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Optimistic vs Pessimistic Moving-Horizon Estimation for Quasi–LPV Discrete-Time Systems

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Abstract: In this paper we focus on estimation for nonlinear plants that can be rewritten under the form of quasi-linear parameter-varying systems with bounded unknown parameters. Moving-horizon estimators are proposed to estimate the state of such systems according to two different formulations, i.e., "optimistic" and "pessimistic." In the former case, we perform estimation by minimizing the least-squares moving-horizon cost w.r.t. both state variables and parameters simultaneously. In the latter, we minimize such a cost w.r.t. the state variables after picking up the maximum w.r.t. the parameters. Under suitable assumptions, the stability analysis of the estimation is proved in both cases. A simple numerical example is provided to compare the proposed approaches.

Keywords: Moving horizon estimation; LPV systems; stability analysis.

1. INTRODUCTION

Linear parameter-varying (LPV) systems have received a special attention from the control community in recent times. Such an interest is motivated by the fact that a linearized model may fail in describing the nonlinear behavior of the underlying real system. A reasonable compromise between complexity and precision can be reached by resorting to an LPV approximation by preserving the linear structure of the state space and using a scheduling parameter vector to account for the nonlinearity in plant. In this paper, we investigate the possibility of applying moving-horizon estimation (MHE) to nonlinear plants rewritten under the form of quasi-LPV discrete-time systems with unknown parameters by focusing on two different formulations, which will be referred to as "optimistic" and "pessimistic" MHE. As shown later in the paper, the considered class of systems encompasses more general families of plants, namely systems with parameter uncertainties, Takagi-Sugeno systems, systems with Lipschitz nonlinearities, LPV systems with known parameters, etc. For instance, the class of quasi-LPV systems investigated in this paper may be obtained by exact transformation of a nonlinear system or by applying the differential variant of the mean value theorem. For such reasons, the proposed MHE strategies are valid for a large class of nonlinear systems.

MHE is pretty popular for its excellent performance in terms of precision and especially when one has to estimate the state of a plant subject to uncertainties and/or nonlinearities. The classical approaches based on the Kalman filter may undergo performance degradation due to poor modeling and insufficient information on the statistics of the noises affecting the dynamic and measurement equations [Jazwinski, 1970, Gelb, 1974, Anderson and Moore, 1979]. First ideas on MHE have been proposed for linear estimation [Rao et al., 2001, Alessandri et al., 2003]. Next developments of MHE have regarded nonlinear systems [Rao et al., 2003, Alessandri et al., 2008, 2011]. Later on, MHE has been extended to switching systems [Ferrari-Trecate et al., 2002, Alessandri et al., 2005b, Guo and Huang, 2013]. Uncertainties have been explicitly considered in several works [Alessandri et al., 2012, Fagiano and Novara, 2013, Alessandri and Awawdeh, 2016, Wan et al., 2016]. Distribution and decentralization have been recently investigated in Farina et al. [2010b,a], Haber and Verhaegen [2013], Schneider et al. [2015].

The combination of the MHE and LPV paradigms has been treated in Xue et al. [2012], where estimation is performed by using a moving horizon strategy and taking into account the loss of information within a stochastic framework, i.e., regarding the probabilities of input and output packet dropouts as time-varying parameters. Similarly, packet dropouts and quantization are dealt all together in Liu et al. [2013] by using MHE.

As compared with the approaches reported in the literature on observers for LPV systems, MHE requires to explicitly include the unknown parameters in the estimation criterion. In this respect, observers and filters for LPV systems are usually constructed by treating such parameters as polytopic uncertainties [Chebotarev et al., 2015, Wang et al., 2015, Krebs et al., 2018, Marx et al., 2019, Ellero et al., 2019] and using linear matrix inequalities (LMIs) [Boyd et al., 1994] for the purpose of design to ensure the stability of the estimation error. In this paper, we regard such parameters as other state variables or by considering their effect separately in the worst case. In practice, the two formulations correspond to either minimizing the least-squares MHE cost w.r.t. both state variables and parameters simultaneously or minimizing the worst-case least-squares MHE cost w.r.t. the state variables (i.e., after picking up the maximum w.r.t. the parameters). Such formulations based on "optimistic" and "pessimistic" point of view will be studied for what concerns the stability of the estimation error and compared through simulations.

The paper is organized as follows. The problems we will address are formulated in Section 2. The stability of the estimation error for the proposed approaches is analyzed in Section 3. Simulation results are reported in Section 4. Finally, conclusions and prospect of future work are given in Section 5.

Given a generic matrix M, as norm of M we rely on $|M| := \left(\lambda_{\max}(M^\top M)\right)^{1/2}$. For a real vector v, $|v| := (v^\top v)^{1/2}$ denotes its Euclidean norm. Given two real vectors x and y, let us define $(x, y) := \left[x^\top, y^\top\right]^\top$. Given the vectors $v_i, v_{i+1}, \ldots, v_j$ for $i < j$, we define $v_i^j := (v_i, v_{i+1}, \ldots, v_j)$. Finally, let us denote by $\mathrm{e}_n(i)$ a vector of dimension n with all zeros except a 1 in the i^{th} position.

2. MHE FOR QUASI–LPV SYSTEMS

2.1 Model description

Consider the quasi–LPV system described by the following set of equations:

$$x_{t+1} = A\left(p(x_t)\right)\, x_t + B\left(p(x_t)\right)\, u_t + w_t \tag{1a}$$

$$y_t = C\left(p(x_t)\right)\, x_t + v_t \tag{1b}$$

where $t = 0, 1, \ldots$ is the time instant, $x_t \in \mathbb{R}^n$ is the state vector, $u_t \in \mathbb{R}^q$ is the control vector, $w_t \in \mathbb{R}^n$ is the system noise vector, $y_t \in \mathbb{R}^m$ is the vector of the measures, and $v_t \in \mathbb{R}^m$ is the measurement noise vector. The matrices in (2) depends on a set of time-varying parameters that in turn depend on x_t with the mapping $x \mapsto p(x) \in \mathbb{R}^r$ unknown. However, we assume to know the image of such a mapping, namely, the compact set $\mathcal{P} \subset \mathbb{R}^r$ to which the parameters belong (i.e., $p(x_t) \in \mathcal{P}$ for all $t = 0, 1, \ldots$). Thus, in principle we may rely also on such information for the purpose of estimation.

The systems given by (1) include many families of plants and dynamic processes such as those briefly described in the following to highlight the potential advantage as compared with the estimation methods reported in the literature for LPV systems.

Uncertain linear systems Linear systems with parameter uncertainties can be viewed as a particular case of (1). Such a family of systems is often encountered in the literature because it leads to complicated stabilization problems, as discussed in Zemouche et al. [2017]. The proposed MHE approach enables to address such issues by estimating simultaneously the uncertain parameters and the state variables.

Multi-model systems The family of multi-model systems under the Takagi-Sugeno fuzzy structure [Guerra et al., 2015] is widely investigated in the literature, especially in the research area of fault diagnosis [Ichalal et al., 2018]. Such systems can be regarded as beloging to a particular class of plants that can be written under the form (1) with bounded unknown parameters. Such unknown parameters, in fuzzy systems, are called premise variables. Generally speaking, when the premise variables are unmeasurable, the estimation and stabilization problems become complicated and from the LMI point of view, the resulting conditions are very conservative. Using the proposed MHE approach, such premise variables can be estimated and then used for the purpose of stabilization.

Lipschitz nonlinear systems One of the major advantages of handling the class of systems given by (1) is the fact that such a class includes nonlinear Lipschitz systems often studied in the context of nonlinear observer design.

2.2 Formulation of MHE problems

Depending on our trust on $\mathcal{P}$, we may formulate either "optimistic" or "pessimistic" MHE problems by considering a least-squares cost, where such a cost is taken in the best case (by minimizing w.r.t. the parameters) or worst case (by maximizing w.r.t. the parameters). In both cases, we will study the stability of the estimation error, i.e., we will prove the exponential boundedness in the presence of the bounded system and measurement noises. For the sake of notational simplicity, from now on we drop the dependence on the control input as, instead of (1), we refer to

$$x_{t+1} = A\left(p_t\right)\, x_t + w_t \tag{2a}$$

$$y_t = C\left(p_t\right)\, x_t + v_t \tag{2b}$$

where $p_t := p(x_t)$. Based on the aforesaid, we will consider the following problem at $t = N, N+1, \ldots$ with given $\bar{x}_{t-N|t} \in \mathbb{R}^n$ and $\mu \geq 0$.

We consider the problem of estimating x_t according to moving-horizon strategies that consist in deriving an estimate $\hat{x}_t$ of x_t at time t by using the information given by $\mathcal{P}$, the measures $y_{t-N}, \ldots, y_t$, and the inputs $u_{t-N}, \ldots, u_{t-1}$. More specifically, in line with [Alessandri et al., 2003, 2005a, 2008] we aim to estimate $x_{t-N}, \ldots, x_t$ on the basis of such information and of a "prediction" $\bar{x}_{t-N|t}$ of the state x_{t-N} at the beginning of the moving window. We denote the estimates of $x_{t-N}, \ldots, x_t$ at time t by $\hat{x}_{t-N|t}, \ldots, \hat{x}_{t|t}$, respectively.

Problem 1. Find $\hat{x}^t_{t-N|t} \in \mathbb{R}^{n\times(N+1)}$ and $\hat{p}^t_{t-N|t} \in \mathcal{P}^{N+1}$ that minimize

$$J_1\left(x^t_{t-N}, p^t_{t-N}\right) = \mu\, \left|\, x_{t-N} - \bar{x}_{t-N|t}\,\right|^2 + \sum_{i=t-N}^{t} \left|y_i - C(p_i)\, x_i\right|^2 \tag{3}$$

under the constraints

$$x_{i+1} = A(p_i)\, x_i\,,\ i = t-N, \ldots, t-1\,. \tag{4}$$

In practice, Problem 1 consists in finding $\hat{x}^t_{t-N|t}, \hat{p}^t_{t-N|t}$ such that

$$J_1\left(\hat{x}^t_{t-N|t}, \hat{p}^t_{t-N|t}\right) \leq J_1\left(x^t_{t-N}, p^t_{t-N}\right)$$

for all $x^t_{t-N} \in \mathbb{R}^{n\times(N+1)}$ and $p^t_{t-N} \in \mathcal{P}^{N+1}$ subject to (4). As an alternative to Problem 1, a worst-case problem can be formulated as follows.

Problem 2. *Find* $\hat{x}^t_{t-N|t} \in \mathbb{R}^{n\times(N+1)}$ *that minimizes*

$$J_2\left(x^t_{t-N}\right) = \max_{p^t_{t-N}\in\mathcal{P}^{N+1}} \mu \left| x_{t-N} - \bar{x}_{t-N|t} \right|^2 + \sum_{i=t-N}^{t} |y_i - C(p_i)\, x_i|^2 \qquad (5)$$

under the constraints

$$x_{i+1} = A(p_i)\, x_i\,,\; i = t-N,\ldots,t-1\,.$$

Section 3 will concern the investigation of exponentially boundedness of the estimation error provided by Problem 1 and Problem 2.

3. STABILITY ANALYSIS OF THE MHE

First of all, we need some assumptions as follows.

Assumption 1. The mappings $p \mapsto A(p) \in \mathbb{R}^{n\times n}$ and $p \mapsto C(p) \in \mathbb{R}^{m\times n}$ are continuous.

We suppose that all the trajectories of the system lie in a compact set.

Assumption 2. There exists a compact set $\mathcal{X} \subset \mathbb{R}^n$ such that $x_t \in \mathcal{X}$ for all $t = 0,1,\ldots$, and let $r_x := \max_{x\in\mathcal{X}} |x|$.

Finally, the disturbances are assumed to be bounded.

Assumption 3. There exist $r_w, r_v > 0$ such that $|w_t| \le r_w$ and $|v_t| \le r_v$ for all $t = 0,1,\ldots$.

Based on the aforesaid, first we deal with Problem 1, which is solved at each t and hence we need to choose the prediction $\bar{x}_{t-N|t}$. Likewise in Alessandri et al. [2003, 2005a, 2008, 2012], a possible choice is

$$\bar{x}_{t-N+1|t+1} = A(\hat{p}_{t-N|t})\, \hat{x}_{t-N|t}\,,\; t = N, N+1, \ldots \qquad (6)$$

with a given initial $\bar{x}_{0|N}$ in (3) at time $t = N$.

To study the stability of the estimation error that results from the solution of Problem 1 together with the prediction update in (6), we have to account for the system evolution over time and introduce some additional assumptions related to observability. Toward this end, note the following:

$$y^t_{t-N+1} = F\left(p^t_{t-N}\right)\, x_{t-N} + H\left(p^t_{t-N}\right)\; w^{t-1}_{t-N} \quad t = N, N+1, \ldots \qquad (7)$$

where

$$F\left(p^t_{t-N}\right) := \begin{pmatrix} C(p_{t-N}) \\ C(p_{t-N+1})A(p_{t-N}) \\ \vdots \\ C(p_t)\prod_{i=1}^{N} A_{t-i} \end{pmatrix}$$

$$H\left(p^t_{t-N}\right) := \begin{pmatrix} 0 & \ldots & 0 \\ C(p_{t-N+1}) & \ldots & 0 \\ C(p_{t-N+2})A(p_{t-N+1}) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ C(p_t)\prod_{i=1}^{N-1} A(p_{t-i}) & \ldots & C(p_t) \end{pmatrix}.$$

Concerning observability, we assume the following.

Assumption 4. The constant

$$\delta := \min_{p^t_{t-N}\in\mathcal{P}^{N+1}} \lambda_{\min}\left(F\left(p^t_{t-N}\right)^\top F\left(p^t_{t-N}\right)\right) \qquad (8)$$

is strictly positive.

Before stating the first theorem providing a solution to Problem 1, we need to introduce the following definition.

Definition 1. A sequence of vector $\{v_t\}$ is said to be exponentially bounded if there exist $a \in (0,1)$ and $b > 0$ such that

$$|v_t| \le |v_0| a^t + b, t = 0,1,\ldots. \qquad (9)$$

Now, we consider the stability of the estimation error given by the solution of Problem 1.

Theorem 1. The estimation error e_{t-N} given by the solution of Problem 1 is exponentially bounded with

$$a = a_1(\mu) := \frac{36\mu}{3\mu + 2\delta} \qquad (10)$$

$$b = b_1(\mu) := \frac{6\left(c + \mu\Delta_A^2 r_x^2 + \mu r_w^2\right)}{3\mu + 2\delta} \qquad (11)$$

if μ is chosen such that $a_1(\mu) < 1$, where $c > 0$ and

$$\Delta_A := \max_{p,q\in\mathcal{P}} |A(p) - A(q)|\,. \qquad (12)$$

Proof 1. . For the sake of brevity, here we adopt the simpler notations $F_{t-N} := F\left(p^t_{t-N}\right)$, $H_{t-N} := H\left(p^t_{t-N}\right)$, $\hat{F}_{t-N} := F\left(\hat{p}^t_{t-N}\right)$, $\hat{H}_{t-N} := H\left(\hat{p}^t_{t-N}\right)$, and $\hat{x}_i := \hat{x}_{i|t}$, $\hat{p}_i := \hat{p}_{i|t}$, $i = t-N,\ldots,t$.

Let us consider the optimal cost that results from the solution of Problem 1, namely,

$$J_1\left(\hat{x}^t_{t-N}, \hat{p}^t_{t-N}\right) = \mu \left| \hat{x}_{t-N} - \bar{x}_{t-N|t} \right|^2 + \left| y^t_{t-N+1} - \hat{F}_{t-N}\, \hat{x}_{t-N} \right|^2. \qquad (13)$$

The proof consists in deriving lower and upper bounds on such a cost, which can be combined to bound the norm on the estimation error. The derivations of such bounds is only sketched for space limitation.

Following [Alessandri et al., 2008, 2012], after defining

$$\Delta_F := \max_{p^t_{t-N}, q^t_{t-N}\in\mathcal{P}^{N+1}} \left| F\left(p^t_{t-N}\right) - F\left(q^t_{t-N}\right) \right|.$$

and

$$r_H := \max_{p^t_{t-N}\in\mathcal{P}^{N+1}} \left| H\left(p^t_{t-N}\right) \right|,$$

we obtain

$$\left| F_{t-N}\, x_{t-N} - y^t_{t-N+1} \right|^2 \le 2r_H^2 (N+1)^2 r_w^2 + 2(N+1)^2 r_v^2$$

and hence

$$\left| y^t_{t-N+1} - \hat{F}_{t-N}\, \hat{x}_{t-N} \right|^2 \ge \frac{1}{3} \left| \hat{F}_{t-N}\; (x_{t-N} - \hat{x}_{t-N}) \right|^2 - \left| \left(\hat{F}_{t-N} - F_{t-N}\right) x_{t-N} \right|^2 - \left| F_{t-N}\, x_{t-N} - y^t_{t-N+1} \right|^2 \ge \frac{\delta}{3} |x_{t-N} - \hat{x}_{t-N}| - c_1, \qquad (14)$$

where

$$c_1 := \Delta_F^2 r_x^2 + 2r_H^2 (N+1)^2 r_w^2 + 2(N+1)^2 r_v^2\,.$$

Thus, we obtain the lower bound

$$J_1\left(\hat{x}_{t-N}^t, \hat{p}_{t-N}^t\right) \geq \frac{\mu}{2}|x_{t-N} - \hat{x}_{t-N}|^2 - \mu|x_{t-N} - \bar{x}_{t-N|t}|^2 + \frac{\delta}{3}|x_{t-N} - \hat{x}_{t-N}| - c_1 \,. \tag{15}$$

Concerning the upper bound of the cost, we have

$$\begin{aligned} J_1\left(\hat{x}_{t-N}^t, \hat{p}_{t-N}^t\right) &\leq J_1\left(x_{t-N}^t, p_{t-N}^t\right) \\ &= \left|y_{t-N+1}^t - F_{t-N}\, x_{t-N}\right|^2 + \left|y_{t-N+1}^t - F_{t-N}\, x_{t-N}\right|^2 \\ &\quad + \mu\, |x_{t-N} - \bar{x}_{t-N|t}|^2 \\ &= \mu\, |x_{t-N} - \bar{x}_{t-N|t}|^2 + \left|H_{t-N}\, w_{t-N}^{t-1} + v_{t-N}^t\right|^2 \\ &\leq \mu\, |x_{t-N} - \bar{x}_{t-N|t}|^2 + 2\left|H_{t-N}\, w_{t-N}^{t-1}\right|^2 + 2\left|v_{t-N}^t\right|^2 \\ &\leq \mu\, |x_{t-N} - \bar{x}_{t-N|t}|^2 + c_2, \end{aligned} \tag{16}$$

where

$$c_2 := 2\, r_H^2 (N+1)^2 r_w^2 + 2(N+1)^2 r_v^2 \,.$$

Using (15) and (16), we have

$$\left(\frac{\mu}{2} + \frac{\delta}{3}\right)\, |x_{t-N} - \hat{x}_{t-N}|^2 \leq 2\mu\, |x_{t-N} - \bar{x}_{t-N|t}|^2 + c_1 + c_2 \tag{17}$$

and let us focus on the term $|x_{t-N} - \hat{x}_{t-N}|$ in the r.h.s. of (17). It follows that

$$\begin{aligned} |x_{t-N} - \bar{x}_{t-N|t}| &= |A(p_{t-N-1})\, x_{t-N-1} + w_{t-N-1} \\ &\quad - A(\hat{p}_{t-N-1})\, \hat{x}_{t-N-1}| \\ &= |(A(p_{t-N-1}) - A(\hat{p}_{t-N-1}))\, x_{t-N-1} \\ &\quad + A(\hat{p}_{t-N-1})\, (x_{t-N-1} - \hat{x}_{t-N-1}) \\ &\quad + w_{t-N-1}| \\ &\leq |(A(p_{t-N-1}) - A(\hat{p}_{t-N-1}))\, x_{t-N-1}| \\ &\quad + |A(\hat{p}_{t-N-1})\, (x_{t-N-1} - \hat{x}_{t-N-1})| \\ &\quad + |w_{t-N-1}| \end{aligned}$$

and with a little algebra, we get

$$\begin{aligned} |x_{t-N} - \bar{x}_{t-N|t}|^2 &\leq 3|(A(p_{t-N-1}) - A(\hat{p}_{t-N-1}))\, x_{t-N-1}|^2 \\ &\quad + 3|A(\hat{p}_{t-N-1})\, (x_{t-N-1} - \hat{x}_{t-N-1})|^2 \\ &\quad + 3|w_{t-N-1}|^2 \,. \end{aligned}$$

Using the definition in (12) and $r_A := \max_{p \in \mathcal{P}} |A\,(p)|$, the previous inequality yields

$$|x_{t-N} - \bar{x}_{t-N|t}|^2 \leq 3\Delta_A^2 r_x^2 + 3r_w^2 + 3r_A^2 |x_{t-N-1} - \hat{x}_{t-N-1}|^2. \tag{18}$$

If we substitute (18) in (17), we finally obtain

$$|x_{t-N} - \hat{x}_{t-N}|^2 \leq \frac{36\mu}{3\mu + 2\delta}|x_{t-N-1} - \hat{x}_{t-N-1}|^2 + \frac{6\left(c + \mu\Delta_A^2 r_x^2 + \mu r_w^2\right)}{3\mu + 2\delta} \tag{19}$$

where $c := c_1 + c_2$, which concludes the proof.

Now let us focus on the stability analysis of the estimation error given by the solution of Problem 2.

Theorem 2. The estimation error e_{t-N} given by the solution of Problem 2 is exponentially bounded with the same definitions of $a_2(\mu)$ in (10) and $b_2(\mu)$ in (11) if μ is chosen such that $a_2(\mu) < 1$.

Proof 2. It is straightforward to follow the same steps of the proof of Theorem 1 by using the cost function

$$J_2(\hat{x}_{t-N|t}^t) = J_1\left(\hat{x}_{t-N|t}^t, \hat{\pi}_{t-N|t}^t\right)$$

where $\hat{\pi}_{t-N|t}^t \in \mathcal{P}$ is the maximizer in the r.h.s. of (5) and with related definitions such as $F_{t-N} := F\left(\pi_{t-N}^t\right)$, $H_{t-N} := H\left(\pi_{t-N}^t\right)$, and so on.

Remark 1. As compared with alternative estimation methods for LPV systems, MHE is able to provide estimates of both system states and unknown parameters, even when such parameters affect the system matrices nonlinearly. In fact, the matrices A and C in (2) may depend nonlinearly on the parameter p_t. Although owing to the boundedness of the parameters, it is always possible to avoid such nonlinearities by introducing a new extended parameter vector, $\bar{p}_t$, with higher dimension and rewrite A and C with linear dependence on $\bar{p}_t$, this would increase the size of the new parameter $\bar{p}_t$. If such a parameter increases significantly, it may happen also to lose the detectability and stabilizability conditions when augmenting the size of the unknown parameter vector as well as the infeasibility of LMI conditions ensuring the stability of the estimation error for the current methods reported in the literature. Moreover, LMI-based techniques may require additional conservative conditions to guarantee existence of an observer. This is the case of adaptive observers with strong equality constraints [Cho and Rajamani, 1997] or unknown input observers with restrictive rank conditions based on singular systems theory [Dai, 1989].

It is worth noting that the bounds of Theorems 1 and 2 are different and conservative in general. Therefore, for a fair evaluation of the effectiveness of the proposed approaches, Section 4 will be focused on a simple simulation example.

4. SIMULATION RESULTS

Consider the discrete-time oscillator model presented in [Turner, 2003, p. 107, (37)], i.e.,

$$x_{t+1} = A(p)\, x_t$$

where $x_t \in \mathbb{R}^2$ and

$$A(p) = \begin{pmatrix} \sqrt{1-p^2} & p \\ -p & \sqrt{1-p^2} \end{pmatrix}$$

with $p \in [0.5, 1]$ and having at disposal the measurements of the first state variable, i.e., $C = (1\ 0)$.

In each simulation run, the initial state and the system and measurement noises have been generated with Gaussian distributions centered at zero and covariances equal to I, $0.01I$, and 0.1. Each simulation run is made of 200 time steps. The unknown parameter p is initially randomly generated in interval $[0.5, 1]$ and is subject to an additive, zero-mean Gaussian noise with variance 0.01, which is lower or upper saturated in case it goes out of the interval $[0.5, 1]$.

We have solved Problem 1 by minimizing a cost function with $\mu = 1$ via the general-purpose Matlab routine *fmincon.* The solution of Problem 2 has been obtained with *fmincon* that calls inside the *fminbnd* routine to maximize. We will refer to the solution of Problem 1 as optimistic MHE (OMHE) and to that of Problem 2 as pessimistic MHE (PMHE). For both OMHE and PMHE, the estimated parameters is taken constant over all the moving horizon.

wind. length	RMSE (x_1)	RMSE (x_2)	RMSE (x_3)	comp. time
$N = 1$	0.10774	0.18035	0.049345	0.010486
$N = 2$	0.10391	0.16046	0.037413	0.012381
$N = 3$	0.10131	0.13999	0.027378	0.012953
$N = 4$	0.09772	0.12650	0.023780	0.013960

Table 1. Medians of RMSEs and computational times of a single optimization round (in s) for the OMHE with different window lengths over 100 simulation runs.

wind. length	RMSE (x_1)	RMSE (x_2)	RMSE (x_3)	comp. time
$N = 1$	0.13120	0.50943	0.17122	0.08124
$N = 2$	0.20043	0.55501	0.17081	0.10850
$N = 3$	0.63928	0.70930	0.16873	0.11723
$N = 4$	0.83602	0.98305	0.16974	0.12239

Table 2. Medians of RMSEs and computational times of a single optimization round (in s) for the PMHE with different window lengths over 100 simulation runs.

The medians of RMSEs and computational times over 100 runs are summarized in Tables 1 and 2. The result of a simulation run is depicted in Fig.s 1, 2, and 3. The OMHE provides better performances not only in terms of computational demand but also of precision. The RMSEs given by the OMHE are lower and they become lower and lower with an increasing window length N. By contrast, the RMSEs of the PMHE degrades if N grows. The PMHE provides an estimate of the time-varying parameter p that corresponds to its "average" value, thus failing to follow the variation of p (see Fig. 3).

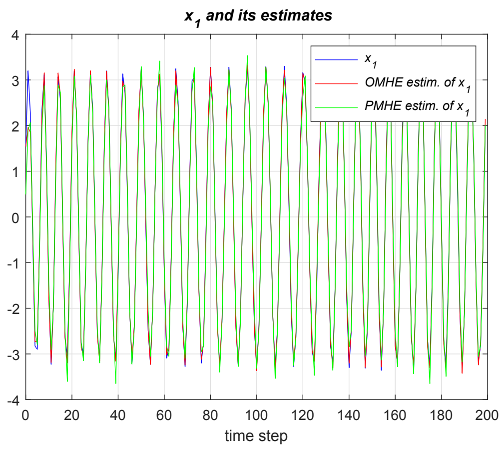

Fig. 1. Plots of x_1 and its OMHE and PMHE estimates with MHE of length $N = 2$.

5. CONCLUSIONS

In this paper, we have presented a novel approach to estimation for quasi-LPV system based on MHE. MHE is accomplished by minimizing a least-squares cost function, thus we may deal with the unknown parameters by either regarding them as state variables (and hence minimizing also w.r.t. them) or by considering them in the worst case, i.e., by solving a *min-max* problem. We have referred to the former as the "optimistic" one, while the latter is the "pessimistic" one. The stability of the estimation error given by the exponential boundedness is proved for both of them. Simulation results have been provided that show the effectiveness of the "optimistic" approach as compared with the "pessimistic" one.

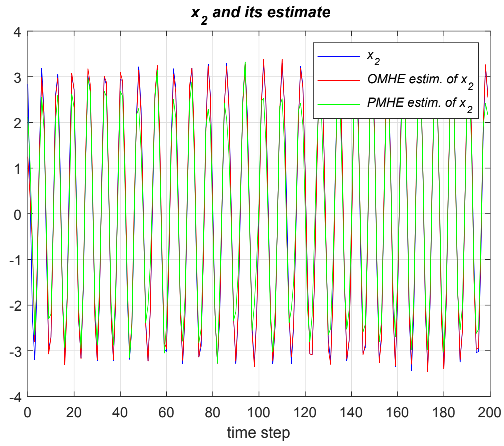

Fig. 2. Plots of x_2 and its OMHE and PMHE estimates with MHE of length $N = 2$.

Fig. 3. Plots of p and its OMHE and PMHE estimates with MHE of length $N = 2$.

Since MHE enables to estimate the unknown parameters belonging to a quasi-LPV structure for nonlinear systems, a comparison with different other alternative estimation methods is deserved, especially for what concerns the relaxation of the stability conditions. In the context of LMI-based techniques, an LPV observer can be designed under more or less conservative conditions to be evaluated. For example, the state of Lipschitz systems may be estimated by considering a constant-gain Luenberger observer, which leads to an exponential number of LMIs to be solved with the same gain. Such issues may be the subject of future work. In addition, we will address the reduction of the computational effort by using fast MHE techniques [Alessandri and Gaggero, 2017]. Another direction of investigation may be the analysis of stochastic stability under suitable assumptions on the statistics of the noises.

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