Average Dwell-Time Conditions for Input-to-State Stability of Impulsive Systems

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Abstract: This paper provides sufficient conditions for input-to-state stability of impulsive control systems on Banach spaces. The derived conditions determine average dwell-time constraints for a candidate Lyapunov function parametrized by a class of nonlinear rate functions in order to guarantee the ISS property. Thereby, we consider a generalized case with unstable continuous flow maps and assume the jumps, rather than the continuous flow to induce a stabilizing influence on the system dynamics of the impulsive system. Compared to some well-known related and recent results in the literature, such as fixed dwell-time conditions, the obtained conditions are more general, while offering a higher flexibility in the choice of candidate Lyapunov functions.

Keywords: Impulsive system, hybrid dynamical system, input-to-state stability, dwell-time condition, Lyapunov function.

1. INTRODUCTION

The present paper is concerned with sufficient stability conditions for the class of impulsive systems. Impulsive systems cover a mathematical framework for modeling dynamical processes that combine continuous and jumping behavior. An introduction to the theory of impulsive systems is given by Samoilenko and Perestyuk (1995).

The other key concept considered in this paper is input-to-state stability (ISS) which was introduced by Sontag (1989) for systems of ordinary differential equations. ISS characterizes the behavior of solutions to control systems with respect to external inputs. Later, it was extended to switched (Mancilla-Aguilar and García, 2001) and hybrid dynamical systems (Cai and Teel, 2005). Studies on ISS for impulsive control systems were initiated by Hespanha et al. (2005, 2008) by suggesting classes of impulsive time sequences that guarantee the ISS property. Thereby, an exponential candidate ISS-Lyapunov function for impulsive systems, parametrized by two linear rate functions has been utilized to derive dwell-time conditions (DTC).

Roughly speaking, dwell-time conditions establish a balance of continuous and discontinuous dynamics in terms of restrictions on the frequency of the jump sequences to guarantee the ISS property of the underlying impulsive system. Note that the main difference between fixed and average DTC is that fixed DTC characterize the minimum/maximum time interval between two consecutive jumps, while the average ones characterize the average jump frequency. Generally, fixed dwell-time conditions are easier to handle, but more conservative than average ones.

Various extensions of the result of Hespanha et al. (2008) to several classes of impulsive systems have appeared recently in the literature. For instance, impulsive systems with delay (Dashkovskiy et al., 2012; Liu et al., 2011; Sun and Wang, 2012; Wu et al., 2016), infinite-dimensionality (Dashkovskiy and Mironchenko, 2013) and stochasticity (Ren and Xiong, 2017; Wu et al., 2016; Yao et al., 2014) have been extensively studied. In particular, Dashkovskiy and Mironchenko (2013) suggest a generalized setting for studying ISS of impulsive control systems by means of a candidate ISS-Lyapunov function with nonlinear rates $\varphi : [0,\infty) \to \mathbb{R}$ and $\psi : [0,\infty) \to [0,\infty)$, rather than the linear rate functions employed by Hespanha et al. (2005). Naturally, the resulting sufficient conditions appear to be less conservative. While these conditions have been initially established in form of fixed DTC only, it was not until recently that average type ISS-DTC in this general setting were proposed by Feketa and Bajcinca (2019a). The interest in the latter is justified by the reduced conservativeness due to the conjunction of the average description of the jump sequences and the general setting with non-linear flow and discontinuous rates of the candidate Lyapunov functions φ and ψ, respectively.

The work of Feketa and Bajcinca (2019a) focuses on the special case with negative definite function φ, indicating stable flow dynamics of the impulsive system at hand.

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Basically, the present paper extends this result to the dual case with a possibly unstable flow (i.e., positive definite rate function φ), while requiring a stabilizing impact of the discontinuous dynamics as governed by the function ψ at the moments of jump. Compared to the result of Feketa and Bajcinca (2019a), we introduce a doubly parametrized sequence of jumps, this enabling us a wider perspective of the devised condition. Interestingly, it turns out that the aforementioned fixed DTC result is a special case of the average ones. Further interpretations are also given in the hope to develop a better intuition.

The paper is organized as follows. Section 2 recaps the preliminaries and defines the notation used in the paper. In Section 3, we state and prove the main result. Section 4 discusses the developed sufficient conditions in view of the existing ones in the literature. Section 5 provides an example. We complete the paper with concluding remarks in Section 6.

2. PRELIMINARIES

We start by recapping some fundamental concepts and notations used in this paper.

Definition 1. We define the following function classes of so-called *comparison functions*, see Sontag (1989).

(1) *Class* $\mathcal{P}$ is the set of all continuous functions $\gamma : [0,\infty) \to [0,\infty)$ which satisfy $\gamma(0) = 0$ and $\gamma(r) > 0$ for all $r > 0$.
(2) *Class* $\mathcal{K}$ is the set of all continuous functions $\gamma : [0,\infty) \to [0,\infty)$ which are strictly increasing and $\gamma(0) = 0$. *Class* $\mathcal{K}_\infty$ is the subset of class $\mathcal{K}$ for which additionally $\gamma(s) \to \infty$ as $s \to \infty$.
(3) *Class* $\mathcal{KL}$ is the set of all continuous functions $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$, for which $\beta(s,r)$ is class $\mathcal{K}$ for every fixed $r \geq 0$, and for each fixed $s > 0$, the mapping $\beta(s,r)$ is strictly decreasing with respect to r and $\beta(s,r) \to 0$ as $r \to \infty$.

Let $(X, \|\cdot\|)$, $(U, \|\cdot\|)$ be two Banach spaces which represent the state space and the input space, respectively. Let $t_0 \in \mathbb{R}$ be the initial time. U_c is the space of bounded functions from $[t_0,\infty)$ to U and we define the norm

$$\|u\|_\infty := \sup_{t\in[t_0,\infty)} \{\|u(t)\|\}$$

on this space. We denote the left limit of a function f at t by $f^-(t)$.

Let $S = (t_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of impulse times in (t_0,∞) without accumulation points.

An impulsive differential equation (IDE) is defined by interacting continuous and discontinuous evolution maps:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(t, x(t), u(t)), & t &\in [t_0,\infty) \setminus S, \\ x(t) &= g_i\big(x^-(t), u(t)\big), & t &= t_i,\ i \in \mathbb{N}, \end{aligned} \tag{1}$$

where $u \in U_c$ and $x(t) \in X$. The closed linear operator A is the infinitesimal generator of a C_0-semigroup on X, $f : [t_0,\infty) \times X \times U \to X$ and $g_i : X \times U \to X$ for all $i \in \mathbb{N}$. The function f is continuous and Lipschitz-continuous on bounded subsets of X, uniformly with respect to the first and third argument, i.e., for every $T \geq t_0$, every $C \geq 0$ and every $D > 0$, there is a constant $L > 0$, such that

$$\|f(t,x,u) - f(t,y,u)\| \leq L\,\|x - y\|$$

holds for all $x, y \in X$ with $\|x\|,\|y\| \leq C$, all $u \in U_c$ and all $t \in [t_0, T]$. We are interested in solutions in the mild sense, i.e., $x \in \mathcal{PC}([t_0,\infty), X)$, where $\mathcal{PC}([t_0,\infty), X)$ is the space of piecewise continuous functions from $[t_0,\infty)$ to X which are right-continuous and the left limit exists for all times $t \in [t_0,\infty)$. We assume that for the given system, a robust forward unique global mild solution exists for every initial condition $x(t_0) = x_0$ and every $u \in U_c$, i.e. for all $C > 0$ and all $T > t_0$

$$\sup\{\|x(t;t_0,x_0,u)\| \mid \|x_0\| \leq C, t \in [t_0,T], u \in U_c\}$$

exists and is finite. Here, we denoted the value of the solution trajectory at time t with the initial condition $x(t_0) = x_0$ and input $u \in U_c$ by $x(t;t_0,x_0,u)$. We shorten the notation by $x(t)$ if the parameters are clear from the context or can be chosen arbitrarily. Sufficient, though rather strict conditions for the existence and uniqueness of the solutions on Banach spaces are given by Liu (1999) and Ahmed (2003).

Definition 2. For a given sequence of impulse times S we call system (1) *input-to-state stable (ISS)* if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial values $(t_*, x_*) \in [t_0,\infty) \times X$ and every input function $u \in \mathcal{PC}([t_*,\infty), U)$, system (1) has a global solution which satisfies

$$\|x(t;t_*,x_*,u)\| \leq \beta(\|x_*\|, t - t_*) + \gamma(\|u\|_\infty)$$

for all $t \in [t_*,\infty)$.

Definition 3. A continuous function $V : X \to \mathbb{R}_0^+$ is called a *candidate ISS-Lyapunov function* for an impulsive differential equation (1), if it fulfills the following conditions:

(1) There exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that
$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$
holds true for all $x \in X$.
(2) There exist such functions $\chi \in \mathcal{K}_\infty$, $\psi \in \mathcal{P}$ and a continuous function $\varphi : [0,\infty) \to \mathbb{R}$ with $\varphi(0) = 0$, that for all $u \in U_c$ and all $x = x(t;t_0,x_0,u)$ whenever
$$V(x) \geq \chi(\|u\|_\infty),$$
the differential and jump inequalities
$$\frac{\mathrm{d}}{\mathrm{d}t}(V(x)) \leq \varphi(V(x)), \tag{2}$$
$$V\big(g_i(x^-, u(t_i))\big) \leq \psi\big(V(x^-)\big) \tag{3}$$
hold true for all $i \in \mathbb{N}$. Here, $\frac{\mathrm{d}}{\mathrm{d}t}(V(x))$ stands for the *Dini-derivative*
$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_*) = \limsup_{s\searrow 0} \frac{1}{s}\left(V(x(s+t;t,x_*,u)) - V(x_*)\right).$$
(3) There exists a function $\alpha_3 \in \mathcal{K}$ such that for all $x \in X$ and all $u \in U$ which satisfy
$$V(x) < \chi(\|u\|),$$
as well as for all $i \in \mathbb{N}$ the jump inequality satisfies
$$V(g_i(x,u)) \leq \alpha_3(\|u\|). \tag{4}$$

Remark 4. We emphasize that Condition 3 in Definition 3 is essential. The results of e.g. Feketa and Bajcinca (2019a) and Dashkovskiy and Mironchenko (2013) neglect to mention this condition, but it is mandatory there as well. We show its necessity in the following example: Let us consider the one-dimensional system

$$\dot{x} = -x, \qquad \forall t \in \mathbb{R}_0^+ \setminus \mathbb{N};$$
$$x = \begin{cases} \frac{1}{2}x^-, & \text{if } |x^-| \geq |u|, \\ 1, & \text{else}, \end{cases} \qquad \forall t \in \mathbb{N}.$$

The function $V(x) = |x|$ satisfies Conditions (1) and (2) with $\chi(s) = s$, $\varphi(s) = \psi(s) = \frac{1}{2}s$, but there exists no continuous $\mathcal{K}_\infty$-function γ such that the system is ISS.

Remark 5. A possible way to dispense with the requirement of global existence of the solution of (1) is by imposing the following condition: Rate function φ is Lipschitz-continuous, i.e., there exists a constant $L > 0$ such that for all $s, r \geq 0$ the estimate $|\varphi(s) - \varphi(r)| \leq L|s - r|$ holds true. The global existence of (7) on intervals $[t_i, t_{i+1})$, $i \in \mathbb{N}_0$ then follows from Picard-Lindelöf theorem, and by Definition 3, Condition (1), $x(t)$ is bounded for each $t \in [t_0, \infty)$. From Pazy (1983, Chapter 6, Theorem 1.4) one can conclude the existence of a unique global solution.

Remark 6. For a compact Banach space, e.g., if X is finite-dimensional, Condition 3 in Definition 3 can be substituted by the requirement that $g_i(0, 0) = 0$ and continuity of g_i. However, this is not sufficient for the non-compact case anymore, as $g_i(\,\cdot\,, u)$ may not be bounded on the ball $B_r = \{x \in X \mid \|x\| < r\}$ for some $r > 0$ and $\|u\| = r$. Apart from this, our proof will not exhibit additional complexity from the use of infinite-dimensional spaces.

Finally, by $N = N(t, s) : [t_0, \infty) \times [t_0, \infty) \to \mathbb{N}_0$, we define the number of elements of S in the interval $(s, t]$, reflecting the number of jumps in the same interval.

3. MAIN RESULT

With the definitions of the last section, we are now able to state our main result.

Theorem 7. Let there exist a candidate ISS-Lyapunov function for the impulsive differential equation (1) with rates $\varphi \in \mathcal{P}$, $\psi \in \mathcal{P}$ defined as in Definition 3 and constants $\rho, N_0 > 0$ such that for S the inequality

$$N(t, s) \geq \rho \cdot (t - s) - N_0 \tag{5}$$

holds for all $t > s \geq t_0$. If there exists some $\delta > 0$, such that for all $a > 0$, the inequality

$$\int_a^{\psi(a)} -\frac{1}{\varphi(s)}\,\mathrm{d}s \geq \frac{1}{\rho} + \delta \tag{6}$$

holds, then the impulsive system (1) is ISS.

Note that (5) and (6) define our Lyapunov-based dwell-time condition. Also for all $a > 0$, the inequality $\psi(a) < a$ needs to be fulfilled because the integral of a negative term can only be strictly positive if the upper limit of the integral in (6) is smaller than the lower limit. We will use this fact several times in the rest of the paper.

Proof. We first assume $V(x(t)) \geq \chi(\|u\|_\infty)$ for all (t, x, u). In this case, by Definition 3 the function V fulfills the inequalities:

$$\frac{\mathrm{d}}{\mathrm{d}t}(V \circ x)(t) \leq \varphi((V \circ x)(t)), \qquad t \in [t_0, \infty) \setminus S, \tag{7}$$
$$(V \circ x)(t) \leq \psi\big((V \circ x)^-(t)\big), \qquad t \in S. \tag{8}$$

If the right hand side of (7) is not equal to zero, we can transform (7) to

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}(V \circ x)(t)}{\varphi((V \circ x)(t))} \leq 1. \tag{9}$$

Hereby, we explicitly exclude the case $\varphi((V \circ x)(t)) = 0$, i.e., $(V \circ x)(t) = 0$ from this inequality. However, we do not need this estimate anyway because trajectories of (7)–(8) that become equal to zero will remain identical to zero.

Integrating (9) over the interval $[t_i, t_*]$, $i \in \mathbb{N}_0$, for some $t_* \in [t_i, t_{i+1})$, we obtain

$$\int_{t_i}^{t_*} \frac{\frac{\mathrm{d}}{\mathrm{d}t}(V \circ x)(t)}{\varphi((V \circ x)(t))}\,\mathrm{d}t \leq t_* - t_i. \tag{10}$$

In particular, with the substitution $s := (V \circ x)(t)$, the left limit $t_* \nearrow t_{i+1}$ reads

$$\int_{(V\circ x)(t_i)}^{(V\circ x)^-(t_{i+1})} \frac{1}{\varphi(s)}\,\mathrm{d}s \leq t_{i+1} - t_i. \tag{11}$$

In contrast to fixed dwell-time, for average dwell-time assertions it is not sufficient to investigate the time between two jumps only to decide whether the system is ISS. Instead, to prove stability one needs to consider a long-term average. To this end, we evaluate the solution after a sufficiently large number of jumps $n \in \mathbb{N}$. Assembling (6) and (11) yields the upper-bound estimate

$$\begin{aligned}
&\int_{(V\circ x)(t_i)}^{(V\circ x)(t_{i+n})} \frac{1}{\varphi(s)}\,\mathrm{d}s \\
&= \sum_{k=i+1}^{i+n} \left(\int_{(V\circ x)(t_{k-1})}^{(V\circ x)^-(t_k)} \frac{1}{\varphi(s)}\,\mathrm{d}s + \int_{(V\circ x)^-(t_k)}^{\psi((V\circ x)^-(t_k))} \frac{1}{\varphi(s)}\,\mathrm{d}s \right) \\
&\leq \sum_{k=i+1}^{i+n} \left(t_k - t_{k-1} - \left(\frac{1}{\rho} + \delta\right) \right) \\
&= t_{i+n} - t_i - n\left(\frac{1}{\rho} + \delta\right).
\end{aligned}$$

The restriction for admissible jump sequences (5) enables us to estimate the length of the time interval (t_i, t_{i+n}) by

$$t_{i+n} - t_i \leq \frac{N(t_{i+n}, t_i) + N_0}{\rho} = \frac{n + N_0}{\rho}. \tag{12}$$

From this, it follows

$$\begin{aligned}
\int_{(V\circ x)(t_i)}^{(V\circ x)(t_{i+n})} \frac{1}{\varphi(s)}\,\mathrm{d}s &\leq t_{i+n} - t_i - n\left(\frac{1}{\rho} + \delta\right) \\
&\leq \frac{n + N_0}{\rho} - n\left(\frac{1}{\rho} + \delta\right) \\
&= \frac{N_0}{\rho} - n\delta.
\end{aligned}$$

If we now set $n \geq 2\frac{N_0}{\delta\rho}$, we obtain

$$\int_{(V\circ x)(t_{i+n})}^{(V\circ x)(t_i)} \frac{1}{\varphi(s)}\,\mathrm{d}s \geq \frac{N_0}{\rho} > 0. \tag{13}$$

Analogously to Dashkovskiy and Mironchenko (2013), we show that from this condition we can conclude that

$(V \circ x)(t)$ goes to zero for increasing t. Hence, we fix $r > 0$ and define the function $F : (0, \infty) \to \mathbb{R}$

$$F(q) := \int_r^q \frac{1}{\varphi(s)}\, \mathrm{d}s.$$

F is strictly increasing for all $q \in (0, \infty)$ because φ is positive. The image of F is an open interval of the form $(-\infty, M)$ for some constant $M \in \mathbb{R} \cup \{\infty\}$ since

$$\begin{aligned}\lim_{q \to 0} \int_r^q \frac{1}{\varphi(s)}\, \mathrm{d}s &= \int_r^{\psi(r)} \frac{1}{\varphi(s)}\, \mathrm{d}s + \int_{\psi(r)}^{\psi^2(r)} \frac{1}{\varphi(s)}\, \mathrm{d}s + \ldots \\ &= \sum_{k=0}^{\infty} \int_{\psi^k(r)}^{\psi^{k+1}(r)} \frac{1}{\varphi(s)}\, \mathrm{d}s \\ &\le \sum_{k=0}^{\infty} \left(-\frac{1}{\rho} - \delta\right).\end{aligned}$$

The last term goes to minus infinity. We have used in the first line that $a > \psi(a) > 0$ for all $a > 0$ as ψ is a $\mathcal{P}$-function. By ψ^k, we mean the k-times composition $\psi \circ \cdots \circ \psi$, where $\psi^0(a) = a$. The upper bound is open as F is strictly increasing. Therefore, F is invertible and $F^{-1} : (-\infty, M) \to (0, \infty)$ is also an increasing function.

Substitution into (10) returns

$$F((V \circ x)(t)) - F((V \circ x)(t_i)) \le t - t_i$$

for all $t \in [t_i, t_{i+1})$ which is equivalent to

$$F((V \circ x)(t)) \le F((V \circ x)(t_i)) + t - t_i \qquad \forall t \in [t_i, t_{i+1}).$$

We show that this inequality is true for all $t > t_i$ by applying the fact that $\psi(a) < a$. Accordingly, the jumps are always stabilizing the system and by only considering the flow, we obtain the upper bound

$$F((V \circ x)(t)) \le F((V \circ x)(t_i)) + t - t_i \qquad \forall t > t_i \quad (14)$$

for the behavior of the system.

After substituting F in (13), we obtain

$$F\big((V \circ x)\big(t_{(k+1)n}\big)\big) \le F((V \circ x)(t_{kn})) - \frac{N_0}{\rho},$$

where $k \in \mathbb{N}_0$. By induction, we get

$$F((V \circ x)(t_{kn})) \le F((V \circ x)(t_0)) - k\frac{N_0}{\rho}. \quad (15)$$

Combining (14) and (15) results in

$$\begin{aligned}F((V \circ x)(t)) &\le F((V \circ x)(t_0)) - k\frac{N_0}{\rho} + t - t_{kn} \\ &\le F((V \circ x)(t_0)) - k\frac{N_0}{\rho} + \frac{n + N_0}{\rho}\end{aligned}$$

for all $t \in [t_{kn}, t_{(k+1)n})$. Additionally, we used (12) here. As $N(t, t_0) \le (k + 1)n$ for such $t \in [t_{kn}, t_{(k+1)n})$, the restriction for admissible jump sequences (5) yields

$$k \ge \frac{1}{n}\left(\rho \cdot (t - t_0) - N_0\right) - 1. \quad (16)$$

Therefore, we have

$$\begin{aligned}&F((V \circ x)(t)) \\ &\le F((V \circ x)(t_0)) - k\frac{N_0}{\rho} + \frac{n + N_0}{\rho} \\ &\le F((V \circ x)(t_0)) - \frac{N_0}{n}(t - t_0) + \frac{N_0^2}{n\rho} + \frac{n + 2N_0}{\rho}.\end{aligned}$$

Note that F might not be invertible for every value of the right hand side. We circumvent this issue by only inverting a domain where $-\frac{N_0}{n}(t - t_0) + \frac{N_0^2}{n\rho} + \frac{n+2N_0}{\rho}$ is negative. We achieve this by choosing

$$t \ge \bar{t} := \frac{n^2}{N_0\rho} + \frac{N_0 + 2n}{\rho} + t_0.$$

Then, the estimate

$$(V \circ x)(t) \le F^{-1}\left(F((V \circ x)(t_0)) - \frac{N_0}{n}\left(t - \bar{t}\right)\right) \quad (17)$$

holds for all $t \ge \bar{t}$.

The solution x of (1) is piecewise bounded by its flow $\dot{x}(t) = Ax(t) + f(t, x(t), u(t))$, which is robustly forward complete and Lipschitz on bounded subsets of X. We apply Lemma 4.6 by Mironchenko and Wirth (2018) here to show that $V \circ x \le \alpha_2(\|x\|)$ is bounded for $t \in \left[t_0, \bar{t}\right]$ by a function which is a $\mathcal{K}$-function in the initial condition. From this and (17) we can define a $\mathcal{KL}$-function $\tilde{\beta} = \tilde{\beta}((V \circ x)(t_0), t - t_0)$ such that

$$(V \circ x)(t) \le \tilde{\beta}((V \circ x)(t_0), t - t_0) \quad (18)$$

is satisfied as long as $V(x(t)) \ge \chi(\|u\|_\infty)$.

Until now, we only dealt with the case $V(x(t)) \ge \chi(\|u\|_\infty)$. Now let us define the set

$$A_1 := \{x \in X \,|\, V(x) < \chi(\|u\|_\infty)\}\,.$$

and investigate the behavior when the trajectory leaves A_1. We fix an initial condition (t_0, x_0). For all t which satisfy $x(t) \notin A_1$ the stability relation (18) is fulfilled. So, we investigate $t^* := \inf\{t \in [t_0, \infty] \,|\, x(t) \in A_1\}$. Then

$$\|x(t; t_0, x_0, u)\| \le \beta(\|x_0\|, t - t_0) \quad (19)$$

holds for $t \in [t_0, t^*]$, $\beta(r, s) := \alpha_1^{-1}\left(\tilde{\beta}(\alpha_2(r), s)\right)$, where $\beta \in \mathcal{KL}$. In case $t^* = \infty$, (19) holds for $t \in [t_0, \infty)$. But, then $x(t) \to 0$ for $t \to \infty$ follows immediately.

We define the sets

$$\begin{aligned}A_2 &:= \{x \in X \,|\, V(x) \le \max\{\alpha_3(\|u\|_\infty), \chi(\|u\|_\infty)\}\}\,, \\ A_3 &:= \{x \in X \,|\, V(x) \le \tilde{\gamma}(\|u\|_\infty)\}\,,\end{aligned}$$

where $\tilde{\gamma} \in \mathcal{K}_\infty$,

$$\tilde{\gamma}(s) = \max\Big\{\alpha_3(s)\,, \chi(s)\,, \tilde{\beta}\left(\max\{\alpha_3(s)\,, \chi(s)\}\,, 0\right)\Big\}\,.$$

We now show that any trajectory starting in A_1 remains in A_3. Obviously, $A_1 \subset A_2 \subset A_3$. All trajectories that leave A_1 by jump are bounded by (4) and do not leave A_2. Trajectories leaving A_1 by flow have to cross the boundary ∂A_1. In both cases there must be a $t' \in [t_0, \infty)$ such that $x(t') \subset A_2 \setminus A_1$. Therefore, we can apply (18) with $t = t_0 = t'$. By construction, all the trajectories that leave A_1 will stay in A_3. We define $\gamma \in \mathcal{K}_\infty$, $\gamma := \alpha_1^{-1} \circ \tilde{\gamma}$ such that

$$\|x(t; t_0, x_0, u)\| \le \gamma(\|u\|_\infty) \quad (20)$$

is satisfied for all $t > t^*$. Adding up the right-hand sides of (19) and (20), we obtain the desired result

$$\|x(t; t_0, x_0, u)\| \le \beta\left(\|x_0\|, t - t_0\right) + \gamma(\|u\|_\infty)$$

for all $t \ge t_0$. □

4. DISCUSSION

Next, we provide several remarks and comments to compare our result with existing ones in the literature.

Remark 8. As the present paper is inspired by and adopts parts of the proof of Feketa and Bajcinca (2019a), we want to point out some key differences. Most importantly, the present paper addresses unstable continuous dynamics and stabilizing jumps rather than stable flows and destabilizing jumps. Furthermore, we discuss the impulsive control systems on Banach spaces, thus covering the case of infinite-dimensional spaces, as well, which is not regarded by Feketa and Bajcinca (2019a).

In particular, with (5) we relax the limit-based definition of the class of impulse sequences given in Equation (5) by Feketa and Bajcinca (2019a), as it turns out that the requested uniformity restriction therein can be impossibly fulfilled by infinite sequences. We show it as follows: Let us assume the limit exists and equals ρ. We choose two sequences $(\tau_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ where

$$\tau_n = t_n + \frac{1}{2n}, \qquad s_n = t_n - \frac{1}{2n}.$$

The sequences are chosen such that $N(\tau_n, s_n) \geq 1$ for each n. Then

$$\rho = \lim_{t\to\infty} \sup_{s\in[t_0,t]} \left|\frac{N(t,s)}{t-s}\right| \geq \lim_{n\to\infty} \left|\frac{N(\tau_n,s_n)}{\tau_n - s_n}\right| \geq \lim_{n\to\infty} \frac{1}{2\frac{1}{2n}},$$

where the last term goes to infinity. This is a contradiction.

While with the new sequence definition (5), the proof steps remain similar, we emphasize that our proof in the present paper covers more details, e.g., in how to get from (13) to (18). Therefore, it is expected (and, in fact, it can be shown by following the lines of proof of our Theorem 7) that the main result of Feketa and Bajcinca can be adopted to:

Theorem 9. Let there exist a candidate ISS-Lyapunov function for the impulsive differential equation (1) with rates $-\varphi \in \mathcal{P}$, $\psi \in \mathcal{P}$ and constants $\rho, N_0 > 0$ such that

$$N(t,s) \leq \rho \cdot (t-s) + N_0 \tag{21}$$

holds for all $t > s \geq t_0$. If for some $\delta > 0$ and all $a > 0$

$$\int_a^{\psi(a)} -\frac{1}{\varphi(s)}\,\mathrm{d}s \leq \frac{1}{\rho} - \delta \tag{22}$$

is true, then the impulsive system (1) is ISS.

Note that here the flow is restricted to be stable.

Remark 10. While the restrictions for the jump sequence (5) and (21) are inherited with a slight modification from

$$-dN(t,s) - (c-\lambda)(t-s) \leq \mu$$

with $c, d \in \mathbb{R}$ and $\mu, \lambda > 0$, which was introduced by Hespanha et al. (2008), the main difference in the present paper refers to the nonlinearity of the Lyapunov rate functions (see also Feketa and Bajcinca, 2019a). Indeed, Hespanha et al. address average DTC for linear rate functions of the form $\varphi(s) = -cs$ and $\psi(s) = e^{-d}s$ only. We show the advantage of nonlinear rates in Example 14.

Remark 11. Let the conditions of Theorem 9 hold, where

$$\tau := \sup_{a>0} \int_a^{\psi(a)} -\frac{1}{\varphi(s)}\,\mathrm{d}s.$$

Theorem 9 claims that impulsive system (1) is ISS, if for some $\rho \in (0, \frac{1}{\tau})$ and some $N_0 > 0$ the condition (21) is fulfilled, although impulses might cause destabilizing effects. Thus, $\frac{1}{\tau}$ can be interpreted as an upper bound to the average frequency of destabilizing jumps ρ, while the parameter N_0 represents the maximum number of additional jumps above average on an arbitrary interval. Similarly, let (1) have stable jumps such that $\psi(a) < a$ holds for all $a > 0$. We further define

$$\tau := \inf_{a>0} \int_a^{\psi(a)} -\frac{1}{\varphi(s)}\,\mathrm{d}s.$$

If there are $\rho > \frac{1}{\tau}$ and $N_0 > 0$ that fulfill condition (5), then, despite the potentially destabilizing flow, the impulsive system (1) is ISS. Here, $\frac{1}{\tau}$ is a lower bound on the average frequency of jumps ρ and N_0 defines a lower bound on the number of jumps under average on any interval.

By substituting the linear rate functions $\varphi(s) = -cs$ and $\psi(s) = e^{-d}s$ in the above equations, we obtain $\tau = \left|\frac{d}{c}\right|$, which matches the outcome of Hespanha et al. (2008).

Remark 12. Condition (5) restricts the admissible impulse sequences, i.e., the maximum interval without impulses is

$$\begin{aligned}\sup_{i\in\mathbb{N}} \{t_i - t_{i-1}\} &= \lim_{\varepsilon\to 0} \sup_{i\in\mathbb{N}} \{(t_i - \varepsilon) - t_{i-1}\} \\ &\leq \lim_{\varepsilon\to 0} \sup_{i\in\mathbb{N}} \left\{\frac{N(t_i-\varepsilon, t_{i-1}) + N_0}{\rho}\right\} = \frac{N_0}{\rho}\end{aligned} \tag{23}$$

as $N(t_i - \varepsilon, t_{i-1}) = 0$ for sufficiently small parameters $\varepsilon > 0$. This is not a lack of generality of the theorem, it is rather inherently impossible to extend the result to impulsive control systems with unstable flows and unrestricted intervals without jumps. Indeed, ISS is by Definition 2 a uniform property with regard to initial time as function β only depends on $t - t_*$. In arbitrarily large regions without jumps the flow cannot be bounded uniformly in time. This means that there cannot be a strictly falling function $\beta(r, \cdot)$ which bounds the trajectories of such a system.

An option to obtain stability results for that kind of jump sequences might be the introduction of a notion of *pointwise ISS* such that function β is also dependent on the initial time. Then, convergence is a pointwise property in the initial time. That must be subject of further studies.

Remark 13. The theorems on average dwell-time given in this paper include the equivalent results for fixed dwell-time by Dashkovskiy and Mironchenko (2013). We show this for ISS with stabilizing flows and potentially unstable jumps. If we bound the interval between two consecutive jumps by $T_{\min} := \inf_{i\in\mathbb{N}} \{t_i - t_{i-1}\}$, we can apply (21) to fix the parameters N_0 and ρ. We set $N_0 = 1$. Then

$$\begin{aligned}T_{\min} &= \lim_{\varepsilon\to 0} \inf_{i\in\mathbb{N}} \{t_i - (t_{i-1} - \varepsilon)\} \\ &\geq \lim_{\varepsilon\to 0} \inf_{i\in\mathbb{N}} \left\{\frac{N(t_i, t_{i-1}-\varepsilon) - N_0}{\rho}\right\} = \frac{1}{\rho}\end{aligned}$$

holds as $N(t_i, t_{i-1} - \varepsilon) = 2$ for sufficiently small $\varepsilon > 0$. This gives exactly the dwell-time restriction

$$\int_a^{\psi(a)} -\frac{1}{\varphi(s)}\,\mathrm{d}s \leq T_{\min} - \delta$$

which resembles Theorem 1 stated by Dashkovskiy and Mironchenko (2013). Analogously, for ISS with stabilizing jumps and possibly unstable flows we fix $N_0 = 1$ and obtain the upper bound $T_{\max} := \sup_{i\in\mathbb{N}} \{t_i - t_{i-1}\} \leq \frac{1}{\rho}$ from (23). If we substitute this estimate into (6), the resulting restriction

$$\int_a^{\psi(a)} -\frac{1}{\varphi(s)}\,\mathrm{d}s \geq T_{\max} + \delta, \tag{24}$$

matches Dashkovskiy and Mironchenko (2013, Thm. 3).

5. EXAMPLE

Theorem 7 provides relaxed conditions for finding a candidate Lyapunov function and rate functions compared to linear rate functions φ, ψ in Hespanha et al. (2008). Besides, our result generalizes Theorem 3 of Dashkovskiy and Mironchenko (2013) with fixed dwell-time conditions. This is illustrated by the following example.

Example 14. Let

$$\begin{aligned} \dot{x} &= \tanh\left(\sqrt{|xu|}\right), & \forall t \in [0,\infty) \setminus S, \\ x &= \begin{cases} \frac{1}{2}(x^-)^3, & \text{if } x^- \in [-1,1], \\ \frac{1}{2}(x^-)^{\frac{1}{3}}, & \text{else}, \end{cases} & \forall t \in S \end{aligned} \tag{25}$$

and $S = \{t_i \,|\, i \in \mathbb{N}\}$, where $t_i \in \left[\frac{4}{5}i - 0.35, \frac{4}{5}i + 0.35\right]$ is chosen independently equally distributed. Obviously, we then have $\rho = \frac{5}{4}$.

Furthermore, we choose the Lyapunov function $V(x) = |x|$ and $\chi(s) = s$ of Definition 3, as well as rate functions $\varphi(s) = \tanh(s)$ and ψ as defined by

$$\psi(s) = \begin{cases} \psi_1(s), & \text{if } s \in [-1,1], \\ \psi_2(s), & \text{else}, \end{cases}$$

where $\psi_1(s) = \frac{1}{2}s^3$ and $\psi_2(s) = \frac{1}{2}s^{\frac{1}{3}}$. Inequality (6)

$$\int_a^{\frac{1}{2}a^3} -\frac{1}{\tanh(x)}\,\mathrm{d}x \geq \ln(e+1) - \frac{1}{2} \approx 0.81 > \frac{4}{5} = \frac{1}{\rho}$$

for $0 < a \leq 1$ and

$$\int_a^{\frac{1}{2}a^{\frac{1}{3}}} -\frac{1}{\tanh(x)}\,\mathrm{d}x \geq \ln(e+1) - \frac{1}{2} \approx 0.81 > \frac{4}{5} = \frac{1}{\rho}$$

for $a > 1$, respectively, holds for all $a > 0$, which means that system (25) is ISS. Condition (24) is not fulfilled as $T_{\max} = 1.5$. As shown in Example 1 of Feketa and Bajcinca (2019b), there are no linear rate functions for the candidate Lyapunov function $V(x) = |x|$ which guarantee global asymptotic stability. Therefore, no statement of ISS with linear rate functions can be made either. On the other hand, finding a suitable candidate Lyapunov function might be difficult or impossible. □

6. CONCLUSION

The present paper provides sufficient ISS conditions for the class of control impulsive systems in form of average DTC expressed in terms of nonlinear Lyapunov rate functions and a doubly parametrized sequence of jumps. It produces a less conservative statement with regard to existing sufficiency conditions defined in terms of average and fixed DTC with linear and nonlinear candidate Lyapunov rate functions, respectively. In particular, we demonstrate that ISS fixed dwell-time conditions can be interpreted as a special case of average dwell-time conditions in our setting. Our work offers space for various extensions. One possibility is to investigate pointwise ISS for systems with unbounded jump-free interval lengths. Another perspective is to extend the result to a setting with stochastic jump times. However, these topics will be reported elsewhere.

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