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Measurement-Induced Boolean Dynamics from Open Quantum Networks ⋆

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Abstract: In this paper, we study the induced probabilistic Boolean dynamics for dynamical quantum networks subject to sequential quantum measurements. In this part of the paper, we focus on closed networks of quits whose states evolutions are described by a continuous Lindblad master equation. When measurements are performed sequentially along such continuous dynamics, the quantum network states undergo random jumps and the corresponding measurement outcomes can be described by a probabilistic Boolean network. First of all, we show that the state transition of the induced Boolean networks can be explicitly represented through realification of the master equation. Next, when the open quantum dynamics is relaxing in the sense that it possesses a unique equilibrium as a global attractor, structural properties including absorbing states, reducibility, and periodicity for the induced Boolean network are direct consequences of the relaxing property. Finally, we show that for quantum consensus networks as a type of non-relaxing open quantum network dynamics, the communication classes of the measurement-induced Boolean networks are encoded in the quantum Laplacian of the underlying interaction graph.

Keywords: quantum networks, open quantum systems, quantum measurements, Boolean networks

1. INTRODUCTION

Quantum systems admit drastically different behaviors compared to classical systems in terms of state representations, evolutions, and measurements, based on which there holds the promise to develop fundamentally new computing and cryptography infrastructures for our society Nielsen and Chuang (2010). Quantum states are described by vectors in finite or infinite dimensional Hilbert spaces; isolated quantum systems exhibit closed dynamics described by Schrödinger equations; performing measurements over a quantum system yields random outcomes and creates back action to the system being measured. When interacting with environments, quantum systems admit more complex evolutions which are often approximated by various types of master equations. The study of the evolution and manipulation of quantum states has been one of the central problems in the fields of quantum science and engineering Altafini and Ticozzi (2012); Jurdjevic and Sussman (1972); Brockett (1972); Brockett and Khaneja (2000); Schirmer et al. (2001); Albertini and D'Alessandro (2003); Li and Khaneja (2009); Tsopelakos et al. (2019).

Qubits, the so-called quantum bits, are the simplest quantum states with a two-dimensional state space. Qubits naturally form networks in various forms of interactions: they can interact directly with each other by coupling Hamiltonians in a quantum composite system; implicitly through coupling with local environments; or through local quantum operations such as measurements and classical communications on the operation outcomes. Qubit networks have become canonical models for quantum mechanical states and interactions between particles and fields under the notion of spin networks Kato and Yamamoto (2014), and for quantum information processing platforms in computing and communication Nielsen and Chuang (2010). In the first part of the paper, we have investigated

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the properties of closed qubit networks in the presence of sequential measurements.

If we connect a network of quantum subsystems such as qubits by a series of local environments, an open quantum network is obtained. Open quantum networks have proven to be a resource for universal quantum computing (Verstraete et al., 2009), and a way of realizing quantum consensus and synchronization at the quantum level (Mazzarella et al., 2015; Shi et al., 2016). In this second part of the paper, we study open quantum networks of qubits with sequential measurements. When measurements are performed sequentially along the continuous dynamics, the quantum network states undergo random jumps and the measurement outcomes are naturally described by a vector of random Boolean variables, forming a probabilistic Boolean network (Shmulevich et al., 2002). The induced recursion of the Boolean networks defines a Markov chain, which is governed both by the master equation of the continuous quantum dynamics, and the basis of the network measurement.

We establish a clear and explicit representation for the state transition of the random measurement outcomes from realification of the master equation. Moreover, for relaxing and non-relaxing quantum network dynamics, respectively, we establish the following results.

(i) When the open quantum dynamics is relaxing, i.e., it possesses a unique equilibrium that is globally asymptotically stable, structural properties including absorbing states, reducibility, and periodicity for the induced Boolean networks are established directly from the relationship between the master equation and the measurement basis. Particularly, we show that as a generic property, relaxing quantum dynamics leads to irreducible and aperiodic chains for the measurement outcomes.

(ii) We show that for quantum consensus networks as a special type of non-relaxing open quantum network dynamics, the communication classes of the measurement-induced Boolean networks are fully encoded in the quantum Laplacian of the underlying interaction graph.

A more complete version of the work with full proofs and more detailed examples can be found in Qi et al. (arXiv). The remainder of the paper is organized as follows. Section 2 presents some graph theory and Markov chain essentials. In Section 3, we introduce the quantum network model, the resulting hybrid quantum network dynamics, and the definition of the induced probabilistic Boolean network from the measurement outcomes. In Section 4, we establish the representation of the state transition of the Boolean network. Then in Sections 5 and 6, we present the results for relaxing and non-relaxing quantum dynamics, respectively. Finally some concluding remarks are presented in Section 7.

2. PRELIMINARIES

2.1 Graph Theory Essentials

A simple undirected graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ consists of a finite set $\mathrm{V} = \{1, \dots, n\}$ of nodes and an edge set E, where an element $e = \{i, j\} \in \mathrm{E}$ denotes an *edge* between two distinct nodes $i \in \mathrm{V}$ and $j \in \mathrm{V}$. Two nodes $i, j \in \mathrm{V}$ are said to be *adjacent* if $\{i, j\}$ is an edge in E. The number of adjacent nodes of v is called its degree, denoted $\deg(v)$. The nodes that are adjacent with a node v as well as itself are called its neighbors. A graph G is called to be *regular* if all the nodes have the same degree. A path between two vertices v_1 and v_k in G is a sequence of distinct nodes $v_1 v_2 \dots v_k$ such that for any $m = 1, \dots, k-1$, there is an edge between v_m and v_{m+1}. A pair of distinct nodes i and j is called to be *reachable* from each other if there is a path between them. A node is always assumed to be reachable from itself. We call graph G *connected* if every pair of distinct nodes in V are reachable from each other. A subgraph of G associated with node set $\mathrm{V}^* \subseteq \mathrm{V}$, denoted as $\mathrm{G}|_{\mathrm{V}^*}$, is the graph $(\mathrm{V}^*, \mathrm{E}^*)$, where $\{i, j\} \in \mathrm{E}^*$ if and only if $\{i, j\} \in \mathrm{E}$ for $i, j \in \mathrm{V}^*$. A connected component (or just component) of G is a connected subgraph induced by some $\mathrm{V}^* \subseteq \mathrm{V}$, which is connected to no additional nodes in $\mathrm{V} \setminus \mathrm{V}^*$.

The (weighted) Laplacian of G, denoted $L(\mathrm{G})$, is defined as (Mesbahi and Egerstedt, 2010)

$$L(\mathrm{G}) = D(\mathrm{G}) - A(\mathrm{G}),$$

where $A(\mathrm{G})$ is the $n \times n$ matrix given by $[A(\mathrm{G})]_{kj} = [A(\mathrm{G})]_{jk} = a_{kj}$ for some $a_{kj} > 0$ if $\{k, j\} \in \mathrm{E}$ and $[A(\mathrm{G})]_{kj} = 0$ otherwise, and $D(\mathrm{G}) = \mathrm{diag}(d_1, \dots, d_N)$ with $d_k = \sum_{j=1, j\neq k}^{N} [A(\mathrm{G})]_{kj}$. It is well known that $L(\mathrm{G})$ is always positive semi-definite, and $\mathrm{rank}(L(\mathrm{G})) = n - C_*(\mathrm{G})$, where $C_*(\mathrm{G})$ denotes the number of connected components of G.

2.2 Markov Chains

Let $\mathscr{S}$ be the finite set $\{1, 2, \dots, m\}$. Let $\mathbf{P}$ be an $m \times m$ non-negative matrix with $\sum_{j=1}^{m} [\mathbf{P}]_{ij} = 1$ for $i \in \mathscr{S}$. A stochastic process $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ with state space $\mathscr{S}$ is called a homogeneous Markov chain with transition matrix $\mathbf{P}$, if there holds

$$\mathbb{P}(\mathbf{x}(t+1) | \mathbf{x}(0), \dots, \mathbf{x}(t)) = \mathbb{P}(\mathbf{x}(t+1) | \mathbf{x}(t))$$

for $t = 0, 1, 2, \dots,$ and

$$\mathbb{P}(\mathbf{x}(t+1) = j | \mathbf{x}(t) = i) = [\mathbf{P}]_{ij},$$

for all $i, j \in \mathscr{S}$.

Let row vector π_0 be the initial distribution of the time homogeneous Markov chain $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ with $[\pi_0]_i = \mathbb{P}(\mathbf{x}(0) = i)$ and $\sum_{i\in\mathscr{S}} [\pi_0]_i = 1$. Let π_t denote the distribution of the chain at time t, i.e., $[\pi_t]_i = \mathbb{P}(\mathbf{x}(t) = i)$. Then there holds

$$[\pi_{t+1}]_j = \sum_{i=1}^{m} [\pi_t]_i [\mathbf{P}]_{ij},$$

or in a compact form, $\pi_{t+1} = \pi_t \mathbf{P}$.

A time homogeneous Markov chain $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ with the state space $\mathscr{S}$ is called irreducible if there exists an integer $l \geq 1$ such that $[\mathbf{P}^l]_{ij} > 0$ for any $i, j \in \mathscr{S}$. The period $d(i)$ of a state $i \in \mathscr{S}$ is defined as the greatest common divisor of all l that satisfy $[\mathbf{P}^l]_{ii} > 0$ and $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ is called aperiodic if all the states have period one. If the chain is both irreducible and aperiodic, then there exist a row vector π_* satisfying

$$\pi_* = \lim_{t\to\infty} \pi_0 \mathbf{P}^t$$

for all initial distribution π_0. In that case π_* is termed the stationary distribution of the Markov chain.

3. PROBLEM DEFINITION

3.1 Qubit Networks

Qubit is the simplest quantum system whose state space is a two-dimensional Hilbert space $\mathcal{H}$ ($:= \mathcal{H}_2$). Consider a quantum network with n qubits, which are indexed by $\mathrm{V} = \{1, \dots, n\}$. The state space of the n-qubit network is denoted as $\mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n}$ (i.e., $\mathcal{H}_{2^n}$). The density operator of the n-qubit network is denoted as ρ. Let there be an observable (or a projective measurement) for a single qubit as

$$\mathsf{M} = \lambda_0 \mathsf{P}_0 + \lambda_1 \mathsf{P}_1,$$

where $\mathsf{P}_m = |v_m\rangle\langle v_m|$ is the projector onto the eigenspace generated by the eigenvector $|v_m\rangle \in \mathcal{H}_2$ with eigenvalue λ_m, $m \in \{0, 1\}$. Then $\mathsf{M}^{\otimes n}$ is an observable of the n-qubit network.

3.2 Open Quantum Networks with Sequential Measurements

Consider the continuous time horizon for $s \in [0, \infty)$. Let the open quantum network state $\rho(s)$ be measured along $\mathsf{M}^{\otimes n}$ from $s = 0$ periodically with a period τ. To be precise, $\rho(s)$ satisfies the following hybrid dynamics

$$\dot{\rho}(s) = \mathscr{L}(\rho(s)), \quad s \in [t\tau, (t+1)\tau), \tag{1a}$$

$$\rho(t\tau) = \rho_{\mathrm{p}}((t\tau)^-), \tag{1b}$$

for $t = 0, 1, 2, \dots$. Here

$$\mathscr{L}(\rho(s)) = -\imath[\mathsf{H}, \rho(s)] + \mathscr{L}_D(\rho(s)), \tag{2}$$

where H is the Hamiltonian as a Hermitian operator over $\mathcal{H}_N$, and $\mathscr{L}_D(\rho) = \sum_d \mathcal{D}[\mathsf{V}_d]\rho$ is the Lindblad operator from environments. The V_d are operators over $\mathcal{H}_N$, and

$$\mathcal{D}[\mathsf{V}_d]\rho(s) = \mathsf{V}_d \rho(s) \mathsf{V}_d^\dagger - \frac{1}{2}[\mathsf{V}_d^\dagger \mathsf{V}_d \rho(s) + \rho(s) \mathsf{V}_d^\dagger \mathsf{V}_d]. \tag{3}$$

Moreover, $\rho(((t+1)\tau)^-)$ represents the quantum network state right before $(t+1)\tau$ along (1a) starting from $\rho(t\tau)$, and $\rho((t\tau)^-)_{\mathrm{p}}$ is the post-measurement state of the network when a measurement is performed at time $s = t\tau$ along $\mathsf{M}^{\otimes n}$, respectively.

We further introduce

$$\xi(t) := \rho((t\tau)^-),$$

$$\xi_{\mathrm{p}}(t) := \rho_{\mathrm{p}}((t\tau)^-),$$

for the pre- and post-measurement network states at the t-th measurement.

3.3 Induced Boolean Networks

The measurement $\mathsf{M}^{\otimes n}$ measures the individual qubit states of the entire network, which yields 2^n possible outcomes $[\lambda_{m_1}, \dots, \lambda_{m_n}], m_j \in \{0, 1\}, j = 1, \dots, n$. We use the Boolean variable $x_i(t) \in \{0, 1\}$ to represent the measurement outcome at qubit i for step t, where $x_i(t) = 0$ corresponds to λ_0 and $x_i(t) = 1$ corresponds to λ_1. We can further define the n-dimensional random Boolean vector

$$\mathbf{x}(t) = [x_1(t), \cdots, x_n(t)] \in \{0, 1\}^n$$

as the outcome of measuring $\xi(t)$ under $\mathsf{M}^{\otimes n}$ at step t.

Clearly, $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ forms a Markov chain as the distribution of $\mathbf{x}(t+1)$ is fully determined by $\rho_{\mathrm{p}}(t+1)$, which depends only on $\mathbf{x}(t)$, e.g., Figure 1. The $\mathbf{x}(t), t = 0, 1, 2, \dots$ therefore falls to the category of classical probabilistic Boolean networks (Shmulevich et al., 2002).

$$\begin{tikzcd}
\xi_{\mathrm{p}}(t) \arrow[r, "e^{\mathscr{L}\tau}"] \arrow[d] & \xi(t+1) \arrow[r, "\text{measurement}"] & \xi_{\mathrm{p}}(t+1) \arrow[d] \\
\mathbf{x}(t) \arrow[rr] & & \mathbf{x}(t+1)
\end{tikzcd}$$

Fig. 1. Induced Boolean network dynamics from the sequential quantum measurements.

3.4 Problems of Interest

In this paper, we are interested in the properties of the induced Boolean network dynamics. Particularly, we aim to address the following questions:

(i) How can we represent the state transition of the $\mathbf{x}(t)$?
(ii) When and how can we characterize the basic properties of $\mathbf{x}(t)$ as a Markov chain, e.g., absorbing states, reducibility and periodicity, communication classes?
(iii) Can we establish a clear relationship between the quantum interaction structure encoded in the $\mathscr{L}$, and structures in the state space of $\mathbf{x}(t)$?

Answers to these questions will add to the understandings of the behaviors of open quantum systems in the presence of sequential measurements.

4. STATE TRANSITION REPRESENTATION

In this section, we establish an explicit representation of the state transition of the chain $\mathbf{x}(t)$. Such a representation is certainly non-unique, and we choose to carry out the analysis under the following standard realification of the master equation (1a) (cf. e.g., Schirmer and Wang (2010)).

Denote $N = 2^n$. Let there be an orthonormal basis $\boldsymbol{\sigma} = \{\sigma_k\}_{k=1}^{N^2}$ for Hermitian operators on $\mathcal{H}^{\otimes n}$ by $\sigma_k = \lambda_{pq}$, $k = p + (q-1)N$ and $1 \le p < q \le N$, where

$$\lambda_{pq} = \frac{1}{\sqrt{2}}(|p\rangle\langle q| + |q\rangle\langle p|),$$

$$\lambda_{qp} = \frac{1}{\sqrt{2}}(-i|p\rangle\langle q| + i|q\rangle\langle p|),$$

$$\lambda_{pp} = \frac{1}{\sqrt{p+p^2}} \left(\sum_{k=1}^{p} |k\rangle\langle k| - p|p+1\rangle\langle p+1| \right).$$

Under the basis $\boldsymbol{\sigma}$, ρ is represented as a real vector $\mathbf{r} = (r_k) \in \mathbb{R}^{N^2}$

$$\rho = \sum_{k=1}^{N^2} r_k \sigma_k = \sum_{k=1}^{N^2} \mathrm{tr}(\rho\sigma_k)\sigma_k.$$

Then the Lindblad master equation (2) can be equivalently expressed as a real differential equation

$$\dot{\mathbf{r}} = \left(\mathbf{L} + \sum_d \mathbf{D}^{(d)} \right) \mathbf{r} := \mathbf{W}\mathbf{r}, \tag{4}$$

where $\mathbf{L}$, $\mathbf{D}^{(d)} \in \mathbb{R}^{N^2\times N^2}$ with entries

$$L_{mn} = \mathrm{tr}(\imath \mathsf{H}[\sigma_m, \sigma_n]), \tag{5a}$$

$$D^{(d)}_{mn} = \mathrm{tr}(\mathsf{V}_d^\dagger \sigma_m \mathsf{V}_d \sigma_n) - \frac{1}{2}\mathrm{tr}(\mathsf{V}_d^\dagger \mathsf{V}_d\{\sigma_m, \sigma_n\}). \tag{5b}$$

4.1 Transition Matrix

Let $\mathcal{V} := \{1, \dots, 2^n\}$. We introduce two mappings:

(i) $\lfloor\cdot\rfloor : \{0,1\}^n \to \mathcal{V}$, where $\lfloor i_1 \cdots i_n \rfloor = \sum_{k=1}^n i_k 2^{n-k} + 1$;

(ii) $\lceil\cdot\rceil : \mathcal{V} \to \{0,1\}^n$ with $\lceil i \rceil = [i_1 \dots i_n]$ satisfying $i = \sum_{k=1}^n i_k 2^{n-k} + 1$.

Let $\mathsf{M}_{\lceil i\rceil} := \mathsf{P}_{i_1} \otimes \cdots \otimes \mathsf{P}_{i_n}$ denote the projector onto the eigenspace generated by $|v_{i_1} \cdots v_{i_n}\rangle$ for $i_k \in \{0,1\}, k = 1, \dots, n$. Upon measuring the network state ρ, the probability of observing $\lceil i\rceil$ is given by

$$p(\lceil i\rceil) = \mathrm{tr}(\mathsf{M}_{\lceil i\rceil}\rho).$$

Given that the outcome $\lceil i\rceil$ occurred, the qubit network state immediately after the measurement is

$$\rho_{\mathrm{p}} = |v_{i_1} \cdots v_{i_n}\rangle\langle v_{i_1} \cdots v_{i_n}|.$$

Then $\mathsf{M}_{\lceil i\rceil}$ is expressed under the basis $\boldsymbol{\sigma}$ as

$$\mathsf{M}_{\lceil i\rceil} = \sum_{k=1}^{N^2} \theta_{i_k}\sigma_k.$$

Denote $\theta_i = [\theta_{i_1}, \dots, \theta_{i_{N^2}}]^\top, i \in \mathcal{V}$. Let $\Theta = [\theta_1, \dots, \theta_N] \in \mathbb{R}^{N^2\times N}$.

The following theorem presents an explicit representation of the state transition characterization for the induced Boolean series $\{\mathbf{x}(t)\}_{t=0}^\infty$.

Theorem 1. Along the quantum system (1a)–(1b), the induced Boolean network dynamics $\{\mathbf{x}(t)\}_{t=0}^\infty$ form a stationary Markov chain over the state space $\{0,1\}^n$, whose state transition matrix is described by

$$\mathbf{P}_\tau = \Theta^\top e^{\mathbf{W}\tau}\Theta,$$

where $[\mathbf{P}_\tau]_{ij}$ is the transition probability from $\lceil i\rceil$ to $\lceil j\rceil$, here $i, j \in \mathcal{V}$.

5. RELAXING QUANTUM DYNAMICS

In this section, we focus on the case where the semigroup $\{e^{\mathscr{L}s}\}_{s\geq 0}$ from (1a) is relaxing (Schirmer and Wang (2010)), i.e., there exists a unique $\rho_\star$ such that

$$\lim_{s\to\infty} e^{\mathscr{L}s}(\rho(0)) = \rho_\star \tag{6}$$

for all $\rho(0)$. Recall that a state in a Markov chain is an absorbing state if it is not possible to leave whenever this chain arrived at this state. We present the following result.

Theorem 2. Suppose the semigroup $\{e^{\mathscr{L}s}\}_{s\geq 0}$ from (1a) is relaxing with a unique steady state $\rho_\star$. Then for the induced Boolean network dynamics $\{\mathbf{x}(t)\}_{t=0}^\infty$ from the quantum system (1a)–(1b), the following statements hold.

(i) If $\rho_\star \in \{\mathsf{M}_{\lceil i\rceil}\}_{i=1}^N$ and τ is sufficiently large, then the chain $\{\mathbf{x}(t)\}_{t=0}^\infty$ has a unique absorbing state.

(ii) If $\mathrm{tr}(\mathsf{M}_{\lceil i\rceil}\rho_\star) > 0$ for all $i = 1, \dots, N$ and τ is sufficiently large, then $\{\mathbf{x}(t)\}_{t=0}^\infty$ is irreducible and aperiodic.

6. NON-RELAXING QUANTUM DYNAMICS: QUANTUM CONSENSUS NETWORKS

In this section, we turn our attention to non-relaxing quantum dynamics (1a). It is clear that various types of master equations could lead to non-relaxing quantum dynamics. Instead of looking into the general form of (1a), we discuss the quantum network dynamics (1a)–(1b) under the so-called consensus master equation (Shi et al., 2016).

6.1 Consensus Master Equation

A permutation of the qubit set $\mathrm{V} = \{1, \dots, n\}$ is a bijective map from V onto itself. We denote by χ such a permutation. Particularly, a permutation χ is called a swapping between j and k, denoted by χ_{jk}, if $\chi(j) = k$, $\chi(k) = j$, and $\chi(l) = l, l \in \mathrm{V} \setminus \{j,k\}$. The set of all permutations of V forms a group, called the n'th permutation group and denoted by $\Upsilon_n = \{\chi\}$. There are $n!$ elements in Υ_n.

Definition 3. Let $\chi \in \Upsilon_n$. We define the unitary operator U_χ over $\mathcal{H}^{\otimes n}$ induced by χ, by

$$\mathsf{U}_\chi\big(|q_1\rangle \otimes \cdots \otimes |q_n\rangle\big) = |q_{\chi(1)}\rangle \otimes \cdots \otimes |q_{\chi(n)}\rangle,$$

where for $i = 1, \dots, n$. Similarly, with slight abuse of notation, we define the action of χ over $\{0,1\}^{\otimes n}$ by

$$\chi\big([i_1 \dots i_n]\big) = [i_{\chi_1} \dots i_{\chi_n}]$$

where $i_k \in \{0,1\}$ for all $k = 1, \dots, n$.

Let the operator $\mathsf{U}_{\chi_{jk}}$ be denoted as U_{jk} for the ease of presentation. Let the network interaction structure be described by an undirected and, without loss of generality, connected graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$. The so-called quantum consensus master equation is described by (Shi et al., 2016)

$$\dot\rho(s) = \mathscr{L}(\rho(s)) = \sum_{\{j,k\}\in\mathrm{E}} \alpha_{jk}\Big(\mathsf{U}_{jk}\rho\mathsf{U}_{jk}^\dagger - \rho\Big), \tag{7}$$

where $\alpha_{jk} > 0$ represents the weight of link $\{j,k\}$.

Define an operator over the density operators of $\mathcal{H}^{\otimes n}$, $\mathscr{P}_*$, by

$$\mathscr{P}_*(\rho) = \frac{1}{n!}\sum_{\chi\in\Upsilon_n} \mathsf{U}_\chi\rho\mathsf{U}_\chi^\dagger. \tag{8}$$

It is known that when the graph G is connected, along the equation (7) there holds

$$\lim_{s\to\infty}\rho(s) = \mathscr{P}_*(\rho_0) \tag{9}$$

with $\rho(0) = \rho_0$. Clearly the master equation (7) is not relaxing as the limiting point depends on the initial quantum state.

6.2 State Transitions

We are now in a place to study the quantum network dynamics (1a)–(1b) when the continuous quantum dynamics (1a) is described by (7).

Recall that $\mathsf{P}_m = |v_m\rangle\langle v_m|$ is the projector onto the eigenspace generated by the eigenvector $|v_m\rangle \in \mathcal{H}_2$ with eigenvalue λ_m, $m \in \{0,1\}$. Let $\big\{|i^\sharp\rangle\langle j^\sharp|\big\}_{i,j=1}^{N^2}$ be a basis of $\mathcal{L}(\mathcal{H}_N)$, where by definition

$$i^\sharp := v_{i_1}v_{i_2}\dots v_{i_n}$$

with $\lceil i \rceil = [i_1, \dots i_n]$. According to the definition of U_χ, we can verify that

$$\mathsf{U}_\chi\big(|i^\sharp\rangle\big) = \big|v_{i_{\chi(1)}} v_{i_{\chi(2)}} \dots v_{i_{\chi(n)}}\big\rangle. \tag{10}$$

As a result, under the basis of $\big\{|i^\sharp\rangle\big\}_{i=1}^{N}$, the matrix representation of U_χ, denoted U_χ, is a real permutation matrix for any $\chi \in \Upsilon_n$. Similarly, we denote U_{jk} as the matrix representation of the operator U_{jk} under the basis $\big\{|i^\sharp\rangle\big\}_{i=1}^{N}$.

Definition 4. (Shi et al., 2016) The quantum Laplacian of G is defined as

$$L_{\rm q}(\mathrm{G}) := - \sum_{\{j,k\}\in \mathrm{E}} \alpha_{jk}\big(U_{jk} \otimes U_{jk} - I\big).$$

Let i_N denoted the $N \times 1$ unit vector with the ith entry being one and all other entries being zero. Then we can establish the following result.

Proposition 5. Consider (1a)–(1b) with (1a) being described by the quantum consensus master equation (7) under qubit interaction graph G. Define

$$E_N := [1_N \otimes 1_N, \dots, N_N \otimes N_N].$$

Then there holds for the $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ under the measurement $\mathsf{M}^{\otimes n}$ that

$$\mathbf{P}_\tau = E_N^\top e^{-\tau L_{\rm q}(\mathrm{G})} E_N.$$

Proposition 5 shows that the exponential of the quantum Laplacian directly characterizes the state transition matrix of the induced probabilistic Boolean dynamics $\{\mathbf{x}(t)\}_{t=0}^{\infty}$. The proof of Proposition 5 follows from a similar process as the proof of Theorem 1, where we only need to notice the following two points:

(i) The consensus master equation (7) can be written as

$$\frac{d}{ds}\mathrm{vec}([\rho_{ij}(s)]) = -L_{\rm q}(\mathrm{G})\mathrm{vec}([\rho_{ij}(s)]). \tag{11}$$

(ii) Under the basis $\big\{|i^\sharp\rangle\langle j^\sharp|\big\}_{i,j=1}^{N^2}$, there holds

$$\mathsf{M}_{\lceil i \rceil} = \mathsf{P}_{i_1} \otimes \cdots \otimes \mathsf{P}_{i_n} = \big|\lfloor i_1 \cdots i_n \rfloor^\sharp\big\rangle\big\langle\lfloor i_1 \cdots i_n \rfloor^\sharp\big|.$$

6.3 Communication Classes

For the Markov chain $\{\mathbf{x}(t)\}_{t=0}^{\infty}$, a state $[p_1 \dots p_n]$ in its state space is said to be *accessible* from state $[q_1 \dots q_n]$ if there is a nonnegative integer t such that

$$\mathbb{P}\big(\mathbf{x}_t = [p_1 \dots p_n] \;\big|\; \mathbf{x}_0 = [q_1 \dots q_n]\big) > 0.$$

It is termed that $[p_1 \dots p_n]$ *communicates* with state $[q_1 \dots q_n]$ if $[p_1 \dots p_n]$ and $[q_1 \dots q_n]$ are accessible from each other. This communication relationship forms an equivalence relation among the states in $\{0, 1\}^n$. The equivalence classes of this relation are called *communication classes* of the chain $\{\mathbf{x}(t)\}_{t=0}^{\infty}$. The following theorem provides a full characterization to the communication classes of $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ under the consensus master equation.

Theorem 6. Consider (1a)–(1b) with (1a) being described by the quantum consensus master equation (7) under qubit interaction graph G. Then the following statements hold for the $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ under the measurement $\mathsf{M}^{\otimes n}$.

(i) There are $n + 1$ different communication classes.

(ii) For any $g = [g_1, \dots, g_n] \in \{0, 1\}^n$, the communication class containing g is given by

$$\mathcal{C}_g = \Big\{\chi(g) : \chi \in \Upsilon_n\Big\}. \tag{12}$$

(iii) The number of states in $\mathcal{C}_g$ is

$$\binom{n}{|g|} = \frac{n!}{|g|!(n - |g|)!}$$

with $|g| = \sum_{i=1}^{n} g_i$.

Theorem 6 is closely related to the notion of generalized graph of the quantum interaction graph introduced in Shi et al. (2016). The generalized graph is the graph that is consistent with the quantum Laplacian, where for an n-qubit network, its generalized graph contains $N^2 = 4^n$ nodes. Particularly, the communication class $\mathcal{C}_g$ essentially coincides with the connected components of the N nodes in the generalized graph corresponding to the diagonal entry of the network density operator.

6.4 Example

We now present a concrete example as an illustration of the established results for quantum consensus networks with sequential measurements. We consider three qubits indexed by 1, 2, and 3. The qubit interaction graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ is assumed to be a path graph as shown in Figure 2.

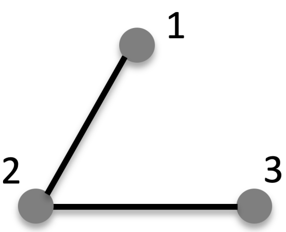

Fig. 2. The interaction graph for the three-qubit network.

The quantum Laplacian $L_{\rm q}(\mathrm{G})$ for this graph is a 64×64 matrix. Let the measurement M be taken under the standard computational basis, without loss of generality, i.e.,

$$\mathsf{M} = \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1|, \tag{13}$$

and the resulting network measurement is $\mathsf{M}^{\otimes 3}$. Let the continuous quantum state follow the evolution described by the quantum consensus master equation (7) with two swapping operators U_{12} and U_{23} as specified from the interaction graph G. Let the measurement $\mathsf{M}^{\otimes 3}$ be carried out periodically with inter-measurement time $\tau = 1$. The measurement outcome for the s'th measurement is recorded as $\mathbf{x}(t) \in \{0, 1\}^3$. Then we can verify the following aspects.

(i) The state transition matrix of the chain $\mathbf{x}(t)$ is given by

$$\mathbf{P}_\tau = E_8^\top e^{-L_{\rm q}(\mathrm{G})} E_8$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.525 & 0.317 & 0 & 0.158 & 0 & 0 & 0 \\ 0 & 0.317 & 0.366 & 0 & 0.317 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.525 & 0 & 0.317 & 0.158 & 0 \\ 0 & 0.158 & 0.317 & 0 & 0.525 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.317 & 0 & 0.366 & 0.317 & 0 \\ 0 & 0 & 0 & 0.158 & 0 & 0.317 & 0.525 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(ii) Let $g = [001]$. Note that $\lfloor 001 \rfloor = 2$. From the second row of $\mathbf{P}_\tau$, clearly the three nonzero entries are $[\mathbf{P}_\tau]_{22} > 0$, $[\mathbf{P}_\tau]_{23} > 0$ and $[\mathbf{P}_\tau]_{25} > 0$. Consequently, the states that are accessible from g are $[001] = \lceil 2 \rceil$, $[010] = \lceil 3 \rceil$, and $[100] = \lceil 5 \rceil$.

On the other hand, we can verify directly that

$$\mathcal{C}_g = \{\chi(g) : \chi \in \Upsilon_3\} = \{[001], [010], [100]\} \tag{14}$$

which is consistent with the communication class that we established directly from $\mathbf{P}_\tau$. This is a validation of Theorem 6.(ii) above.

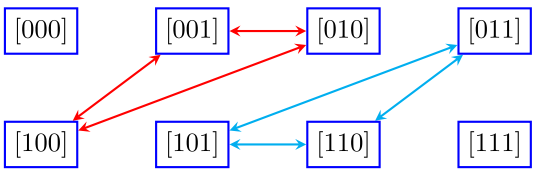

Fig. 3. The state transition map from $\mathbf{P}_\tau$ for the measurement outcomes $\mathbf{x}(t)$.

(iii) We can also establish from $\mathbf{P}_\tau$ (see the resulting state transition map in Figure 3) that the communication classes of $\mathbf{x}(t)$ are

$$\{[000]\};$$
$$\{[001], [010], [100]\};$$
$$\{[011], [101], [110]\};$$
$$\{[111]\}.$$

The number of the communication classes and the size of each communication class are clearly consistent with Theorem 6.(i) and Theorem 6.(iii).

7. CONCLUSIONS

Open quantum networks, as a proven resource for universal quantum computation, are networked quantum subsystems such as qubits with the interconnections established by local environments. Their state evolutions can be described by structured master equations, and in the presence of sequential quantum measurements, the network states undergo random jumps with the measurement outcomes form a probabilistic Boolean network. We showed that the the state transition of the random measurement outcomes can be explicitly represented from the master equation. It was also shown that structural properties including absorbing states, reducibility, and periodicity for the induced Boolean dynamics can be made clear directly when the quantum dynamics is relaxing. For quantum consensus networks as a type of non-relaxing open quantum network dynamics, we showed that the communication classes of the measurement-induced Boolean networks arise from the quantum Laplacian of the underlying interaction graph.

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