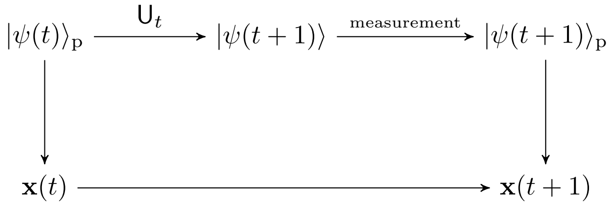

Fig. 1. Induced Boolean network dynamics.

We are interested in the interplay between the underlying quantum state evolution and the induced probabilistic Boolean network dynamics.

3.2 Global Measurement: Markovian PBN

Transition Characterizations We first analyze the behaviors of the induced probabilistic Boolean network dynamics under global qubit network measurements. Let δ_N^i be the i-th column of identify matrix I_N. Denote $\Delta_N = \{\delta_N^i | i = 1, \ldots, N\}$, and particularly $\Delta := \Delta_2$ for simplicity. Identify $\{0,1\} \simeq \Delta$ under which $0 \sim \delta_2^1$ and $1 \sim \delta_2^2$. Let $\mathbf{x} = [x_1, \ldots, x_n] \in \{0,1\}^n$ be associated with

$$\mathbf{x}^\sharp := \delta_2^{x_1+1} \otimes \cdots \otimes \delta_2^{x_n+1} = \delta_{2^n}^{\sum_{i=1}^n x_i 2^{n-i}+1}, \quad (3)$$

where $\otimes$ represents the Kronecker product. In this way, we have identified $\{0,1\}^n \simeq \Delta_{2^n}$. For the ease of presentation, we also denote $\lfloor \mathbf{x} \rfloor := \sum_{i=1}^n x_i 2^{n-i}+1$, and consider $\mathbf{x}$, $\lfloor \mathbf{x} \rfloor$, and $\mathbf{x}^\sharp = \delta_{2^n}^{\lfloor \mathbf{x} \rfloor}$ interchangeable without further mentioning. Recall $\mathcal{S}$ as the set containing all $(2^n)^{2^n}$ Boolean mappings from $\{0,1\}^n$ to $\{0,1\}^n$. Each element in $\mathcal{S}$ is indexed by $f_{[\alpha_1,\ldots,\alpha_{2^n}]} \in \mathcal{S}$ with $\alpha_i = 1, \ldots, 2^n, i = 1, \ldots, 2^n$, where

$$f_{[\alpha_1,\ldots,\alpha_{2^n}]}(s_i) = s_{\alpha_i}, \quad s_i \in \{0,1\}^n, \ i = 1, \ldots, 2^n. \quad (4)$$

In this way, the matrix $f_{[\alpha_1,\ldots,\alpha_{2^n}]} = [\delta_{2^n}^{\alpha_1}, \ldots, \delta_{2^n}^{\alpha_{2^n}}]$ serves as a representation of $f_{[\alpha_1,\ldots,\alpha_{2^n}]}$ since

$$f_{[\alpha_1,\ldots,\alpha_{2^n}]} \delta_{2^n}^i = \delta_{2^n}^{\alpha_i}, \quad i = 1, \ldots, 2^n. \quad (5)$$

Recall the observable $\mathsf{M} = \lambda_0 \mathsf{P}_0 + \lambda_1 \mathsf{P}_1$ for one qubit. We choose $\{|0\rangle, |1\rangle\}$ as the standard orthonormal basis of $\mathcal{H}$, and denote $\mathsf{Q}_0 = |0\rangle\langle 0|$, $\mathsf{Q}_1 = |1\rangle\langle 1|$. Then there exists a unitary operator $\mathsf{u} = |v_0\rangle\langle 0| + |v_1\rangle\langle 1| \in \mathfrak{L}_*(\mathcal{H})$, whose representation under the chosen basis $\{|0\rangle, |1\rangle\}$ is $u \in \mathbb{C}^{2\times 2}$ which is a unitary matrix, such that $\mathsf{P}_0 = \mathsf{u}\mathsf{Q}_0\mathsf{u}^\dagger$ and $\mathsf{P}_1 = \mathsf{u}\mathsf{Q}_1\mathsf{u}^\dagger$.

Let $\{|0\rangle, |1\rangle\}^{\otimes n}$ be the standard computational basis of the n-qubit network. We denote for $i = 1, \ldots, 2^n$ that

$$|b_i\rangle = |b_{i_1} \cdots b_{i_n}\rangle \quad (6)$$

where $|b_{i_1} \cdots b_{i_n}\rangle \in \{|0\rangle, |1\rangle\}^{\otimes n}$ with $b_{i_j} \in \{0,1\}$, $j = 1, \ldots, n$. Now we can sort the elements of $\{|0\rangle, |1\rangle\}^{\otimes n}$ by the value of $\lfloor b_i \rfloor$ in an ascending order. Let U_t have the representation $U_t \in \mathbb{C}^{2^n \times 2^n}$ under such an ordered basis. Note that $\mathsf{u} \otimes \cdots \otimes \mathsf{u}$ has its matrix representation as $u \otimes \cdots \otimes u$ under the same sorted basis. Define

$$U_t^{\mathsf{M}} = (u \otimes \cdots \otimes u)^\dagger U_t (u \otimes \cdots \otimes u). \quad (7)$$

For the induced Boolean series $\{\mathbf{x}(t)\}_{t=0}^\infty$, the following result holds, whose proof is omitted as it is a direct verification of quantum measurement postulate.

Proposition 4. Let Assumption 3 hold. With global measurement, the $\{\mathbf{x}(t)\}_{t=0}^\infty$ form a Markov chain over the state space $\{0,1\}^n$, whose state transition matrix $\mathbf{P}_t$ at time t is given by

$$[\mathbf{P}_t]_{i,j} = \mathbb{P}\Big(\mathbf{x}(t+1)\Big|\mathbf{x}(t)\Big) = \Big|[U_t^{\mathsf{M}}]_{j,i}\Big|^2,$$

for $i = \lfloor \mathbf{x}(t) \rfloor, j = \lfloor \mathbf{x}(t+1) \rfloor \in \{1, 2, \ldots, 2^n\}$, where $[\cdot]_{i,j}$ stands for the (i,j)-th entry of a matrix. In fact, there holds $\mathbf{P}_t = (U_t^{\mathsf{M}})^\dagger \circ (U_t^{\mathsf{M}})^\top$, where $\circ$ stands for the Hadamard product.

The following theorem establishes an algebraic representation of the recursion for $\{\mathbf{x}(t)\}_{t=0}^\infty$.

Theorem 5. Let Assumption 3 hold. The recursion of $\{\mathbf{x}(t)\}_{t=0}^\infty$ can be represented as a random linear mapping

$$\mathbf{x}^\sharp(t+1) = F_t \mathbf{x}^\sharp(t), \quad (8)$$

where $\langle F_t \rangle$ is a series of independent random matrices in $\mathbb{R}^{2^n \times 2^n}$. Moreover, the distribution of F_t is described by

$$\mathbb{P}(F_t = f_{[\alpha_1,\ldots,\alpha_{2^n}]}) = \prod_{i=1}^{2^n} \Big|[U_t^{\mathsf{M}}]_{\alpha_i,i}\Big|^2.$$

3.3 Local Measurement: Non-Markovian PBN

We now turn to the local measurement case, where at time t, $\mathsf{M}^{\mathsf{V}_*} = \mathsf{M}^{\otimes k} \otimes \mathsf{I}^{\otimes(n-k)}$ is performed over $|\psi(t)\rangle$ and produces outcome $\mathbf{x}_{\mathrm{k}}(t) = [x_1(t), \ldots, x_k(t)]$. Therefore, without loss of generality, we assume that $\mathsf{M} = \lambda_0 \mathsf{P}_0 + \lambda_1 \mathsf{P}_1 = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1|$.

Given $\mathbf{x}_{\mathrm{k}}(t)$, the post-measurement state $|\psi(t)\rangle_{\mathrm{p}}$ depends on $\mathbf{x}_{\mathrm{k}}(0), \ldots, \mathbf{x}_{\mathrm{k}}(t-1)$ due to the local measurement effect as $\mathbf{x}_{\mathrm{k}}(t)$ alone is not enough to determine $|\psi(t)\rangle$. Therefore $\{\mathbf{x}_{\mathrm{k}}(t)\}_{t=0}^\infty$ is no longer Markovian. Let $\mathrm{r}: \ \mathscr{X}_{\mathrm{k}}(0), \ldots, \mathscr{X}_{\mathrm{k}}(t)$ be a path of measurement realization. Define

$$\begin{aligned} \mathscr{P}_{\mathrm{r}}(0) &:= \mathbb{P}(\mathscr{X}_{\mathrm{k}}(0)) \\ \mathscr{P}_{\mathrm{r}}(1) &:= \mathbb{P}(\mathscr{X}_{\mathrm{k}}(1)|\mathscr{X}_{\mathrm{k}}(0)) \\ &\vdots \\ \mathscr{P}_{\mathrm{r}}(t+1) &:= \mathbb{P}(\mathscr{X}_{\mathrm{k}}(t+1)|\mathscr{X}_{\mathrm{k}}(t), \ldots, \mathscr{X}_{\mathrm{k}}(0)). \end{aligned}$$

We aim to provide a recursive way of calculating the above transition probabilities. Recall from (6) that $\{|0\rangle, |1\rangle\}^{\otimes n} = \{|b_i\rangle, i = 1, \ldots, 2^n\}$ is a sorted basis for $\mathcal{H}^{\otimes n}$. Let

$$|\psi(0)\rangle = \sum_{i=1}^{2^n} a_i |b_i\rangle$$

with $\sum_{i=1}^{2^n} |a_i|^2 = 1$ be the state of the quantum network at time $t = 0$. Let U_t be the matrix representation of U_t under the chosen basis for $t = 0, 1, 2, \cdots$. Recall $\lfloor \mathfrak{X}_k(t) \rfloor := \sum_{i=1}^{k} x_i(t) 2^{k-i} + 1$, and $\mathfrak{X}_k^\sharp(t) := \delta_{2^k}^{\lfloor \mathfrak{X}_k(t) \rfloor}$. Then we have the following theorem.

Theorem 6. Let Assumption 3 hold and $\mathsf{M} = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1|$. Let $\mathrm{r} : \mathfrak{X}_k(0), \dots, \mathfrak{X}_k(t)$ be a realization of the random measurement outcomes. Then there exist $\beta^{\mathrm{r}}(t) \in \mathbb{C}^{2^{n-k}}$ with $\beta^{\mathrm{r}}(t) = [\beta_1^{\mathrm{r}}(t), \dots, \beta_{2^{n-k}}^{\mathrm{r}}(t)]^\top$ for $t = 0, 1, 2, \dots$, such that $\mathcal{P}_{\mathrm{r}}(t) = \|\beta^{\mathrm{r}}(t)\|^2$ for all $t \geq 0$, where $\beta^{\mathrm{r}}(t)$ satisfies the recursion

$$\beta^{\mathrm{r}}(t+1) = \Big(\big(\mathfrak{X}_k^\sharp(t+1) \big)^\top \otimes I^{\otimes(n-k)} \Big) U_t \Big(\mathfrak{X}_k^\sharp(t) \otimes I^{\otimes(n-k)} \Big) \frac{\beta^{\mathrm{r}}(t)}{\|\beta^{\mathrm{r}}(t)\|} \tag{9}$$

with $\beta_i^{\mathrm{r}}(0) = a_{(\lfloor \mathfrak{X}_k(0) \rfloor - 1) 2^{n-k} + i}$, $i = 1, \dots, 2^{n-k}$.

4. CONCLUSIONS

We have studied dynamical quantum networks subject to sequential local or global measurements leading to probabilistic Boolean recursions which represent the measurement outcomes. With global measurements, such resulting Boolean recursions were shown to be Markovian, while with local measurements, the state transition probability at any given time depends on the entire history of the sample path. Under the bilinear control model for the Schrödinger evolution, we showed that the measurements in general enhance the controllability of the quantum networks. The global or local measurements were assumed to be prescribed in the current framework.

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