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The J-Orthogonal Square-Root Euler-Maruyama-Based Unscented Kalman Filter for Nonlinear Stochastic Systems ⋆

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Abstract: This paper addresses the issue of square-rooting in the Unscented Kalman Filtering (UKF) methods. Since their discovery the UKF is considered to be among the most valued state estimation algorithms because of its outstanding performance in numerous real-world applications. However, the main shortcoming of such a technique is the need for the Cholesky decomposition of predicted and filtering covariances derived in all time and measurement update steps. Such a factorization is time-consuming and highly sensitive to round-off and other errors committed in the course of calculation, which can result in losing the covariance's positivity and, hence, in failing the Cholesky decomposition. The latter problem is usually overcome via square-root filtering implementations, which propagate not the covariance itself but only its square root (Cholesky factor). Unfortunately, negative weights arising in applications of the UKF schemes to large stochastic systems preclude from designing conventional square-root UKF methods. So, we resolve it with a hyperbolic QR factorization used for yielding J-orthogonal square roots. Our novel square-root filter is grounded in the Euler-Maruyama discretization of order 0.5. It is justified theoretically and examined and compared numerically to the conventional (non-square-root) UKF in an aircraft's coordinated turn scenario with ill-conditioned measurements.

Keywords: Continuous-discrete nonlinear stochastic model, unscented Kalman filter, square-root implementation, radar tracking, maneuvering target, ill-conditioned measurements.

1. INTRODUCTION

Many state estimation tasks rely on continuous stochastic systems with discrete measurements, which take the form

$$dX(t) = F\big(X(t)\big)dt + GdW(t), \quad t > 0, \tag{1}$$

$$Z_k = h(X_k) + V_k, \quad k \geq 1. \tag{2}$$

The process model (1) is supposed to be an Itô-type *Stochastic Differential Equation* (SDE), in which the unknown stochastic process $X(t)$ represents the state of the plant of size n at time t, the known nonlinear vector-function $F : \mathbb{R}^n \to \mathbb{R}^n$ describes its dynamic behavior, the diffusion matrix G is assumed to be time-invariant and of size $n \times n$ in the driving noise used, and the random disturbance $\{W(t), t > 0\}$ is a multivariate Wiener process with independent zero-mean Gaussian increments $dW(t)$ having a covariance of the form Qdt of size $n \times n$ where the matrix Q is positive definite and fixed in time. The initial state can also be a Gaussian variable $X(0) \sim \mathcal{N}(\bar{X}_0, \Pi_0)$ with mean $\bar{X}_0$ and covariance $\Pi_0 > 0$ in SDE (1). Next, the discrete-time measurement model (2) with k being a discrete time index (i.e. X_k means $X(t_k)$) establishes a nonlinear in general link $h : \mathbb{R}^n \to \mathbb{R}^m$ between the distribution of the state X_k in the dynamic process at hand and its measurement Z_k of size m corrupted by a zero-mean Gaussian variable $\{V_k, k \geq 1\}$ with its covariance $R_k > 0$ at every sampling instant t_k. The measurement information Z_k arrives uniformly and with the sampling rate $\delta = t_k - t_{k-1}$ in our setting. This time interval δ is also known as the sampling period in filtering theory. Furthermore, all realizations of the noises $dW(t)$, V_k and the initial state $X(0)$ are taken from mutually independent Gaussian distributions. The continuous-discrete state estimation scenarios are often encountered in practical modeling and motivated in Jazwinski (1970); Särkkä (2007).

In 1995, Julier et al. (1995) gave rise a new class of state estimation algorithms termed the *Unscented Kalman Filter* (UKF), which were found to be a successful alternative to the traditional *Extended Kalman Filter* (EKF) in state and parameter estimation and machine learning tasks by Julier et al. (2000); Julier and Uhlmann (2004); Wan and Van der Merwe (2001) and many others. Certainly, some more advanced EKF versions elaborated in Gustafsson and Hendeby (2012) can be comparable and even outperform the UKF, but these are beyond the scope of our study.

At the heart of all UKF-based techniques lies the notion of *Unscented Transform* (UT). Following Julier et al. (1995, 2000); Julier and Uhlmann (2004), this notion refers to a method for evolution of the mean and covariance of a random variable X of size n, whose first two moments $\hat{X}$

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and P_X are supposed to be known, under a sufficiently smooth nonlinear transformation $F(X)$ also of size n. In other words, it computes asymptotically sound estimates to the mean and covariance of the n-dimensional random variable $Y := F(X)$ provided that all higher moments of the distribution X can be neglected, as explained below.

First, for a given random vector X of size n with mean $\hat{X}$ and covariance matrix P_X, one calculates a set of $2n+1$ deterministically selected vectors called *Sigma Points* (SP)

$$\mathcal{X}_0 := \hat{X},\ \mathcal{X}_i := \hat{X} + \sqrt{3}S_X e_i,\ \mathcal{X}_{i+n} := \hat{X} - \sqrt{3}S_X e_i, \quad (3)$$

$i = 1, 2, \ldots, n$, where e_i denotes the i-th unit coordinate vector in $\mathbb{R}^n$ (i.e. the i-th column in the identity matrix of size n) and S_X stands for the covariance matrix *Square Root* (SR). Below, the concept of SR refers to a square matrix of size n that satisfies the following condition:

$$P_X = S_X S_X^\top \quad (4)$$

where $S_X^\top$ denotes the transposed version of the SR S_X. Note that the SR presentation (4) of covariance P_X is not unique and any product $S_X Q$ with an orthogonal factor Q gives another SR. Historically, the SR S_X is commonly taken to be the lower triangular Cholesky factor. Theoretically, the Cholesky covariance factorization always exists and can be fulfilled because of the positive definiteness of covariance matrices utilized in stochastic modeling.

Second, the SP-set (3) is supplied with scalars $w_i^{(m)}$ and $w_i^{(c)}$, $0, 1, \ldots, 2n$, which are referred to as *UT weights*. Here, we deal with their classical parametrization

$$w_0^{(m)} = w_0^{(c)} := (3-n)/3,\ w_i^{(m)} = w_i^{(c)} := 1/6,\ i = 1, \ldots, 2n. \quad (5)$$

Certainly, other SP-sets and UT parameterizations are possible and surveyed in Menegaz et al. (2015). However, we restrict ourselves to their particular case (3) and (5) considered in this paper. The main property of the UT is that it calculates the given mean and covariance as follows:

$$\hat{X} = \sum_{i=0}^{2n} w_i^{(m)} \mathcal{X}_i \text{ and } P_X = \sum_{i=0}^{2n} w_i^{(c)} (\mathcal{X}_i - \hat{X})(\mathcal{X}_i - \hat{X})^\top. \quad (6)$$

Third, the UT based on the SP-set (3) and weights (5) with condition (6) computes the mean and covariance of the transformed random variable $Y := F(X)$ by the formulas

$$\hat{Y} \approx \sum_{i=0}^{2n} w_i^{(m)} F(\mathcal{X}_i),\ P_Y \approx \sum_{i=0}^{2n} w_i^{(c)} (F(\mathcal{X}_i) - \hat{Y})(F(\mathcal{X}_i) - \hat{Y})^\top. \quad (7)$$

Eqs (7) underlie any UKF employed in practice. Note that these formulas enjoy the true approximation of the first three moments in the Taylor expansion of the mean vector $\hat{Y}$ and covariance matrix P_Y around the given mean $\hat{X}$ for any Gaussian-distributed random variable X, as evidenced by Julier et al. (1995, 2000); Julier and Uhlmann (2004); Wan and Van der Merwe (2001). That is why the UKF grounded in such an UT outperforms usually the classical EKF, which enjoys only the first order of approximation to the mean and covariance, in treating nonlinear continuous-discrete Gaussian systems of the form (1) and (2).

Nevertheless, the mentioned UKF suffers severely from the problem of negative UT weights, which are encountered in parametrization (5) for state estimation scenarios of large size, i.e. when $n > 3$. Julier et al. (2000) indicate that this can lead to indefinite covariance matrices arisen. Note that the non-positivity of computed covariance halts immediately any Cholesky-based UKF because the Cholesky factorization of nonpositive argument matrix may *not* be fulfilled and, hence, SP-set (3) is *not* available then. Such a covariance positivity lost may also stem from round-off operations of finite-digit arithmetic and other disturbances.

The best and commonly accepted solution to the issue of this covariance matrix positivity lost, which is encountered in a number of KF implementations and may result in a disastrous outcome, lies in square-rooting the filters themselves, as elaborated in Andrews (1968); Arasaratnam and Haykin (2009); Arasaratnam et al. (2010); Bellantoni and Dodge (1967); Dyer and McReynolds (1969); Kaminski et al. (1971); Wan and Van der Merwe (2001). It implies that, instead of the covariance matrix P_X, its SR S_X from formula (4) must be evaluated and propagated in the time and measurement update steps of such filters. The latter resolves completely the issue of symmetry and positivity of the covariances computed in practice because the product $S_X S_X^\top$ is always symmetric and positive semi-definite.

Within the above UKF, the search for an SR solution is a nontrivial task because of the negativity of the weights $w_0^{(m)}$ and $w_0^{(c)}$ in parametrization (5) when $n > 3$. So, Arasaratnam and Haykin (2009) count the *unavailability of an SR solution in the UKF with negative weights* among three most important reasons for devising their *Cubature Kalman Filtering* (CKF). It worthwhile to mention that an attempt for constructing the SR-UKF was fulfilled by means of the one-rank Cholesky factor update in Wan and Van der Merwe (2001). Unfortunately, that filter does not solve the issue of the covariance positivity lost because it demands the positivity of the downdating matrix, anyway. That is why Arasaratnam and Haykin (2009) call the latter state estimator as a *pseudo SR* version of the UKF.

Below, we present an actual SR solution within the UKF rooted in the SP-set (3) and weights (5), which is expected to succeed in treating both continuous-discrete stochastic scenarios with stiff SDE models (1) discussed by Kulikov and Kulikova (2017b,c, 2018a,b) and those with ill-conditioned measurements (2), as studied in Kulikov and Kulikova (2017d, 2018c). This solution employs the *Euler-Maruyama* (EM) discretization, which is a popular method for implementation of the UKF as evidenced by Kulikov and Kulikova (2018a); Knudsen and Leth (2019), etc. In addition, our square-rooting technique uses the concept of J-orthogonal transforms implemented by hyperbolic QR factorizations because of potential negativity of some weights in parametrization (5). Note that J-orthogonal QR decompositions are commonly utilized in the realm of H_∞ filtering and other tasks with indefinite inner products. Here, we stick to the J-orthogonal QR factorization of Bojanczyk et al. (2003), which combines the Householder reflections and hyperbolic rotations and considered to be numerically robust and efficient. That technique is used for square-rooting the EM-based DD-UKF presented by Kulikov and Kulikova (2018a). The performance of our novel *J-orthogonal Square-Root Discrete-Discrete Unscented Kalman Filtering* (JSR-DD-UKF) method and its superiority to the non-SR predecessor is assessed and confirmed within the target tracking scenario of Arasaratnam et al. (2010), in which an aircraft executes a coordinated turn, but employed with ill-conditioned measurements.

2. J-ORTHOGONAL SQUARE-ROOT DD-UKF

For yielding this state estimator, we have to square-root the time and measurement update steps in the EM-based DD-UKF designed by Kulikov and Kulikova (2018a). We consider first the time update step in the JSR-DD-UKF.

2.1 The Time Update Step in the JSR-DD-UKF

As we already mentioned above, our new filter enjoys the EM-based discretization of strong convergence order 0.5. With use of an equidistant mesh consisting of $L-1$ equally spaced subdivision nodes (with a user-supplied prefixed quantity L) introduced in each sampling interval $[t_{k-1}, t_k]$ of size δ, the EM method casts SDE (1) into the corresponding discrete-time stochastic system of the form

$$X_{k-1}^{l+1} = X_{k-1}^{l} + \tau F\big(X_{k-1}^{l}\big) + G\tilde{W}_{k-1}^{l} \tag{8}$$

where the drift function $F(\cdot)$ and diffusion matrix G come from SDE (1), and where the discretized noise $\tilde{W}_{k-1}^{l} \sim \mathcal{N}(0, \tau Q)$. We emphasize that equation (8) gives the L-step discretization scheme and, hence, its step size is $\tau = \delta/L$.

Our further intention is to establish mean and covariance time-propagation methods for the nonlinear stochastic process (8). In other words, given the mean $\hat{X}_{k-1|k-1}^{l}$ and covariance SR $S_{k-1|k-1}^{l}$ of the random variable X_{k-1}^{l} (i.e. $P_{k-1|k-1}^{l} = [S_{k-1|k-1}^{l}][S_{k-1|k-1}^{l}]^\top$), we have to advance a step in our discretized process model and compute the mean $\hat{X}_{k-1|k-1}^{l+1}$ and covariance SR $S_{k-1|k-1}^{l+1}$ of the random variable X_{k-1}^{l+1} derived by equation (8) whose time-updated covariance satisfies $P_{k-1|k-1}^{l+1} = [S_{k-1|k-1}^{l+1}][S_{k-1|k-1}^{l+1}]^\top$.

In line with the UKF algorithm, we set first the SP-matrix

$$\mathcal{X}_{k-1|k-1}^{l} := \Big[\, \mathcal{X}_{1,k-1|k-1}^{l} \;\ldots\; \mathcal{X}_{2n,k-1|k-1}^{l} \;\; \mathcal{X}_{0,k-1|k-1}^{l} \,\Big] \tag{9}$$

whose columns are the SP defined by formulas (3) in which $\hat{X} = \hat{X}_{k-1|k-1}^{l}$ and $S_X = S_{k-1|k-1}^{l}$. We recall that the mean $\hat{X}_{k-1|k-1}^{l}$ and covariance SR $S_{k-1|k-1}^{l}$ are assumed to be known at time $t_{k-1}^{l} := t_{k-1} + l\tau$, $l = 0, 1, \ldots, L-1$.

Then, following Särkkä (2007), we organize coefficients (5) in the form of the vector w_m and matrix $\mathcal{W}$ as follows:

$$w_m := \big[\, 1/6 \;\ldots\; 1/6 \;\; (3-n)/3 \,\big]^\top, \tag{10}$$

$$\mathcal{W} := \big(I_{2n+1} - \mathbf{1}^\top \otimes w_m\big)\mathrm{diag}\big\{1/6, \ldots, 1/6, (3-n)/3\big\} \times \big(I_{2n+1} - \mathbf{1}^\top \otimes w_m\big)^\top, \tag{11}$$

where I_{2n+1} stands for the identity matrix of size $2n+1$, $\mathbf{1} := [1, 1, \ldots, 1]^\top \in \mathbb{R}^{2n+1}$, $\mathrm{diag}\big\{1/6, \ldots, 1/6, (3-n)/3\big\}$ denotes the diagonal matrix with the given entries on its main diagonal and $\otimes$ refers to the Kronecker tensor product coded as the built-in function `kron` in MATLAB. We point out that the SP $\mathcal{X}_{0,k|k-1}$ has been put into the last column of matrix (9) as well as the entry $(3-n)/3$ has been moved to the end of vector (10) and of the diagonal in the diagonal matrix of formula (11) because of its potential negativity and specific requirements of the hyperbolic QR factorization code implemented. This is explained below.

Eventually, we modify the SP-matrix (9) to the form

$$\mathcal{Y}_{k-1|k-1}^{l+1} := \Big[\, \mathcal{Y}_{1,k-1|k-1}^{l+1} \;\ldots\; \mathcal{Y}_{2n,k-1|k-1}^{l+1} \;\; \mathcal{Y}_{0,k-1|k-1}^{l+1} \,\Big] \tag{12}$$

by the discretized drift function in equation (8), i.e.

$$\mathcal{Y}_{i,k-1|k-1}^{l+1} := \mathcal{X}_{i,k-1|k-1}^{l} + \tau F(\mathcal{X}_{i,k-1|k-1}^{l}). \tag{13}$$

Following Kulikov and Kulikova (2018a), the UT propagates the mean and covariance of the random variables from the stochastic process (8) in line with the rule

$$\hat{X}_{k-1|k-1}^{l+1} = \mathcal{Y}_{k-1|k-1}^{l+1} w_m, \tag{14}$$

$$P_{k-1|k-1}^{l+1} = [\mathcal{Y}_{k-1|k-1}^{l+1}]\mathcal{W}[\mathcal{Y}_{k-1|k-1}^{l+1}]^\top + \tau GQG^\top. \tag{15}$$

Note that the mean evolution (14) takes its final form, whereas that of covariance (15) is to be square-rooted.

First of all we need an SR of matrix (11). Moreover, taking into account the negativity of the last entry $(3-n)/3$ when $n > 3$, we replace it with its magnitude and arrive at the modified coefficient matrix SR defined by the formula

$$|\mathcal{W}|_0^{1/2} := \big(I_{2n+1} - \mathbf{1}^\top \otimes w_m\big) \times \mathrm{diag}\big\{\sqrt{1/6}, \ldots, \sqrt{1/6}, \sqrt{|3-n|/3}\big\}. \tag{16}$$

Also, we determine the corresponding signature matrix

$$\mathcal{J} := \mathrm{diag}\,\{1, \ldots, 1, \mathrm{sgn}\{(3-n)/3\}\} \tag{17}$$

where the function $\mathrm{sgn}\{(3-n)/3\}$ returns the sign of the last diagonal entry in line with the rule: $\mathrm{sgn}\{(3-n)/3\} = 1$ if $n \le 3$ and $\mathrm{sgn}\{(3-n)/3\} = -1$ if $n > 3$. We remark that formulas (11), (16) and (17) entail the obvious equality

$$\mathcal{W} = |\mathcal{W}|_0^{1/2}\mathcal{J}|\mathcal{W}|_0^{\top/2} \tag{18}$$

where $|\mathcal{W}|_0^{\top/2}$ stands for the transpose of the SR $|\mathcal{W}|_0^{1/2}$.

Next, we apply the Cholesky decomposition to the disctretized process noise covariance for deriving its factorization $\tau Q = [\sqrt{\tau}Q^{1/2}][\sqrt{\tau}Q^{1/2}]^\top$ and assembling the array

$$A := \Big[\, \sqrt{\tau}GQ^{1/2} \;\; \mathcal{Y}_{k-1|k-1}^{l+1}|\mathcal{W}|_0^{1/2} \,\Big]. \tag{19}$$

Array (19) and formulas (15) and (18) entail the identity

$$P_{k-1|k-1}^{l+1} \equiv AJA^\top \quad \text{with} \quad J := \mathrm{diag}\{I_n, \mathcal{J}\}. \tag{20}$$

Further, the concept of J-orthogonality is crucial in our approach to square-rooting the covariance evolution equation (15). Higham (2003) defines a J-orthogonal matrix as follows: A square matrix Θ of size $n \times n$ is said to be *J-orthogonal* with a signature matrix $J := \mathrm{diag}\{\pm\mathbf{1}^\top\}$ of size n, i.e whose diagonal entries equal 1 or -1, when

$$\Theta^\top J\Theta = \Theta J\Theta^\top = J. \tag{21}$$

The J-orthogonality is vital in the hyperbolic QR factorization used in our square-rooting method, below. As already said above, we apply the method of Bojanczyk et al. (2003), which is implemented for J-orthogonal QR decompositions with signatures of the form $J = \mathrm{diag}\{I_p, -I_s\}$, i.e. when all positive entries are placed in the beginning of its main diagonal and the remaining negative ones complete it. That is why we have permuted rows and/or columns in vector (10) and matrices (9), (11) and (12).

This hyperbolic QR decomposition code applied to the transposed matrix of array (19) with the signature matrix from equation (20) returns the lower triangular post-array

$$R^\top = \begin{bmatrix} S^{l+1}_{k-1|k-1} & \mathbf{0} \end{bmatrix}, \tag{22}$$

which is of size $n \times (3n+1)$, with the notation $\mathbf{0}$ standing for the zero-block of size $n \times (2n+1)$. Finally, we read-off the square block $S^{l+1}_{k-1|k-1}$ of size n, which constitutes the requested time-updated covariance matrix SR because formulas (19)–(22) prove the SR condition (4) as follows:

$$\begin{aligned} P^{l+1}_{k-1|k-1} &\equiv AJA^\top = R^\top Q^\top JQR = R^\top JR \\ &= [S^{l+1}_{k-1|k-1}][S^{l+1}_{k-1|k-1}]^\top. \end{aligned} \tag{23}$$

The signature matrix J in (23) becomes the identity one after multiplication of its negative part with zero entries in post-array (22) and, hence, vanishes. This completes the time update in the JSR-DD-UKF designed. For convenience of practical use, we summarize the *time update* of this filter in the following condensed algorithmic form:

Given $\hat{X}_{k-1|k-1}$ and $S_{k-1|k-1}$ at time t_{k-1}, compute the predicted state mean $\hat{X}_{k|k-1}$ and covariance SR $S_{k|k-1}$ at time t_k. Set the local initial values $\hat{X}^0_{k-1|k-1} := \hat{X}_{k-1|k-1}$ and $S^0_{k-1|k-1} := S_{k-1|k-1}$ and fulfil the L-step time-update procedure with $\tau := (t_k - t_{k-1})/L$ as follows:
For $l = 0, 1, \ldots, L-1$ do;
1) Assemble the SP-matrix (9);
2) Set the modified SP-matrix (12) with columns (13);
3) Compute the updated mean $\hat{X}^{l+1}_{k-1|k-1}$ by formula (14);
4) Set the time-updated predicted covariance array (19);
5) Apply the J-orthogonal QR factorization of the transposed array (19) with the signature J from formula (20);
6) Read-off the covariance SR $S^{l+1}_{k-1|k-1}$ in post-array (22).

We point out that the time-invariant coefficient vector (10) and matrices (11) and (16) with signature (17) as well as the disctretized-process-noise-covariance-Cholesky-factorization $\tau Q = [\sqrt{\tau}Q^{1/2}][\sqrt{\tau}Q^{1/2}]^\top$ are computed only once and before the state estimation run itself starts off. The predicted state mean vector $\hat{X}_{k|k-1} := \hat{X}^L_{k-1|k-1}$ and covariance matrix SR $S_{k|k-1} := S^L_{k-1|k-1}$ are further utilized in the measurement update step of our novel JSR-DD-UKF method as explained in the next section.

2.2 The Measurement Update Step in the JSR-DD-UKF

First of all we assemble the following predicted SP-matrix:

$$\mathcal{X}_{k|k-1} := \begin{bmatrix} \mathcal{X}_{1,k|k-1} & \ldots & \mathcal{X}_{2n,k|k-1} & \mathcal{X}_{0,k|k-1} \end{bmatrix} \tag{24}$$

whose columns are the SP defined by formulas (3) in which $\hat{X} = \hat{X}_{k|k-1}$ and $S_X = S_{k|k-1}$. We recall that the mean vector $\hat{X}_{k|k-1}$ and covariance matrix SR $S_{k|k-1}$ come from the time update step elaborated in Sec. 2.1.

Then, the SP-matrix (24) is modified to the form

$$\mathcal{Z}_{k|k-1} := \begin{bmatrix} \mathcal{Z}_{1,k|k-1} & \ldots & \mathcal{Z}_{2n,k|k-1} & \mathcal{Z}_{0,k|k-1} \end{bmatrix} \tag{25}$$

with columns $\mathcal{Z}_{i,k|k-1} := h(\mathcal{X}_{i,k|k-1})$, $i = 0, 1, \ldots, 2n$, i.e. these are transformed by the measurement function $h(\cdot)$ from the measurement model (2). The coefficient matrix (11) and SP-matrices (24) and (25) contribute to computation of the innovations, cross- and filtering covariances in line with the following commonly-used rules:

$$P_{zz,k|k-1} := \mathcal{Z}_{k|k-1}\mathcal{W}\,\mathcal{Z}^\top_{k|k-1} + R_k, \tag{26}$$

$$P_{xz,k|k-1} := \mathcal{X}_{k|k-1}\mathcal{W}\,\mathcal{Z}^\top_{k|k-1}, \tag{27}$$

$$P_{k|k} = P_{k|k-1} - \mathbb{W}_k P_{zz,k|k-1}\mathbb{W}_k^\top \tag{28}$$

where R_k stands for the covariance of the measurement noise in model (2) and the Kalman gain obeys the formula

$$\mathbb{W}_k := P_{xz,k|k-1}P^{-1}_{zz,k|k-1}. \tag{29}$$

It is commonly accepted to square-root all the covariance matrices calculated by formulas (26)–(28) in the form of a single coupled pre-array, which is assembled as follows:

$$B := \begin{bmatrix} R_k^{1/2} & \mathcal{Z}_{k|k-1}|\mathcal{W}|_0^{1/2} \\ \mathbf{0} & \mathcal{X}_{k|k-1}|\mathcal{W}|_0^{1/2} \end{bmatrix} \tag{30}$$

where $R_k^{1/2}$ refers to the lower triangular Cholesky factor (SR) of the measurement noise covariance, i.e. $R_k = R_k^{1/2}R_k^{\top/2}$. Similar to the time update presented in Sec. 2.1, the above-mentioned hyperbolic QR decomposition code is applied to the transposed matrix of pre-array (30) with the signature matrix $J := \mathrm{diag}\{I_m, \mathcal{J}\}$, in which I_m is the identity matrix of size m and $\mathcal{J}$ obeys formula (17). The latter factorization returns the lower triangular post-array

$$R^\top = \begin{bmatrix} P^{1/2}_{zz,k|k-1} & \mathbf{0} & \mathbf{0} \\ \bar{P}_{xz,k|k-1} & S_{k|k} & \mathbf{0} \end{bmatrix}, \tag{31}$$

which is of size $(m+n) \times (m+2n+1)$, with the notation $\mathbf{0}$ standing for zero-blocks of proper size and the matrix $\bar{P}_{xz,k|k-1}$ denoting the modified cross-covariance. This matrix allows the Kalman gain (29) to be amended to the form

$$\mathbb{W}_k = \bar{P}_{xz,k|k-1}P^{-1/2}_{zz,k|k-1}. \tag{32}$$

We further read-off the lower triangular $(m+n)\times(m+n)$-block S in the outcome covariance post-array (31), i.e.

$$S := \begin{bmatrix} P^{1/2}_{zz,k|k-1} & \mathbf{0} \\ \bar{P}_{xz,k|k-1} & S_{k|k} \end{bmatrix}, \tag{33}$$

which contains the innovations covariance SR $P^{1/2}_{zz,k|k-1}$, the modified cross-covariance $\bar{P}_{xz,k|k-1}$ and the filtering covariance SR $S_{k|k}$. These are read-off from matrix (33).

Next, we find the measurement mean by the inner product

$$\hat{Z}_{k|k-1} := \mathcal{Z}_{k|k-1}w_m \tag{34}$$

and complete this measurement update with calculating

$$\hat{X}_{k|k} = \hat{X}_{k|k-1} + \mathbb{W}_k(Z_k - \hat{Z}_{k|k-1}). \tag{35}$$

We recall that the predicted state mean $\hat{X}_{k|k-1}$ comes from the time update elaborated in Sec. 2.1. For convenience of practical use, we present this *measurement update* of JSR-DD-URK in the following condensed algorithmic form:

Given the predicted state mean $\hat{X}_{k|k-1} := \hat{X}^L_{k-1|k-1}$ and covariance matrix SR $S_{k|k-1} := S^L_{k-1|k-1}$, compute the filtering state mean $\hat{X}_{k|k}$ and covariance matrix SR $S_{k|k}$ based on the measurement Z_k fulfilled at time t_k as follows:
1) Assemble the predicted SP-matrix (24);
2) Assemble the measurement-modified SP-matrix (25);

3) Set the innovations and filtering covariance array (30);
4) Apply the J-orthogonal QR factorization of the transposed array (30) with the signature $J := \mathrm{diag}\{I_m, \mathcal{J}\}$;
5) Compute the Kalman gain $\mathbb{W}_k$ by formula (32);
6) Compute the filtering mean $\hat{X}_{k|k}$ by formulas (34), (35);
7) Read-off the filtering covariance SR $S_{k|k}$ in array (33).

Further, we examine the novel JSR-DD-UKF technique presented in Sec. 2.1 and 2.2 and compare it to its non-SR predecessor published by Kulikov and Kulikova (2018a) in severe conditions of tackling a radar tracking problem of Arasaratnam et al. (2010), where an aircraft executes a coordinated turn, but with an ill-conditioned measurement function in our stochastic scenario described below.

3. AIR TRAFFIC CONTROL SCENARIO WITH ILL-CONDITIONED MEASUREMENTS

The flight control scenario under consideration is a famous one in nonlinear filtering theory, which has been published with all particulars by Arasaratnam et al. (2010); Kulikov and Kulikova (2016, 2017a), etc. So, the interested reader is referred to the cited papers for more details. We simulate the turning aircraft dynamics for 150 s and set its angular velocity $\omega = 3^\circ$/s. The performance of our novel SR filter and its non-SR predecessor presented by Kulikov and Kulikova (2018a) is assessed in the sense of the *Accumulated Root Mean Square Errors* in position (ARMSE_p) and in velocity (ARMSE_v), which are defined as follows:

$$\mathrm{ARMSE}_p := \Big[\frac{1}{100K}\sum_{mc=1}^{100}\sum_{k=1}^{K}\big(x_{mc}^{\mathrm{true}}(t_k) - \hat{x}_{k|k}^{mc}\big)^2 + \big(y_{mc}^{\mathrm{true}}(t_k) - \hat{y}_{k|k}^{mc}\big)^2 + \big(z_{mc}^{\mathrm{true}}(t_k) - \hat{z}_{k|k}^{mc}\big)^2\Big]^{1/2},$$

$$\mathrm{ARMSE}_v := \Big[\frac{1}{100K}\sum_{mc=1}^{100}\sum_{k=1}^{K}\big(\dot{x}_{mc}^{\mathrm{true}}(t_k) - \hat{\dot{x}}_{k|k}^{mc}\big)^2 + \big(\dot{y}_{mc}^{\mathrm{true}}(t_k) - \hat{\dot{y}}_{k|k}^{mc}\big)^2 + \big(\dot{z}_{mc}^{\mathrm{true}}(t_k) - \hat{\dot{z}}_{k|k}^{mc}\big)^2\Big]^{1/2}$$

where $x_{mc}^{\mathrm{true}}(t_k)$, $y_{mc}^{\mathrm{true}}(t_k)$, $z_{mc}^{\mathrm{true}}(t_k)$ and $\dot{x}_{mc}^{\mathrm{true}}(t_k)$, $\dot{y}_{mc}^{\mathrm{true}}(t_k)$, $\dot{z}_{mc}^{\mathrm{true}}(t_k)$ stand for the aircraft's position and velocity simulated by the the Euler-Maruyama method with the small step size $\tau := 0.0005$ at time t_k in the mc-th Monte Carlo run (out of 100 independent simulations), $\hat{x}_{k|k}^{mc}$, $\hat{y}_{k|k}^{mc}$, $\hat{z}_{k|k}^{mc}$ and $\hat{\dot{x}}_{k|k}^{mc}$, $\hat{\dot{y}}_{k|k}^{mc}$, $\hat{\dot{z}}_{k|k}^{mc}$ denote the aircraft's position and velocity estimated by each filtering algorithm, k means the particular sampling time t_k and K refers to the total number of samples in the simulation interval $[0, 150\,\mathrm{s}]$. The sampling rate is limited to $\delta = 1\,\mathrm{s}$ in this flight control task.

In contrast to Arasaratnam et al. (2010), for provoking numerical instabilities in the filters under examination, we utilize the artificial measurement equation of the form

$$Z_k = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1+\sigma \end{bmatrix} X_k + V_k \qquad (36)$$

where the aircraft's state $X_k := [\, x_k \ \dot{x}_k \ y_k \ \dot{y}_k \ z_k \ \dot{z}_k \ \omega_k \,]^\top$ is estimated at time t_k and σ denotes a small positive real number determining ill-conditioning of model (36). Here, we address the cases of $\sigma = $ 1.0e-01, 1.0e-02, …, 1.0e-10. Also, each measurement Z_k is supposed to be corrupted by a normally distributed noise $V_k \sim \mathcal{N}(0, R_\sigma)$ with the covariance $R_\sigma = \sigma^2 I_2$ depending on the ill-conditioning parameter σ. Measurements (36) with the measurement noise covariance matrix R_σ are typical means in numerical stability studies of various KF including the continuous-discrete and discrete-discrete methods presented by Dyer and McReynolds (1969); Grewal and Andrews (2001); Kulikov and Kulikova (2017d, 2018c). These correspond to the third reason of ill-conditioning elaborated by Grewal and Andrews (2001) because the matrix inversions in the Kalman gain computations (29) and (32) become increasingly ill-conditioned in line with the vanishing scalar σ.

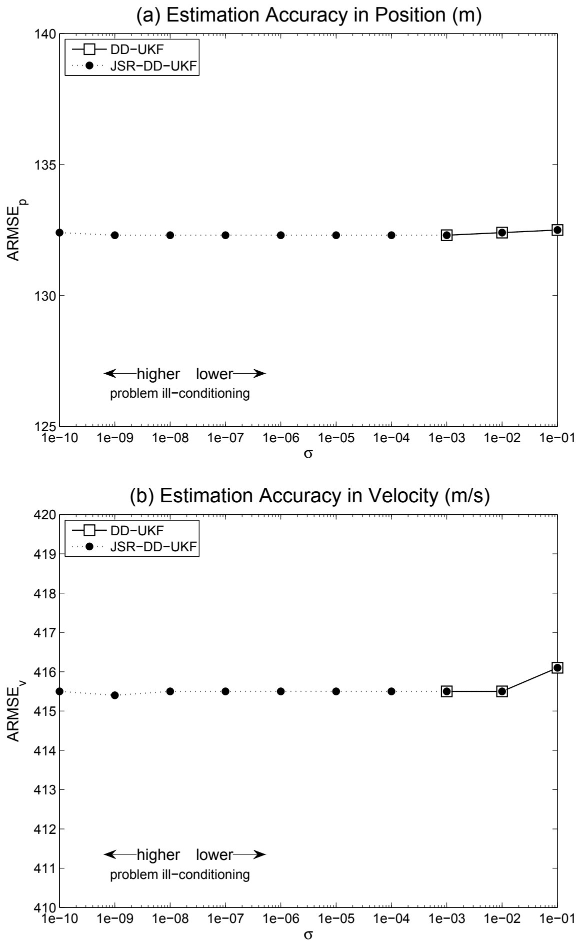

Fig. 1. The ARMSE_p and ARMSE_v observed in the target tracking scenario with the measurement model (36).

We abbreviate our novel SR filter to `JSR-DD-UKF` and its non-SR predecessor published by Kulikov and Kulikova (2018a) to `DD-UKF`, respectively. These methods are coded and run in MATLAB. The state estimators under consideration enjoy $L = 512$ subdivision steps in each sampling period. Parametrization (5) includes negative weights, i.e. $w_0^{(m)} < 0$ and $w_0^{(c)} < 0$, in the target tracking scenario in use. This serves for the effective examination of `JSR-DD-UKF` and its valued comparison to `DD-UKF` in the presence of the ill-conditioned measurement model (36).

Fig 1 exhibits that the SR filter and its non-SR predecessor work identically and expose the same ARMSE_p and ARMSE_v when the ill-conditioning parameter $\sigma \geq$ 1.0e-03, i.e. when our air traffic control scenario is rather well-conditioned. Then, we see that the non-SR `DD-UKF` fails at $\sigma =$ 1.0e-04 because its covariance matrix computed loses the positivity and, hence, the Cholesky factorization may not be fulfilled. In contrast, the `JSR-DD-UKF` succeeds in producing the decent state estimates for all the values of the ill-conditioning parameter σ accepted in our case study. This confirms the sound numerical robustness of the filtering algorithm presented in Sec. 2 and establishes a solid background for its successful applications in practice.

4. CONCLUSION

This paper has yielded a square-root version of the Euler-Maruyama-based Discrete-Discrete Unscented Kalman Filter designed by Kulikov and Kulikova (2018a). Taking into account the negativity of some UT weights in continuous-discrete stochastic scenarios of large size, we have applied the hyperbolic QR decomposition for devising our novel J-orthogonal square-root state estimator, which has been examined in severe conditions of tackling a radar tracking problem, where an aircraft executes a coordinated turn, in the presence of ill-conditioned measurements. The sound state estimation potential of this filter has been proven theoretically and confirmed numerically within the mentioned stochastic flight control scenario.

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