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Performance Bounds for Continuous Trading on Balancing Power Markets ⋆

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Abstract: In power systems the system frequency is a good indicator of the networks resilience to major disturbances. In many deregulated markets, *e.g.* the Nordic power market, the system operator controls the system frequency manually by calling off bids handed in to a market, called the balancing power market.
In this paper we consider the problem of optimal bid call-off on the balancing market, that the system operator is faced with each operating period. We formulate the problem as a stochastic optimal control problem of impulse type.
When searching for numerical solutions a complicating factor is the structure of the balancing power market, where the overall marginal price applies to all bids. To retain numerical tractability we propose computationally efficient upper and lower bounds for the value function in the dynamic programming algorithm.

Keywords: Frequency control, optimal switching, power systems, stochastic optimal control.

1. INTRODUCTION

We consider the problem of tracking the demand for electric power in power system operation, known as frequency control. The frequency control is a multi-layer control system where, in a fully deregulated setting, the top layer is incorporated through a market called the *balancing power market* van der Veen and Hakvoort [2016]. The deregulation of electricity markets has, in this context, often led to a more inflexible generation control as independent system operators do not, themselves, operate the production units. A bid to the balancing market is a block-bid on alteration of injected power at a given price per MW of produced power, that is binding in a specific operation period. One example is the Nordic market with operating periods of one hour, where the balancing market for each operating period closes 45 minutes before the start of the period.

In this setting the problem of finding an economically efficient frequency control scheme is a problem of timing (when to call-off a bid) and choosing the optimal bid to call-off. The problem is further complicated by the structure of the balancing power market, where the market price of upward and downward balancing power equals the corresponding marginal prices. That is, the marginal price of the most expensive upward (downward) bid that has been called off during the period is assigned to all upward (downward) bids that were called off.

As will become apparent later the operator's problem is a stochastic optimal control problem of switching type. The general optimal switching problem (sometimes referred to as starting and stopping problem) has been thoroughly investigated in the last decades after being popularised in Brennan and Schwartz [1985]. In Djehiche et al. [2009] existence of a unique solution to the multi-modes optimal switching problem was shown.

Although some work on optimal bid call-off in real-time operation of power systems exist (see Perninge and Söder [2014, 2012], Perninge [2015]), the main focus has been on maintaining power flow feasibility. Exceptions are Perninge and Eriksson [2017a], where the market model is limited by not allowing reversion of call-offs and Perninge and Eriksson [2017b, 2018] where the structure of the balancing power market is not fully modelled.

The aim of the present article is to close this gap by formulating an optimal call-off problem for a general market model with the possibility of reversing call-offs. As the resulting problem is of high dimension a straightforward application of dynamic programming is prevented by inflated computational times and memory requirements. To remedy this we propose computationally tractable upper and lower bounds for the value function to the operators problem.

The lower bound is based on using a coarser discretization of the state and the upper bound is based on a reduction of the dimension of the state-space.

2. FREQUENCY CONTROL IN DEREGULATED POWER SYSTEMS

The frequency control in power systems is generally a multi-layered control structure where the first layer, denoted *primary control*, is an automated control system under which the production of certain power plants respond

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to changes in system frequency by altering the production output proportionally to the deviation of the frequency from its nominal value. To this is sometimes added a second layer of automated reserves (called *secondary control*) designed to keep transmission on tie-lines within acceptable bounds. In a completely deregulated setting, for example in the Nordic power system, a third layer consists of a manual control where the system operator controls the output in a set of units manually by calling-off bids handed in to the balancing power market. The objective of this layer, termed *tertiary control*, is to restore the reserves in the primary and secondary controls, thus restoring the system frequency to its nominal value.

Bids handed in to the balancing power market can be either upward balancing bids, which correspond to an increase in injected power, or downward balancing bids that, if called-off, have the effect of reducing the injected power.

During operation the system operator has the opportunity to, at any time, contact actors responsible for bids on the balancing market to purchase the power specified in the contract. This is what we refer to as *calling-off* the bid. Furthermore, the system operator can reverse any prior call-offs. Call-offs can thus be made of several bids simultaneously and each bid can, due to the possibility of reversing call-offs, be called-off several times during one operation period. Similar to the ahead markets (day ahead and intra-day trading), the price of electricity on the balancing market is set by the marginal price of the most expensive accepted bid.

In this setting the problem of finding an economically efficient frequency control scheme can be formulated as a multi-modes optimal switching problem.

3. PROBLEM FORMULATION

3.1 Net demand profile

The demand to be tracked by the tertiary control is the *net demand*, $(X_t : 0 \le t \le T)$, that is the total demand minus the planned production. We will assume that X is the strong solution to a stochastic differential equation (SDE) as follows (see *e.g.* Perninge and Eriksson [2017a], Nowicka-Zagrajek and Weron [2002], Olsson et al. [2010] for motivation of the model)

$$\begin{aligned} dX_t &= a(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t \in [0, T], \\ X_0 &= x_0 \end{aligned}$$

where $(W_t; 0 \le t \le T)$ is a one-dimensional Brownian motion that generates the completed filtration $(\mathcal{F}_t; 0 \le t \le T)$, $x_0 \in \mathbb{R}$ and $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are two continuous functions that satisfy

$$|a(t, x) - a(t, x')| + |\sigma(t, x) - \sigma(t, x')| \le C|x - x'|,$$

for some constant $C > 0$.

For all $t \in [0, T]$ and $x \in \mathbb{R}$ we define the process $(X_s^{t,x}; 0 \le s \le T)$ as the strong solution to

$$\begin{aligned} dX_s^{t,x} &= a(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dW_s, \quad \forall s \in [t, T], \\ X_s^{t,x} &= x, \quad \forall s \in [0, t]. \end{aligned}$$

3.2 Tertiary frequency control

We will assume that prior to the start of the specific operating period $[0, T]$ a sequence $\{B_i\}_{i=1}^n$ of bids has been handed in to the balancing market. Here, each bid can be summarized by the 4-tuple $B_i = (\mathit{Vol}_i, c_i^{\text{off}}, c_i^g)$, where:

- $\mathit{Vol}_i \in \mathbb{R} \setminus \{0\}$ is the volume of bid i,
- $c_i^{\text{off}} > 0$ is a fixed cost [1] associated with reversing a call-off of bid i,
- $c_i^g :\in \mathbb{R}$ is the marginal production cost [2] of bid i.

We let [3] $\mathcal{J}^{\text{up}} := \{j \in \{1, \dots, n\} : \mathit{Vol}_j > 0\}$ and $\mathcal{J}^{\text{down}} := \{j \in \{1, \dots, n\} : \mathit{Vol}_j < 0\}$, so that $\mathcal{J}^{\text{up}}$ and $\mathcal{J}^{\text{down}}$ are the sets of indexes for upward and downwards bids, respectively.

The state of the tertiary frequency control is a process $(\xi_t : 0 \le t \le T)$, referred to as the *operating mode*, that take values in $\mathcal{I} := \{0, 1\}^n$. We will let the i^{th} component of ξ_t represent the state of the i^{th} bid, with 0 and 1 representing "off" and "on" respectively.

A control strategy for the operator will be of the type $u = (\tau_1, \dots, \tau_N; \beta_1, \dots, \beta_N)$, where τ_j is the time of the j^{th} intervention (we refer to the call-off of a bid or the reversion of a prior call-off as an intervention) and $\beta_j \in \mathcal{I}^{-\beta_{j-1}}$ (with $\mathcal{I}^{-\mathbf{b}} := \mathcal{I} \setminus \{\mathbf{b}\}$) indicates which bids that are in the "on"-mode following the j^{th} intervention. The number of interventions N is random and decided as part of the control.

The control defines the operating mode as [4]

$$\xi_t^{\mathbf{b}} := \mathbf{b}1_{[0,\tau_1)}(t) + \sum_{j=1}^{N} \beta_j 1_{[\tau_j, \tau_{j+1})}(t)$$

where $\mathbf{b} := (b_1, \dots, b_n) \in \mathcal{I}$ is the initial operating mode and using the convention that $\tau_{N+1} = \infty$. From the operating mode we can easily extract the production in the different units corresponding to the bids as the $\mathbb{R}^n$-valued process $(\Delta P_t^{\mathbf{b}} : 0 \le t \le T)$ with

$$\Delta P_t^{\mathbf{b}} := \text{diag}(\mathit{Vol})\xi_t^{\mathbf{b}}.$$

where $\text{diag}(\mathit{Vol})$ is the $n \times n$-diagonal matrix with $[\text{diag}(\mathit{Vol})]_{ii} = \mathit{Vol}_i$.

To define the operation cost we need to keep track of the total amount of electricity produced by upward and downward bids respectively. To obtain this we let Γ be the $2 \times n$ matrix with $\Gamma_j^i := \mathit{Vol}_j 1_{[(-1)^{i+1} \mathit{Vol}_j > 0]}$. We then introduce the controlled process $Z_s^{\mathbf{b}}$ given by

$$Z_t^{\mathbf{b}} := \int_0^t \Gamma \xi_r^{\mathbf{b}} dr,$$

so that $[Z_t^{\mathbf{b}}]_1$ is the total amount of electricity produced by upward balancing bids during $[0, t]$, and $[Z_t^{\mathbf{b}}]_2$ is the total

[1] Although there is usually no explicit cost for reversing a call-off, we may assume that there is a implicit cost as frequent reversions may lead to a reluctance for producers to participate in the balancing market.
[2] Generally we have $c_i^g > 0$ for upward balancing bids and $c_i^g < 0$ for downward balancing bids.
[3] Throughout we use := to emphasise when the left-hand-side is defined to equal the right-hand-side.
[4] We let 1_A denote the indicator function for the set A.

amount of electricity produced by downward balancing bids during the same period.

The cost for trading on the balancing power market is then

$$J_{TC}(u) := \mathbb{E}\left[\gamma(\beta_1,\ldots,\beta_N)Z_T^{\mathbf{b}} + \sum_{j=1}^{N} K_{\beta_{j-1},\beta_j}\right],$$

where

$$\gamma(\mathbf{b}^1,\ldots,\mathbf{b}^l) := \begin{bmatrix} \max\limits_{j\in\{1,\ldots,l\}} \max\limits_{i\in\mathcal{J}^{\mathrm{up}}} b_i^j c_i^g \\ \min\limits_{j\in\{1,\ldots,l\}} \min\limits_{i\in\mathcal{J}^{\mathrm{down}}} b_i^j c_i^g \end{bmatrix}^\top$$

is the vector of market prices for upward and downward balancing power given the sequence $\mathbf{b}^1,\ldots,\mathbf{b}^l$ of interventions by the operator and

$$K_{\mathbf{b},\mathbf{b}'} := \sum_{i=1}^{n} 1_{[b_i'<b_i]} c_i^{\mathrm{off}}$$

are the switching costs.

3.3 System frequency

On the time-scale that we are interested in, the system frequency, f, can be seen as a continuous function of the net imbalance, $X_t - \Delta P(t)$. When $X_t - \Delta P(t) = 0$ the system is at nominal frequency $f = f_0$.

To incentivize an efficient frequency control we assume that frequency deviations, $f - f_0$, are penalized. We assume that the penalization takes the form of an additional cost

$$J_F(u) := \mathbb{E}\left[\int_0^T \psi_{\xi_t^{\mathbf{b}}}(X_t)dt + h_{\xi_T^{\mathbf{b}}}(X_T)\right]$$

where for all $\mathbf{b} \in \mathcal{I}$ the running cost $\psi_{\mathbf{b}} : \mathbb{R} \to \mathbb{R}$ and the terminal cost $h_{\mathbf{b}} : \mathbb{R} \to \mathbb{R}$ are both locally Lipschitz continuous and of polynomial growth. For example, we may take $\psi_{\mathbf{b}}(x) = (x - \sum_{i=1}^n [\mathrm{diag}(\mathit{Vol})\mathbf{b}]_i)^2$ and $h_{\mathbf{b}}(x) = (x - \sum_{i=1}^n [\mathrm{diag}(\mathit{Vol})\mathbf{b}]_i)^2$ for quadratic penalization.

3.4 The optimization problem

Putting the above costs together yields the following cost functional for our problem:

$$\begin{aligned} J^{\mathbf{b}}(u) := \mathbb{E}\bigg[&\int_0^T \psi_{\xi_t^{\mathbf{b}}}(X_t)dt \\ &+ \gamma(\mathbf{b},\beta_1,\ldots,\beta_N)Z_T^{\mathbf{b}} \\ &+ h_{\xi_T^{\mathbf{b}}}(X_T) + \sum_{j=1}^N K_{\beta_{j-1},\beta_j}\bigg], \end{aligned}$$

where $\beta_0 := \mathbf{b}$. Furthermore, we assume that for all $\mathbf{b} \in \mathcal{I}$

$$h_{\mathbf{b}}(x) < \min_{\mathbf{b}'\in\mathcal{I}^{-\mathbf{b}}} \{K_{\mathbf{b},\mathbf{b}'} + h_{\mathbf{b}'}(x)\},$$

for all $x \in \mathbb{R}$.

The set of admissible controls, $\mathcal{U}$, is the set of all $u := (\tau_1,\ldots,\tau_N;\beta_1,\ldots,\beta_N)$, where $\tau_1 \le \tau_2 \le \cdots \le \tau_N$ are $(\mathcal{F}_t)_{0\le t\le T}$–stopping times and β_j is $\mathcal{F}_{\tau_j}$–measurable. We get the following problem:

Problem 1. Find $u^* \in \mathcal{U}$, such that

$$J^{\mathbf{b}}(u^*) = \inf_{u\in\mathcal{U}} J^{\mathbf{b}}(u). \tag{1}$$

4. SOLUTION BY DYNAMIC PROGRAMMING

To arrive at a dynamic programming equation for Problem 1 we need to introduce state space models for Z and γ. We thus let $Z_s^{t,\mathbf{b},z}$ be given by

$$Z_s^{t,\mathbf{b},z} := z + \int_t^s \Gamma\xi_r^{\mathbf{b}} dr$$

and define, for each $c = [c_1\ c_2] \in \cup_{\mathbf{b}\in\mathcal{I}}\gamma(\mathbf{b})$,

$$\gamma^c(\mathbf{b}^1,\ldots,\mathbf{b}^l) := \begin{bmatrix} \max(c_1, [\gamma(\mathbf{b}^1,\ldots,\mathbf{b}^l)]_1) \\ \min(c_2, [\gamma(\mathbf{b}^1,\ldots,\mathbf{b}^l)]_2) \end{bmatrix}^\top.$$

For $c \in \mathbb{R}^2$ we define $\mathcal{I}_c := \{\mathbf{b} \in \mathcal{I} : [\gamma(\mathbf{b})]_1 \le c_1,\ [\gamma(\mathbf{b})]_2 \ge c_2\}$, so that $\mathcal{I}_c$ is the subset of $\mathcal{I}$ with corresponding marginal costs not exceeding c. We note here that if, at time t, we have $\gamma(\mathbf{b},\beta_1,\ldots,\beta_{N_t}) = c$ (where $N_t := \max\{j : \tau_j \le t\}$), then $Z_t^{\mathbf{b}}$ takes values in $\mathcal{D}_c^z(t) := [0, t\max_{\mathbf{b}\in\mathcal{I}_c}\Gamma^1\mathbf{b}] \times [t\min_{\mathbf{b}\in\mathcal{I}_c}\Gamma^2\mathbf{b}, 0]$, where Γ^i is the i^{th} row of the matrix Γ, for $i = 1, 2$.

For each pair $(\mathbf{b}, c)$ in the extended set of modes [5] $\bar{\mathcal{I}} := \cup_{\mathbf{b}'\in\mathcal{I}}\{\mathbf{b}'\} \times \cup_{\mathbf{b}''\ge\mathbf{b}'}\{\gamma(\mathbf{b}'')\}$ we then define the extended cost functional for Problem 1 as

$$\begin{aligned} J^{\mathbf{b},c}(t,x,z;u) := \mathbb{E}\bigg[&\int_t^T \psi_{\xi_s^{\mathbf{b}}}(X_s^{t,x})ds \\ &+ \gamma^c(\beta_1,\ldots,\beta_N)Z_T^{t,\mathbf{b},z} \\ &+ h_{\xi_T^{\mathbf{b}}}(X_T^{t,x}) + \sum_{j=1}^N K_{\beta_{j-1},\beta_j}\bigg], \end{aligned}$$

for all $(t,x,z) \in \mathcal{D}_c := \cup_{\in[0,T]}\{t\} \times \mathbb{R} \times \mathcal{D}_c^z(t)$.

For each $(\mathbf{b}, c) \in \bar{\mathcal{I}}$, we define the value function $v_{\mathbf{b},c} : \mathcal{D}_c \to \mathbb{R}$ as

$$v_{\mathbf{b},c}(t,x,z) := \inf_{u\in\mathcal{U}_t} J^{\mathbf{b},c}(t,x,z;u), \tag{2}$$

where $\mathcal{U}_t := \{(\tau_1,\ldots,\tau_N;\beta_1,\ldots,\beta_N) \in \mathcal{U} : \tau_1 \ge t\}$.

With the above assumptions, the results of Perninge [2018] trivially extend to the present case and show that for each $(\mathbf{b}, c) \in \bar{\mathcal{I}}$, the value function $v_{\mathbf{b},c}$ exist as a member of $\mathcal{C}(\mathcal{D}_c \to \mathbb{R})$ and is of at most polynomial growth in x. Furthermore, the family $(v_{\mathbf{b},c})_{(c,\mathbf{b})\in\bar{\mathcal{I}}}$ satisfies the recursion

$$\begin{aligned} v_{\mathbf{b},c}(t,x,z) = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_s} \mathbb{E}\bigg[&\int_t^{\tau\wedge T} \psi_{\mathbf{b}}(X_s^{t,x})dr + 1_{[\tau\ge T]}(\\ &c(z + (T-t)\Gamma\mathbf{b}) + h_{\mathbf{b}}(X_T^{t,x})) + 1_{[\tau<T]} \min_{\beta\in\mathcal{I}^{-\mathbf{b}}} \{ \\ &K_{\mathbf{b},\beta} + v_{\beta,\gamma^c(\beta)}(\tau, X_\tau^{t,x}, z + (\tau-t)\Gamma\mathbf{b})\}\bigg] \end{aligned} \tag{3}$$

where $\mathcal{T}_t$ is the set of $\{\mathcal{F}_t\}_{0\le t\le T}$-stopping times $\tau \ge t$. From this relation an optimal feedback control for Problem 1 can then be extracted as:

$$\begin{aligned} \tau_j^* := \inf\{s \ge \tau_{j-1}^* : v_{\beta_{j-1}^*,\gamma(\mathbf{b},\beta_1^*,\ldots,\beta_{j-1}^*)}(s, X_s, Z_s^{\mathbf{b}}) = \\ \min_{\beta\in\mathcal{I}^{-\beta_{j-1}^*}} \{K_{\beta_{j-1}^*,\beta} + v_{\beta,\gamma(\mathbf{b},\beta_1^*,\ldots,\beta_{j-1}^*,\beta)}(s, X_s, Z_s^{\mathbf{b}})\}\} \end{aligned}$$

and

[5] That is, $\bar{\mathcal{I}}$ is the set of all pairs $(\mathbf{b}, c)$, where $\mathbf{b} \in \mathcal{I}$ and c is a possible vector of marginal prices for balancing power given that we have visited mode $\mathbf{b}$.

$$\beta_j^* \in \arg\min_{\beta \in \mathcal{I}^{-\beta_{j-1}^*}} \{K_{\beta_{j-1}^*,\beta} + v_{\beta,\gamma(\mathbf{b},\beta_1^*,\dots,\beta_{j-1}^*,\beta)}(\tau_j^*, X_{\tau_j^*}, Z_{\tau_j^*}^{\mathbf{b}})\}\}.$$

Of importance to obtaining numerical algorithms is the following result that, due to its relevance, we state as a proposition:

Proposition 1. For each $(\mathbf{b}, c) \in \bar{\mathcal{I}}$ the function $(t, x, z) \to v_{\mathbf{b},c}(t, x, z)$ is concave in z.

Proof. Fix $(t, x, z) \in \mathcal{D}_c$ and let $u^\diamond$ be an optimal control in (2) that we know exists and satisfies

$$J^{\mathbf{b},c}(t, x, z; u^\diamond) = \inf_{u \in \mathcal{U}_t} J^{\mathbf{b},c}(t, x, z; u).$$

For each $z' \in \mathbb{R}^2$ we have that

$$\begin{aligned} v_{\mathbf{b},c}(t, x, z') &\le J^{\mathbf{b},c}(t, x, z'; u^\diamond) \\ &= J^{\mathbf{b},c}(t, x, z; u^\diamond) \\ &\quad + \mathbb{E}\big[\gamma^c(\beta_1^\diamond, \dots, \beta_{N^\diamond}^\diamond)\big](z' - z). \end{aligned}$$

We thus have that $v_{\mathbf{b},c}(t, x, z')$ is bounded from above by a hyperplane passing through the point $v_{\mathbf{b},c}(t, x, z)$ and the assertion follows. □

5. NUMERICAL APPROXIMATION

In this section we will consider numerical approximation of the recursion in (3). To obtain numerical schemes we assume that interventions are restricted to the times [6] $\Pi := \{t_1, \dots t_{N_\Pi}\}$, where $t_k = (k-1)\Delta t$ and $t_{N_\Pi} = T$. We get the discrete time cost functional

$$\begin{aligned} \bar{J}^{\mathbf{b},c}(t_k, x, z; u) := \mathbb{E}\Big[&\sum_{j=k}^{N_\Pi - 1} \hat{\psi}_{\xi_s^{\mathbf{b}}}(X_{t_j}^{t_k,x})\Delta t \\ &+ \gamma^c(\beta_1, \dots, \beta_N) Z_T^{t_k,\mathbf{b},z} \\ &+ h_{\xi_T^{\mathbf{b}}}(X_T^{t_k,x}) + \sum_{j=1}^{N} K_{\beta_{j-1},\beta_j} \Big] \end{aligned}$$

for each $u \in \bar{\mathcal{U}}_{t_k} := \{u \in \mathcal{U}_{t_k} : \tau_j \in \Pi, \forall j \in \{1, \dots, N\}\}$, where $\hat{\psi}_{\mathbf{b}} := \Delta t \psi$, and the corresponding Bellman equation

$$\begin{aligned} \hat{v}_{\mathbf{b},c}(T, x, z) &= h_{\mathbf{b}}(x) + c^\top z \\ \hat{v}_{\mathbf{b},c}(t_k, x, z) &= \min_{\mathbf{b}' \in \mathcal{I}} \{\hat{\psi}_{\mathbf{b}'}(x) + K_{\mathbf{b},\mathbf{b}'} \\ &\quad + \mathbb{E}[\hat{v}_{\mathbf{b}',\gamma^c(\mathbf{b}')}(t_{k+1}, X_{t_{k+1}}^{t_k,x}, z + \hat{\Gamma}\mathbf{b}')]\}, \quad (4) \end{aligned}$$

where $\hat{\Gamma} := \Delta t \Gamma$. As the literature on estimating the conditional expectation in (4) when the uncertainty is modelled by an SDE is vast (see *e.g.* Carmona and Ludkovski [2008], Aïd et al. [2014] for results in the context of multi-modes optimal switching), focus will be on the role of the z variable.

5.1 An exact scheme

We can obtain an exact solution scheme by noting that in the extended mode $(\mathbf{b}, c) \in \bar{\mathcal{I}}$ the random variable Z_{t_k} can only take values in the discrete set

$$\hat{\mathcal{D}}_c^z(t_k) := \bigcup_{(\mathbf{b}^1,\dots,\mathbf{b}^{k-1}) \in \mathcal{I}_c^{k-1}} \{\sum_{j=1}^{k-1} \hat{\Gamma}\mathbf{b}^j\}.$$

Now, as the cardinality of the set $\hat{\mathcal{D}}_c^z(t_k)$ is bounded by $N_\Pi^2 2^n$ an exhaustive algorithm can be implemented to solve (4). The numerical complexity of such a scheme is $\mathcal{O}(2^{3n} N_\Pi^3)$. Although, ad-hoc rules can be applied to substantially reduce the number of modes (*e.g.* always activating the cheapest available bid to restore frequency) the cubic complexity in N_Π still prevents implementation.

5.2 A lower bound

To get a lower bound we introduce a less dense discretization of the domain of Z_{t_k} that we denote by $\hat{\mathcal{D}}_c^z(t_k)$. We assume that a triangulation of $\hat{\mathcal{D}}_c^z(t_k)$ partitions $\mathcal{D}_c^z(t_k)$ and set

$$\begin{aligned} \underline{v}_{\mathbf{b},c}(t_k, x, z) &= \min_{\mathbf{b}' \in \mathcal{I}} \{\hat{\psi}_{\mathbf{b}'}(x) + K_{\mathbf{b},\mathbf{b}'} \\ &\quad + \mathbb{E}[\underline{v}_{\mathbf{b}',\gamma^c(\mathbf{b}')}(t_{k+1}, X_{t_{k+1}}^{t_k,x}, z + \hat{\Gamma}\mathbf{b}')]\}, \quad (5) \end{aligned}$$

where $\underline{v}_{\mathbf{b},c}(t_k, x, \cdot)$ is extended to $\mathcal{D}_c^z(t_k)$ by linear interpolation on $(\underline{v}_{\mathbf{b},c}(t_k, x, z_j))_{z_j \in \hat{\mathcal{D}}_c(t_k)}$. We have:

Proposition 2. For each $(\mathbf{b}, c) \in \bar{\mathcal{I}}$, $k = 1, \dots, N_\Pi$ and $(x, z) \in \mathbb{R} \times \mathcal{D}_c(t_k)$,

$$\underline{v}_{\mathbf{b},c}(t_k, x, z) \le \hat{v}_{\mathbf{b},c}(t_k, x, z).$$

Proof. Note that, since $\hat{v}_{\mathbf{b},c}(T, x, z)$ is linear in z, the inequality holds trivially for $k = N_\Pi$. Now assume that the inequality holds for t_k for some $k \in \{1, \dots, N_\Pi\}$, then

$$\begin{aligned} \underline{v}_{\mathbf{b},c}(t_{k-1}, x, z) &= \min_{\mathbf{b}' \in \mathcal{I}} \{\hat{\psi}_{\mathbf{b}'}(x) + K_{\mathbf{b},\mathbf{b}'} \\ &\quad + \mathbb{E}[\underline{v}_{\mathbf{b}',\gamma^c(\mathbf{b}')}(t_k, X_{t_{k+1}}^{t_k,x}, z + \hat{\Gamma}\mathbf{b}')]\} \\ &= \min_{\mathbf{b}' \in \mathcal{I}} \{\hat{\psi}_{\mathbf{b}'}(x) + K_{\mathbf{b},\mathbf{b}'} \\ &\quad + \mathbb{E}[\sum_{j=1}^{3} \alpha_j(\mathbf{b}')\underline{v}_{\mathbf{b}',\gamma^c(\mathbf{b}')}(t_k, X_{t_{k+1}}^{t_k,x}, z_j(\mathbf{b}'))]\} \\ &\le \min_{\mathbf{b}' \in \mathcal{I}} \{\hat{\psi}_{\mathbf{b}'}(x) + K_{\mathbf{b},\mathbf{b}'} \\ &\quad + \mathbb{E}[\sum_{j=1}^{3} \alpha_j(\mathbf{b}')\hat{v}_{\mathbf{b}',\gamma^c(\mathbf{b}')}(t_k, X_{t_{k+1}}^{t_k,x}, z_j(\mathbf{b}'))]\} \end{aligned}$$

where $z_j(\mathbf{b}') \in \hat{\mathcal{D}}_{\gamma^c(\mathbf{b}')}^z$ and $\sum_{j=1}^3 \alpha_j = 1$. By the above and concavity of $\hat{v}$ in z we thus have

$$\begin{aligned} \underline{v}_{\mathbf{b},c}(t_{k-1}, x, z) &\le \min_{\mathbf{b}' \in \mathcal{I}} \{\hat{\psi}_{\mathbf{b}'}(x) + K'_{\mathbf{b},\mathbf{b}'} \\ &\quad + \mathbb{E}[\hat{v}_{\mathbf{b}',\gamma^c(\mathbf{b}')}(t_k, X_{t_{k+1}}^{t_k,x}, z + \hat{\Gamma}\mathbf{b}')]\} \\ &= \hat{v}_{\mathbf{b},c}(t_{k-1}, x, z) \end{aligned}$$

and the result follows by backward induction. □

5.3 An upper bound

We will now exploit the linearity of the terminal cost in z to build a computationally efficient upper bound for the value function. First, note that the cost functional can be written

[6] An investigation of the error convergence rate for time discretization in optimal switching problems was carried out in Aïd et al. [2014].

$$J^{\mathbf{b},c}(t,x,z;u) := \mathbb{E}\bigg[\int_t^T (\psi_{\xi_s^{\mathbf{b}}}(X_s^{t,x}) + \mathbb{E}[\gamma^c(\beta_1,\ldots,\beta_N)|\mathcal{F}_s]\Gamma\xi_s^{\mathbf{b}})ds + \gamma^c(\beta_1,\ldots,\beta_N)z + h_{\xi_T^{\mathbf{b}}}(X_T^{t,x}) + \sum_{j=1}^N K_{\beta_{j-1},\beta_j}\bigg]. \quad (6)$$

In a backward induction scheme, as that employed in dynamic programming, the running cost $\psi_{\mathbf{b}}(x) + \mathbb{E}[\gamma^c(\beta_1,\ldots,\beta_N)|\mathcal{F}_{t_k}]\Gamma\mathbf{b}$ is known in step $N_\Pi - k$ as it only depends on forward quantities. We can thus compute the optimal feedback control, $\hat{\xi}$, in (4) as

$$\hat{\xi}^{\mathbf{b},c}_{t_k,x,z} \in \arg\min_{\mathbf{b}'\in\mathcal{I}}\{\hat{\psi}_{\mathbf{b}'}(x) + \hat{\gamma}^{\mathbf{b}',\gamma^c(\mathbf{b}')}_{t_k,x,z}(z+\hat{\Gamma}\mathbf{b}') + K_{\mathbf{b},\mathbf{b}'} + \mathbb{E}[\tilde{v}_{\mathbf{b}',\gamma^c(\mathbf{b}')}(t_{k+1}, X^{t_k,x}_{t_{k+1}}, z+\hat{\Gamma}\mathbf{b}')]\},$$

where $\hat{\gamma}^{\mathbf{b},c}_{t_k,x,z} := \mathbb{E}[\gamma^c(\hat{\beta}_1,\ldots,\hat{\beta}_{\hat{N}})|(X,Z,\xi)_{t_k} = (x,z,\mathbf{b})]$ and $\tilde{v}_{\mathbf{b}',\gamma^c(\mathbf{b}')}$ is the cost to go in (6) under the optimal control

$$\tilde{v}_{\mathbf{b},c}(T,x,z) = h_{\mathbf{b}}(x)$$

$$\tilde{v}_{\mathbf{b},c}(t_k,x,z) = \hat{\psi}_{\hat{\xi}^{\mathbf{b},c}_{t_k,x,z}}(x) + \hat{\gamma}^{\hat{\xi}^{\mathbf{b},c}_{t_k,x,z},\gamma^c(\hat{\xi}^{\mathbf{b},c}_{t_k,x,z})}_{t_k,x,z}\hat{\Gamma}\hat{\xi}^{\mathbf{b},c}_{t_k,x,z} + K_{\mathbf{b},\hat{\xi}^{\mathbf{b},c}_{t_k,x,z}} + \mathbb{E}[\tilde{v}_{\hat{\xi}^{\mathbf{b},c}_{t_k,x,z},\gamma^c(\hat{\xi}^{\mathbf{b},c}_{t_k,x,z})}(t_{k+1}, X^{t_k,x}_{t_{k+1}}, z+\hat{\Gamma}\hat{\xi}^{\mathbf{b},c}_{t_k,x,z})].$$

Now, to get a numerically tractable algorithm we replace $Z^{\mathbf{b}}_{t_k}$ in the feedback control $\hat{\xi}^{\mathbf{b},c}_{t_k,X_{t_k},Z^{\mathbf{b}}_{t_k}}$ with an estimate $\hat{\zeta}^{\mathbf{b},c}$ and get

$$\bar{v}_{\mathbf{b},c}(T,x) = h_{\mathbf{b}}(x)$$

$$\bar{v}_{\mathbf{b},c}(t_k,x) = \hat{\psi}_{\bar{\xi}^{\mathbf{b},c}_{t_k,x}}(x) + \bar{\gamma}^{\bar{\xi}^{\mathbf{b},c}_{t_k,x},\gamma^c(\bar{\xi}^{\mathbf{b},c}_{t_k,x})}_{t_k,x}\hat{\Gamma}\bar{\xi}^{\mathbf{b},c}_{t_k,x} + K_{\mathbf{b},\bar{\xi}^{\mathbf{b},c}_{t_k,x}} + \mathbb{E}[\tilde{v}_{\bar{\xi}^{\mathbf{b},c}_{t_k,x},\gamma^c(\bar{\xi}^{\mathbf{b},c}_{t_k,x})}(t_{k+1}, X^{t_k,x}_{t_{k+1}})],$$

where this time $\bar{\xi}$ is a measurable selection of

$$\bar{\xi}^{\mathbf{b},c}_{t_k,x} \in \arg\min_{\mathbf{b}'\in\mathcal{I}}\{\hat{\psi}_{\mathbf{b}'}(x) + \bar{\gamma}^{\mathbf{b}',\gamma^c(\mathbf{b}')}_{t_k,x}(\zeta^{\mathbf{b},c}_{t_k,x} + \hat{\Gamma}\mathbf{b}') + K_{\mathbf{b},\mathbf{b}'} + \mathbb{E}[\bar{v}_{\mathbf{b}',\gamma^c(\mathbf{b}')}(t_{k+1}, X^{t_k,x}_{t_{k+1}})]\}.$$

We get that $\bar{v}_{\mathbf{b},c}(0,x) = \hat{J}^{\mathbf{b},c}(0,x,0;\bar{u})$ where $\bar{u}$ is the control corresponding to the operation mode $\bar{\xi}$. We thus trivially get the following:

Proposition 3. The functions $\bar{v}$ bound $\hat{v}$ from above at time 0 in the sense that for each $(\mathbf{b},c) \in \bar{\mathcal{I}}$ and $x \in \mathbb{R}$ we have,

$$\hat{v}_{\mathbf{b},c}(0,x,0) \le \bar{v}_{\mathbf{b},c}(0,x).$$

6. NUMERICAL EXAMPLE

In the numerical example we will study a situation where the operator seeks an optimal call-off strategy for an operating period of $T = 60$ minutes, with a balancing market that has received ten bids (see Table 1). Note that the negative production costs for Bids 6-10 lead to a positive cost for downward regulation and is thus natural. We will assume that the net load can be accurately modelled as the solution to the linear SDE

$$dX_t = \left(\frac{dm(t)}{dt} + \alpha(m(t) - X_t)\right)dt + \sigma dW_t, \quad \forall t \in [0,T]$$
$$X_0 = m(0).$$

Table 1. Bids in the Example.

i	Vol_i	c_i^{off}	$c_{g,i}$
1	150	200	2
2	125	200	3
3	100	150	4
4	75	100	4
5	50	100	3
6	−50	100	−2
7	−75	100	−3
8	−100	150	−2
9	−125	200	−1
10	−150	200	−1

with coefficients $\alpha > 0$, $\sigma > 0$ and forecasted trajectory $(m(t) : 0 \le t \le T)$ (for parameter estimation in this model from actual consumption data see Perninge et al. [2011]). We will try out the two different shapes of $m(t)$ (denoted $m_1(t)$ and $m_2(t)$) plotted in Fig. 1. We will assume a

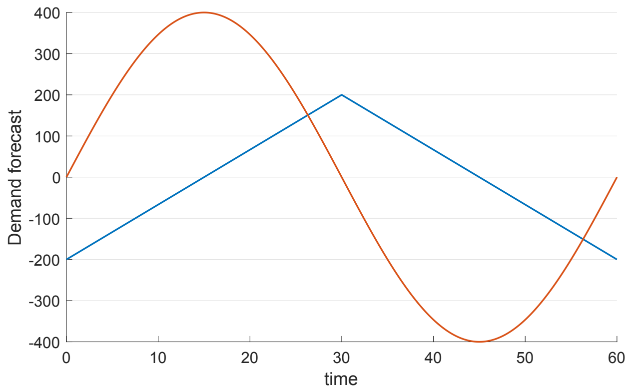

Fig. 1. Different forecasts investigated in the example.

cost functional with quadratic penalization of frequency deviations and let

$$J_F(u) := \mathbb{E}\bigg[c_f\int_0^T (X_t - \mathbf{1}^\top\Delta P^{\mathbf{b}}_t)^2 dt + c_{f,T}(X_T - \mathbf{1}^\top\Delta P^{\mathbf{b}}_T)^2\bigg],$$

where $\mathbf{1}$ is a column-vector of 1's. Parameters used throughout the example are given in Table 2.

Table 2. Parameter values.

α	σ	c_f	$c_{f,T}$
0.01	10	0.1	0.3

We will consider five different balancing power markets $M_1,\ldots,M_5$, where market M_i contains the subset of bids, $\{1,\ldots,i,11-i,\ldots,10\}$. The problem will be solved by discretizing time and state-space. We use a grid with 201 points to discretize the state space of X_t. For the time discretization we use 121 grid points. When computing the lower bound we used 10 equally spaced grid points for each dimension of Z when computing $M_1,\ldots,M_4$, while memory requirements prevented the use of more than 5 grid points for M_5.

The prediction of $Z^{\mathbf{b}}_t$ was taken to be

$$\hat{\zeta}^{\mathbf{b},c}_t := \int_0^t \begin{bmatrix} \min(m_i^+(s), \max_{\mathbf{b}'\in\mathcal{I}_c}\Gamma^1\mathbf{b}') \\ \max(-m_i^-(s), \min_{\mathbf{b}'\in\mathcal{I}_c}\Gamma^2\mathbf{b}') \end{bmatrix}^\top ds.$$

where $m_i = m_i^+ - m_i^-$ is the decomposition of m_i into its positive and negative parts.

Computation times on a standard Intel Core i5 laptop in the different settings are given in Table 3.

Table 3. Computational times [s].

	M_1	M_2	M_3	M_4	M_5
$\underline{v}$	16.4	69	451	2368	5272
$\bar{v}$	0.2	1.4	23	257	3240

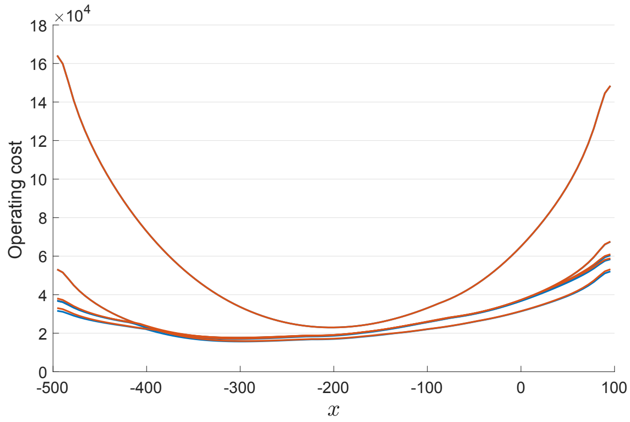

Fig. 2. The lower (blue) and upper (red) bounds on the value function at time 0 in extended operation mode $(0, 0)$ for $M_1, \ldots, M_5$, with forecast m_1.

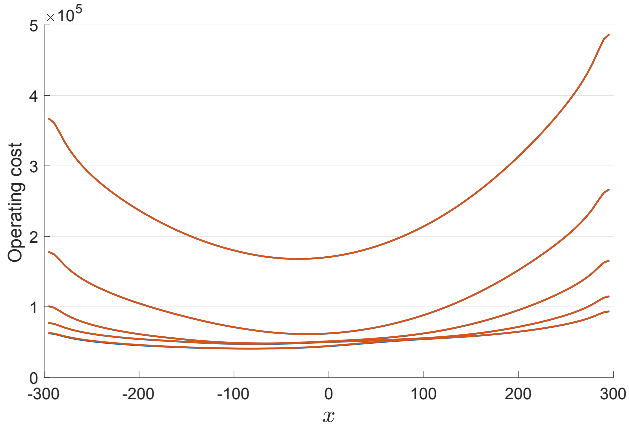

Fig. 3. The lower (blue) and upper (red) bounds on the value function at time 0 in extended operation mode $(0, 0)$ for $M_1, \ldots, M_5$, with forecast m_2.

7. CONCLUSION

In this paper we propose a stochastic control formulation of the frequency control problem that the system operator is faced with each operating period. Characteristics of the problem are the multitude of bids available to the operator and the market structure where the marginal price of the most expensive bid to be called-off applies to all called-off bids.

The main objective of the work is to overcome the numerical tractability issues often encountered when trying to solve high dimensional stochastic optimal control problems. To this purpose we develop computationally efficient upper and lower bounds for the value function.

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