Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Improved Event-Triggering Scheme for Uncertain Systems ⋆

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Abstract: This paper develops a new event-triggering approach for a class of nonlinear systems. This approach goes beyond the control of certain systems by developing sporadic feedback control laws to solve unstructured uncertainty problem. It guarantees asymptotic stability of a broad range of uncertain systems under sample and hold implementation while it efficiently reduces the communication samples. Under the proposed scheme, in order to handle the uncertain dynamics, the sampling time might become smaller than that in the approach of the certain systems. Using linear matrix inequalities, the controller gains and the event triggering parameters are obtained simultaneously. In addition to addressing new scheme for uncertain systems, we propose another triggering mechanism for affine nonlinear systems. Simulation results illustrate that the increase of triggered messages due to the nonlinearity and uncertainty is kept small enough that can be neglected when compared with conventional event-triggering policy.

Keywords: Event-triggering control, Uncertain systems, Robust stability, Networked systems.

1. INTRODUCTION

An important aspect in the implementation of networked control systems is the data congestion due to the controller-plant communication. Several practical applications concerning event triggering control have appeared in literature, see Hendricks et al. (1994); Heemels et al. (1999); Henningsson and Cervin (2009). In recent years, event and self triggering are investigated for real time control over networks, see Zhang et al. (2019) and references therein. Under event-triggering, the state is transmitted when some error exceeds a predefined certain threshold Tabuada (2007). Many researchers considered this idea in a similar manner for probabilistic contained filtering of nonlinear systems Tian et al. (2019). Although several theoretical aspects of event-triggered control have been studied extensively, see Henningsson et al. (2008); Miskowicz (2006), relatively few results exist that treat the robustness behavior under event-triggered control, see Xing et al. (2016); Wang and Han (2016); Xing et al. (2018).

Compared with classical time-triggered control, event-based control methodologies are more preferred since signals are transmitted through the network only when a certain condition is violated Zhang and Yang (2019). Due to its complexity, it is less preferable from synthesis and analysis standpoints particularly for uncertain dynamics Huang and Liu (2019). Typically, obtaining an event-triggering policy for a pre-designed stabilizing controllers wastes the robustness of the control law. On the other hand, designing a controller that depends on the event-triggered policy is an awkward task that increases the restrictions on stabilizing controllers. In addition, it usually decreases the synthesis flexibility which is essential for perturbed systems.

The challenging problem of implementing event-triggering control for uncertain systems appears when the triggering policy and the control law are designed simultaneously. Moreover, the available results about designing such controllers might not provide flexible structure and may lead to a unacceptable increase in the number of transmitted signals through the network Abdelrahim et al. (2018). A robust stabilization methodology of nonlinear systems using event-triggered output feedback is investigated in Abdelrahim et al. (2017). In addition, choosing an appropriate event triggering mechanism for nonlinear systems is also difficult and cumbersome even for certain systems. In this regard, it might be better to consider alternatives to the classical triggering mechanisms in order to handle a wide range of plant structures effectively. Extra transmissions may become inevitable to stabilize uncertain systems. This allows us to make a trade off between the communication congestion on the one hand and robustness on the other hand.

It is worth mentioning that most of the prior work about event-triggering methodologies concentrates on designing controllers for certain systems Heemels et al. (2015); Zhang et al. (2016); Liuzza et al. (2016). Only few results have been dedicated to study the robustness of event-triggered

⋆ This work is supported by the Deanship of Scientific Research (DSR) at KFUPM through Distinguished Professorship Award Project no. (IN 161065)

control systems, see Abdelrahim et al. (2017) and the references therein. This motivates us to investigate a new event-triggering policy based on linear matrix inequality (LMIs). Moreover, the majority of research on event-triggering is based on the same structure of event-triggering policy turns out to be a special case of the proposed one in this paper. To truly realize the attractive advantages of the new mechanism, one would need a remote controller, in which the controller-gain and its event policy are solely obtained by simple LMIs conditions based on the uncertainty structure and plant dynamics.

This paper considers continuous time-invariant dynamical systems that compromises unstructured uncertainties. The proposed approach provides an improvement to the event-triggering mechanism to efficiently control uncertain plants. The problem under consideration makes full sense for plant being controlled over networks, where event-triggering can be used. Moreover, theoretical results are established for affine nonlinear systems. Simulation results emphasized that the increase of triggered messages is kept small enough that can be neglected when compared with conventional event-triggering policy.

Notations: In the sequel, $\mathbb{R}^n$ denote the n-dimensional vector space equipped with the Euclidean norm. We use W^T and W^{-1} to denote the transpose and the inverse of any square matrix W, respectively. We use $W > 0$ to denote a symmetric positive definite matrix W and I to denote the $n \times n$ identity matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. In symmetric block matrices or complex matrix expressions, we use the symbol $\bullet$ to represent a term that is induced by symmetry.

Fact 1. Mahmoud (2011): *For any positive definite matrix Z and any real matrices X and Y, with appropriate dimensions,*

$$X^T Y + Y^T X \leq X^T Z X + Y^T Z^{-1} Y$$

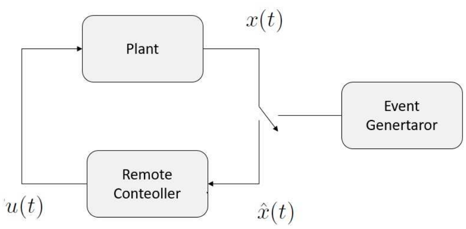

Fig. 1. "Remote controller" and "Event generator" diagram

2. PRELIMINARIES

Consider the event triggering scheme depicted in Fig. 1, where the plant dynamics is given by the following nonlinear time invariant system:

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control. Let the control law in (1) of the remote controller be given by $u(t) = k(\hat{x}(t))$ where $\hat{x}(t) = x(t_k)$ denotes the last transmitted state via the network at $t = t_k$ and $k(.) : \mathbb{R}^n \to \mathbb{R}^n$ is a static mapping that defines the remote control structure. Let the error between the actual and the transmitted state be $e(t) = x(t) - \hat{x}(t)$, then the closed loop system can be rewritten as follows:

$$\dot{x} = f(x, k(x - e)) \tag{2}$$

Theorem 2. Khalil and Grizzle (2002): *Consider the system $\dot{x} = f(x, u)$, where $f(x, u)$ is Lipschitz in x and u. Let $V(t, x)$ be a continuously differentiable function. Then the system is input to state stable if there exist positive definite function $W(.)$ and a class $\mathcal{K}$ functions $\alpha_1(.)$, $\alpha_2(.)$, and $\rho(.)$ such that the following conditions*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{3}$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, u) \leq -W(x), \quad \forall \|x\| \geq \rho(\|u\|) \geq 0$$

are satisfied $\forall (t, x, u) \in \{[0, \infty), \mathbb{R}^n, \mathbb{R}^m\}$

Definition 3. : A continuously differentiable positive definite function $V(x)$ is a control Lyapunov function (CLF) for the system $\dot{x} = f(x) + g(x)u$ if

$$\frac{\partial V}{\partial x} g(x) = 0, x \neq 0 \implies \frac{\partial V}{\partial x} f(x) < 0$$

Control Lyapunov Function (CLF) and Sontag's formula will be used to generate the control law for affine nonlinear systems. The existence of a CLF ensures that its time derivative is negative when the control law vanishes. The following theorem is a well-known method used to stabilize affine nonlinear systems.

Theorem 4. (Sontag (1989)). : *Let $V(x)$ be a CLF for the dynamical system in Definition (3), then the origin can be stabilized by the following control law*

$$u(x) = \begin{cases} -\dfrac{\frac{\partial V}{\partial x} f + \sqrt{(\frac{\partial V}{\partial x} f)^2 + (\frac{\partial V}{\partial x} g)^4}}{\frac{\partial V}{\partial x} g(x)}, & \dfrac{\partial V}{\partial x} g \neq 0 \\ 0, & \dfrac{\partial V}{\partial x} g = 0 \end{cases}$$

To this end, Taylor expansion and error bounds of the prescribed control law are needed to design the triggering policy. However, traditional notations for partial derivatives become rather cumbersome for derivatives of order higher than two, and they make it rather difficult to write Taylor's theorem in an intelligible fashion. However, a multi-index notation, which is now in common usage in the literature of partial differential equations, is available. A multi-index is an n-tuple of nonnegative integers. Multi-indexes are generally denoted by the Greek letters α or β: $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_j \in \{0, 1, 2, ...\}$. For any n-dimensional multi-index α, we define
$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$,
$\alpha! = \alpha_1! \alpha_2! ... \alpha_n!$,
$x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, and

$$\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_n}^{\alpha_n}}$$

For example, $\partial^{(4,0,3)} f = \frac{\partial^7 f}{\partial x_1{}^4 \partial x_3{}^3}$ and $\partial^{(0,4,0)} f = \frac{\partial^4 f}{\partial x_2^4}$.

Lemma 5. (Taylor's Theorem in Several Variables): *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is of class $\mathcal{C}^{k+1}$, i.e. it is continuously* *differentiable function up to order of $k+1$, on an open convex set $\mathcal{S}$. If $x \in \mathcal{S}$ and $x+e \in \mathcal{S}$, then*

$$f(x+e) = \sum_{|\alpha|\leq k} \frac{\partial^\alpha f(x)}{\alpha!} e^\alpha + R_{x,k}(e)$$

where the remainder is given in Lagrange's form:

$$R_{x,k}(e) = \sum_{|\alpha|=k+1} \partial^\alpha f(x+ce)\frac{e^\alpha}{\alpha!}, \quad \text{for some} c \in (0,1)$$

Corollary 6. : If $f : \mathbb{R}^n \to \mathbb{R}$ is of class $\mathcal{C}^{k+1}$ on $\mathcal{S}$ and $|\partial^\alpha f(x)| \leq M$ for $x \in \mathcal{S}$ and $|\alpha| = k+1$, then the remainder function is bounded by the following inequality

$$|R_{x,k}(e)| \leq \frac{M}{(k+1)!}\|e\|_w^{k+1},$$

where $\|e\|_w = |e_1| + |e_2| + ... + |e_n|$

3. MAIN RESULTS

Transmitting states is considered to decrease the communication congestion in distributed control systems. Since it activates the transmission depending on a certain condition/rule, in our case the rule is defined in terms of the sign of a predefined mapping $\phi(x,e)$ with specific properties. If the function $\phi(x,e)$ becomes negative a transmission is required in order to maintain it positive. The following assumption is made to identify the characteristics of the proposed event triggering policy.

Assumption 7. The event triggering rule $\phi(x,e)$ satisfies the following

- $\phi(x,0) \geq 0 \quad \forall x \in \mathbb{R}^n$
- *There exists a class $\mathcal{K}_\infty$ function $\rho(.)$ such that $\|x\| \geq \rho(\|e\|)$ whenever $\phi(x,e) \geq 0$,* i.e. there exists a positive scalar γ such that the following inequality holds:
$$\|x\| - \rho(\|e\|) + \gamma\phi(x,e) \geq 0$$
for all $x, e \in \mathbb{R}^n$.
- *The state transmission is activated when the event triggering rule $\phi(x,e)$ becomes negative*

Remark 8. The triggering $\phi(x,e)$ described in Assumption 7 generalizes the triggering-policy in literature, and they become equivalent if the negativity of $\phi(x,e)$ violates $\|x\| \geq \rho(\|e\|)$. In general, $\|x\| \geq \rho(\|e\|)$ does not imply that $\phi(x,e) \geq 0$. This means that transmission may be still activated even if the time derivative of the Lyapunov function in Theorem 2 is negative. The generalized form allows us to use the cross terms between the state and the transmission error which is difficult even if it is not possible using the special case. Despite of the fact that the new criteria might increase the number of transmitted samples to handle unmodeled dynamics, it increases the robustness and flexibility of the controllers as can be seen in the sequel.

The following result is established:

Lemma 9. Consider the triggering rule $\phi(x,e)$ and assume that Assumption7 holds. Then, if there exist positive definite functions $V(x)$ and $W(x)$ and a class $\mathcal{K}_\infty$ function $\rho(.)$ such that the broadcasting error $e(t)$ satisfies the following conditions

$$t_{k+1} = \inf\{t > t_k : \phi(x,e) \leq 0\} \tag{4}$$

$$\frac{\partial V}{\partial x} f(x, k(x-e)) \leq -W(x) - \phi(x,e), \quad \forall \|x\| \geq \rho(\|e\|) \geq 0$$

Then system (2) remains asymptotically stable.

Proof: Let $V(x)$ be a Lyapunov function candidate. Using Assumption 7 and Equation (4), the event triggering rule $\phi(x,e)$ is always maintained to be positive, and the following inequality holds

$$\dot{V}(t) = \frac{\partial V}{\partial x} f(x, k(x-e)) \leq -W(x), \quad \forall \|x\| \geq \rho(\|e\|) \geq 0$$

which is simply the time derivative of $V(x)$ along the trajectory (2). Then it follows by *Theorem* (2) that the stability is guaranteed. □

3.1 Event-triggering for affine nonlinear systems

Consider the following single input affine dynamical system:

$$\dot{x}(t) = f(x) + g(x)u(x) \tag{5}$$

where $x(t) \in \mathbb{R}^n$ and $u(x) : \mathbb{R}^n \to \mathbb{R}$. It is assumed that the state $x(t)$ is measurable and transmitted with a single packet. In addition, the sensor is event driven while the controller is time driven with zero order hold. The purpose of this note is to design a nonlinear event-triggerd controller, such that the dynamics of (5) becomes asymptotically stable. Sontag's formula can not be applied directly to event driven systems since the event mechanism may lead the dynamics to instability. Obviously, checking the sign of $\frac{\partial V}{\partial x}g$ as an event triggering condition does not guarantee the stability. In addition, the formula itself is somehow complicated to generate a triggering condition. With this in mind, a slightly change to the control law is needed to facilitate our design methodology. Similar to Sontag's theorem, let $V(x)$ be any CLF function such that the control law $u(x)$ is given by the following:

$$u(x) = \begin{cases} -\dfrac{\frac{\partial V}{\partial x} f(x) + k_1 x^T x}{\frac{\partial V}{\partial x} g(x)}, & \dfrac{\partial V}{\partial x} g \neq 0 \\ 0, & \dfrac{\partial V}{\partial x} g = 0 \end{cases} \tag{6}$$

then the time derivative of the Lyapunov function becomes

$$\dot{V}(x) = \frac{\partial V}{\partial x}\left[f(x) + g(x)u(x)\right] = -k_1 x^T x \tag{7}$$

$\dot{V}(x)$ becomes negative if k_1 is strictly positive.

Theorem 10. Let $V(x)$ be a CLF for the dynamical system (5), and the triggering samples are given by the following rule:

$$t_{k+1} = \inf\{t > t_k : k_2\|e(t)\| + k_3\|e(t)\|_w^2 > \beta\|x\|^2\} \tag{8}$$

where

$$k_2 = \max_{x\in\Omega}\left\{\frac{\partial V}{\partial x} g(x) \|\frac{\partial u}{\partial x}\|\right\}$$

$$k_3 = \max_{x\in\Omega}\left\{\frac{\partial V}{\partial x} g(x)\right\}\frac{M}{2}$$

$$\partial^2 u(x) \leq M.$$

If there exists a positive constant k_1 such that $0 < \beta < k_1$ then the control law:

$$u(\hat{x}) = \begin{cases} -\dfrac{\frac{\partial V}{\partial x} f(\hat{x}) + k_1 \hat{x}^T \hat{x}}{\frac{\partial V}{\partial x} g(\hat{x})}, & \dfrac{\partial V}{\partial x} g(\hat{x}) \neq 0 \\ 0, & \dfrac{\partial V}{\partial x} g(\hat{x}) = 0 \end{cases} \quad (9)$$

asymptotically stabilizes the dynamical system (5).

Proof: Taking the time derivative of $V(x)$ alongside the system (5), we obtain

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x}\left[f(x) + g(x)u(\hat{x})\right] \\ &= \frac{\partial V}{\partial x}\left[f(x) + g(x)u(x)\right] + \frac{\partial V}{\partial x} g(x)\left[u(\hat{x}) - u(x)\right] \end{aligned} \quad (10)$$

Substitute equation (7) in (10) and replace $\hat{x}$ by $x - e$ to get the following

$$\dot{V}(x) \leq -k_1 x^T x + \frac{\partial V}{\partial x} g(x)\left[u(x-e) - u(x))\right] \quad (11)$$

Applying the multi-variable Taylor series on $u(x)$ with $k = 1$ as illustrated in Theorem. 5 and Corollary. 6 gives:

$$\begin{aligned} \dot{V} &\leq -k_1 x^T x + \frac{\partial V}{\partial x} g(x)\left[\frac{\partial u}{\partial x} e + R_{k,x}(e)\right] \\ &\leq -k_1 x(t)^T x(t) + k_2\|e(t)\| + k_3\|e(t)\|_w^2 \\ &\leq -(k_1 - \beta)\|x\|^2 - \beta\|x\|^2 + k_2\|e\| + k_3\|e\|_w^2 \end{aligned} \quad (12)$$

The proposed event triggering policy maintains $-\beta\|x\|^2 + k_2\|e\| + k_3\|e\|_w^2$ to be negative, then

$$\dot{V} \leq -(k_1 - \beta)\|x\|^2$$

Since $0 < \beta < k_1$, then $\dot{V}$ is strictly negative which completes the proof.

4. LINEAR TIME INVARIANT SYSTEMS

Using the event-triggering strategy presented in the previous section, we specialize it to linear systems. Consider the following linear-time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (13)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ with appropriate matrices A and B. It is assumed that the system is controllable. Let the control law is a static feedback of the last transmitted state, that is, $u(t) = K\hat{x}(t)$. The control system in (13) could be rewritten as

$$\dot{x}(t) = (A + BK)x(t) - BKe(t) \quad (14)$$

where $e(t)$ is the transmission error between the last transmitted signal and the actual one, i.e. $e(t) = x(t) - \hat{x}(t)$. An event-triggering rule using the general policy indicated in *Assumption* (7) can be cast into the quadratic form:

$$\phi(x,e) = \left[\, x^T \;\; e^T \,\right] \begin{bmatrix} M & L \\ L^T & N \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad (15)$$

It must be kept in mind that this quadratic policy is much more general than the old version presented in Tabuada (2007) due to the existence of cross terms between the error and states. Although cross term is considered in Heemels et al. (2012), the rule is still not general since it does not introduce a new variable. Therefore the appearance of L-matrix in Equation (15) makes it more general and much more flexible to chose appropriate controller parameters. On the other hand, the number of events increases since an interaction exists between the error and plant's states that may cause a change in the event condition sign . Moreover, the quadratic structure can be replaced by another policy to overcome this problem but the change may increase the complexity of controller design. A brilliant strategy is to use the structure of quadratic form along side with decreasing the interaction effect, that is, decreasing the maximum singular value of the matrix L, to get red of the increase in transmitted signals.

The following theorem characterizes the usefulness of the quadratic event triggering policy. The theorem uses a set of LMIs to obtain an appropriate controller gain that stabilizes plants in event-triggered environment (1).

Theorem 11. Consider the event triggering structure (15). It stabilizes the linear system (13) asymptotically if there exist a set of matrices $Y > 0$, $S > 0$, $W > 0$, $U > 0$, $R \geq 0$, $V \geq 0$, and $X \in \mathbb{R}^{m\times n}$ such that the following LMIs are feasible,

$$\begin{bmatrix} \Pi & V - BX^T \\ \bullet & -W \end{bmatrix} \leq 0 \quad (16)$$

$$\begin{bmatrix} -W & V \\ \bullet & -U \end{bmatrix} < 0 \quad (17)$$

$$\Pi = SA^T + SA + R + XB^T + BX^T + Y \quad (18)$$

Moreover, the control gain $K = X^T S^{-1}$, and the event-triggering rule is given by

$$\phi(x,e) = \left[\, x^T \;\; e^T \,\right] \begin{bmatrix} S^{-1}YS^{-1} & S^{-1}VS^{-1} \\ \bullet & -S^{-1}WS^{-1} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad (19)$$

Proof: Consider a quadratic Lyapunov function $V(x) = x^T Px$. Simple manipulation on the event triggering policy $\phi(x,e)$ and using fact (1), it becomes

$$\begin{aligned} \phi(x,e) &= x^T Mx + x^T Le + e^T L^T x + e^T Ne \\ &\leq x^T(M + Z)x + e^T(L^T Z^{-1} L + N)e \end{aligned} \quad (20)$$

where Z is any positive definite matrix. To satisfy assumptions in Assumption 7, the following inequalities must be hold

$$M \geq 0, \qquad \begin{bmatrix} N & L^T \\ \bullet & -Z \end{bmatrix} < 0 \quad (21)$$

Multiply both sides of (21) by $diag[S,\ S]$ and use a new variable $U = SZS$ gives the following inequality

$$\begin{bmatrix} -W & V \\ \bullet & -U \end{bmatrix} < 0$$

which corresponds to (17). Using the Lyapunov function $V(x) = x^T Px$, it follows from *Lemma* 9 that

$$\begin{aligned} &x^T(A^T P + PA + Q + K^T B^T P + PBK + M)x \\ &-2e^T PBKx + 2e^T L^T x + e^T Ne \leq 0 \end{aligned} \quad (22)$$

To convexify (22), we substitute the following $S = P^{-1}$, $X = SK^T$, $R = SQS$, $Y = SMS$, $V = SLS$, $W = -SNS$ alongside with simple manipulation and using (20), then

$$x^T(SA^T + SA + R + XB^T + BX^T + Y)x \\ +2x^T(V - BX^T)e - e^T We \leq 0 \tag{23}$$

This inequality is equivalent to (16). This completes the proof. □

4.1 Uncertain systems with event-triggering

This section concerning about the stability of uncertain plants remotely. One of the challenges that faces the control of plants using event triggered policies is the uncertainty that corrupts in plant models. In the sequel, we direct attention to stabilize uncertain dynamical systems using the quadratic policy (7). Consider uncertain linear dynamical system

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) \tag{24}$$

where the uncertain part of the system is written as

$$[\,\Delta A \;\; \Delta B\,] = EF(t)\,[\,D_0 \;\; D_1\,], \quad F^T(t)F(t) \leq 1$$

The main result is established by the following theorem:

Theorem 12. The dynamical system (24) is robustly stable under event triggering policy if there exist a set of matrices $Y > 0$, $S > 0$, $W > 0$, $U > 0$, $R \geq 0$, and $X \in \mathbb{R}^{m\times n}$ such that the following LMIs are feasible,

$$\begin{bmatrix} \Pi + \rho EE^T & V - BX & \bar{D}_0 & \bar{D}_1 & \bar{D}_2 \\ \bullet & -W & 0 & 0 & 0 \\ \bullet & \bullet & -\lambda_1 I & 0 & 0 \\ \bullet & \bullet & \bullet & -\lambda_2 I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\lambda_3 I \end{bmatrix} \leq 0$$

$$\begin{bmatrix} -W & V \\ \bullet & -U \end{bmatrix} \leq 0$$

where

$$\Pi = SA^T + SA + R + XB^T + BX^T + Y,$$
$$\bar{D}_0 = \begin{bmatrix} SD_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{D}_1 = \begin{bmatrix} XD_1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\bar{D}_2 = \begin{bmatrix} 0 & 0 \\ XD_1 & 0 \end{bmatrix}, \quad \rho = \lambda_1 + \lambda_2 + \lambda_3$$

Moreover, the control gain $K = X^T S^{-1}$, and the event-triggering rule is (19)

Proof: Let

$$\Phi = \begin{bmatrix} \Pi & V - BX^T \\ \bullet & -W \end{bmatrix} \leq 0$$

then Equation (16) with the uncertainty of system (24) becomes

$$\Phi + \Phi_a + \Phi_a^T \leq 0 \tag{25}$$

where

$$\Phi_a = \begin{bmatrix} E \\ 0 \end{bmatrix} F\,[\,D_0 S + D_1 X^T \;\; -D_1 X^T\,] \tag{26}$$

Simple manipulations alongside with Fact 1 and Schur complements complete the proof. □

Remark 13. The feasibility of the LMIs in Theorem (11) and Theorem (12) depends on the system's dynamics as well as on the LMI variables. To decrease the conservativeness of the freely chosen variables. Introducing new variables is crucial to obtain event-triggering controllers. One of the new variables introduced in this note is the cross term in the event triggering rule, i.e. the matrix L. If L is assumed to be zero, the LMI conditions might become infeasible or leads to a unacceptable increase in the data transmissions depending on the dynamics and the uncertainty of the system.

5. SIMULATION RESULTS

In this note, the proposed method has been tested on a four-cart coupled with soft mass spring system as shown in Figure 2. The nonlinearity is considered as a norm bounded uncertainty and its linearized model is used to obtain the control law and the event triggering rule. The first order model of the considered nonlinear mass-spring system is given in Wang and Lemmon (2010); Chen et al. (1995).

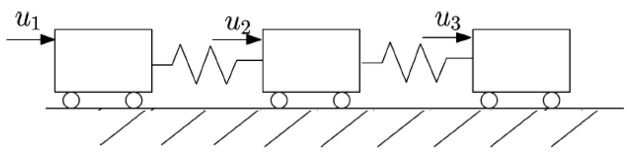

Fig. 2. Four carts coupled with soft springs.

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) \tag{27}$$

where

$$A = \begin{bmatrix} 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \end{bmatrix}, \quad B = \begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 0&0&0 \\ 0&1&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix}$$

and the uncertainty matrices are given by:

$$E = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_0 = \begin{bmatrix} 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1 \\ 0.3 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0.1 & 0 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0.2 \\ 0 & 0.2 & 0.1 \\ 0 & 0 & 0 \\ 0.1 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

Solving the LMIs (16) and (17) in Theorem 11 gives the following gain matrix:

$$K = \begin{bmatrix} -3.736 & -3.919 & -2.301 & -1.706 & -1.887 & -1.066 \\ -2.654 & -1.763 & -3.935 & -4.065 & -2.243 & -1.506 \\ -1.785 & -1.125 & -2.314 & -1.658 & -2.934 & -3.724 \end{bmatrix}$$

Consider the initial condition $x_0 = [-0.1, 1.5, 2, 0.5, -2, 1]$. Using the triggering mechanism (19) and the gain matrix K, then the uncertain system (27) is asymptotically stable. Figure 3 shows the the evolution of the states and Figure 4 plots the sample intervals that are generated by the proposed robust event triggered scheme. The smooth trajectories of the simulated example demonstrates the effectiveness of the developed strategy. The control laws of this system are shown in Figure 5. It can be seen that the inputs are constants between any two consecutive samples due to the the zero order hold.

To demonstrate the importance of the new event triggering structure, assume the cross term L in the event triggering rule to be zero. The same procedures in Theorem (12) are considered in order to extract the variables of the controller and the event-triggering rule. The LMIs problem might becomes infeasible in some cases when the cross term L is zero. However, for this simulation example, given $L = 0$ leads to an increase of samplings, 83 triggers when $L = 0$ compared to 52 samples only using our proposed scheme. This indicates that the traditional event triggering policy is worse and might be deficient for specific systems especially for uncertain systems.

Another simple simulation example is devoted for affine nonlinear system. Consider the following first order nonlinear system

$$\dot{x} = x^2 + xu$$

then $V(x) = x^2$ is clearly a control Lyapunov function.

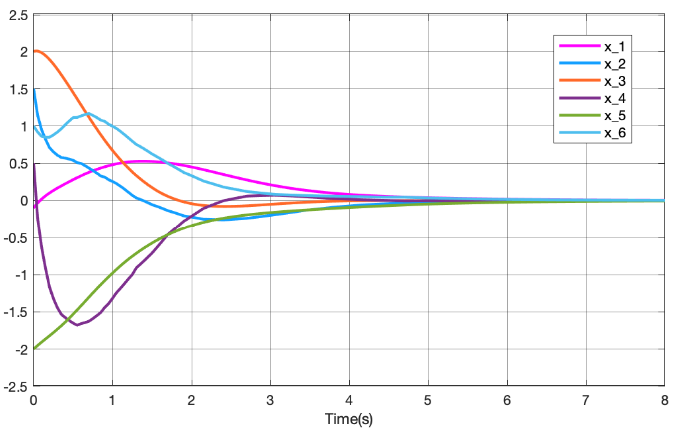

Fig. 3. States evolution of the four-cart dynamical system.

Applying theorem 10 for initial conditions $\|x(t_0)\| \leq 2$ gives the constants: $M = 0$, $k_3 = 0$, and $k_2 = 8$. We choose periodic checking of the event condition with sampling period less than 0.01 second. Since $k_1 = 1$ let $\beta = 0.5 < k_1$. The number of samples when $k_1 = 1$ equals to just 15 samples. The transmission error and the state response are shown in Fig 6 and 7, respectively.

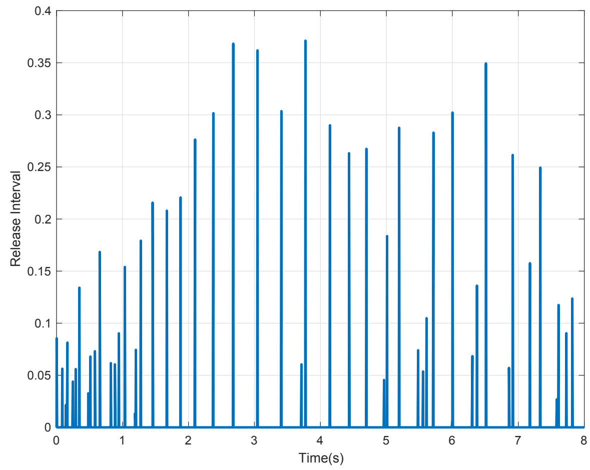

Fig. 4. Triggering instants and release intervals for the uncertain coupled cart.

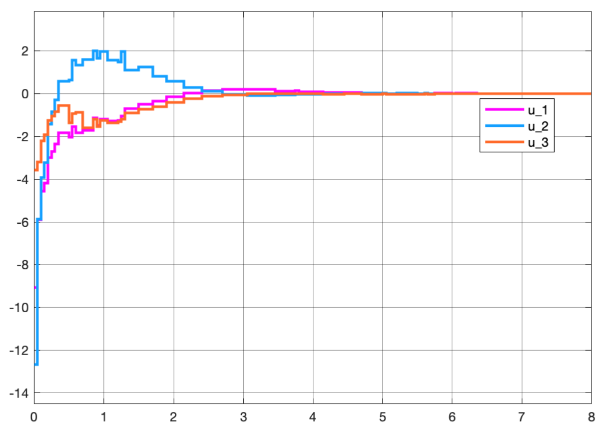

Fig. 5. The control laws of the four-cart system

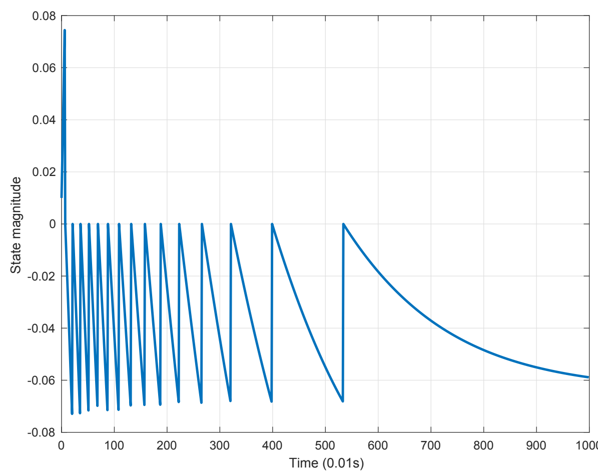

Fig. 6. Transmission error of the affine nonlinear system.

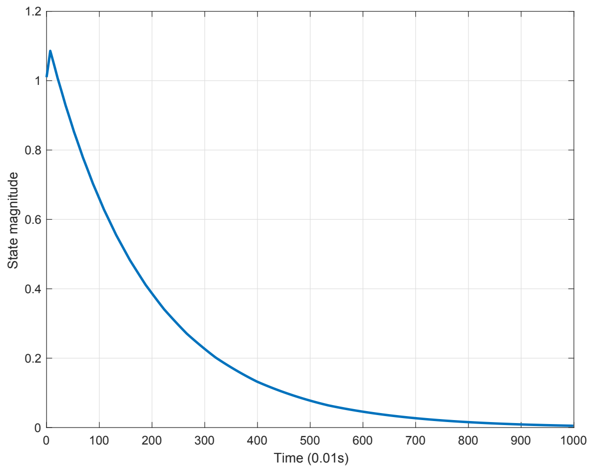

Fig. 7. State evolution of the closed loop affine nonlinear system

6. CONCLUSIONS

A novel event-triggering control scheme is proposed for uncertain and nonlinear systems. The proposed scheme is more general than the traditional event triggering presented in literature and more fixable to obtain control laws. By using linear matrix inequality, the controller gains and the event triggering parameters are obtained simultaneously. Simulation results using an example of four-cart system demonstrate the ability of the proposed method to stabilize uncertain systems while it decreases the transmitted samples sufficiently. Another nonlinear example is given to test the stability of the triggering scheme.

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