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A Lyapunov-Razumikhin Condition of ISS for Switched Time-Delay Systems Under Average Dwell Time Commutation ⋆

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Abstract: A condition of ISS is proposed for nonlinear time-delay systems based on the Lyapunov-Razumikhin theory, which allows the rate of convergence to be evaluated. Then, this condition is used for ISS analysis of switched nonlinear time-delay systems with average dwell time switching. Finally, one example is given to verify the effectiveness of theoretical findings.

Keywords: Switched nonlinear time-delay systems, Exponential stability, Razumikhin approach

1. INTRODUCTION

Time delays are frequently used in technical systems to model the features of the information transmission and actuator constraints. Appearance of a time delay is usually related with a degradation of the quality of the transients (Dugard, 1998; Gu et al., 2003; Richard, 2003), while a large delay can lead to instability of the systems. There are two main frameworks to deal with stability analysis of time-delay systems using Lyapunov-Razumikhin functions (Hale and Lunel, 2013; Teel, 1998; Myshkis, 1995; Mao, 1996) and Lyapunov-Krasovskii functionals (Dugard, 1998; Gu et al., 2003; Hale and Lunel, 2013). The latter approach is necessary and sufficient (Pepe and Jiang, 2006), and it can be used to establish the stability conditions (Moon et al., 2001; Wu et al., 2004); whereas the former method is typically applied for analysis of delay-independent stability (Fridman and Shaked, 2003; Jiao and Shen, 2005). It should be pointed out that usually the asymptotic stability is considered when using Lyapunov-Razumikhin functions. These observations leave a space for improvements of the Lyapunov-Razumikhin approach.

Switched systems are a widely distributed subclass of hybrid dynamics. There are plenty of analysis and design results obtained for such a kind of systems (El-Farra et al., 2005; Zhao and Dimirovski, 2004; Zhao et al., 2015; Fainshil et al., 2009; Zhai et al., 2007; Liberzon and Morse, 1999; Hespanha and Morse, 1999). Almost these results are derived under (average) dwell-time condition imposed on the commutation logic. A prerequisite to use the dwell-time approaches is that the exponential stability of each subsystem should be achieved (Efimov et al., 2008). Such a constraint imposes additional obstructions for utilization of Lyapunov-Razumikhin approach for analysis of switched time-delay systems. In Yan and Özbay (2008), the delay-independent asymptotic stability of switched linear time-delay systems was addressed using Lyapunov-Razumikhin criterion integrated with minimum dwell time. In Ren and Xiong (2019), for stochastic systems an average dwell-time switching was studied via the Lyapunov-Krasovskii approach and only asymptotic stability under a dwell-time switching was analyzed via Lyapunov-Razumikhin functionals. The work Jiang et al. (2013) considered the asymptotic stability of switched time-delay systems, where an average dwell-time switching law was integrated with small-gain conditions. Furthermore, the input-to-state stability (ISS) under presence of unstable subsystems was discussed in Jiang et al. (2016) (the authors used Lyapunov functions and small-gain conditions for stability analysis, but it is not in the framework of Lyapunov-Razumikhin approach).

Following the existing literature, this paper first presents a Lyapunov-Razumikhin criterion on ISS of nonlinear time-delay systems. It is verified that the corresponding systems without input exhibit exponential stability (in terms of Lyapunov-Razumikhin functions). Using this result, ISS of switched nonlinear time-delay systems is explored. For the systems without input, the exponential stability is reached and an average dwell time switching rule is provided. The remainder of the paper is organized as follows. Section 2 introduces the preliminaries, Section 3 presents an extension of ISS Lyapunov-Razumikhin conditions, the main results are formulated in Section 4, one example is provided in Section 5, and Section 6 concludes the paper.

Notation: Let $\Re$ (or $\Re_+$), $\Re^n$ (or $\Re^n_+$), $\Re^{n\times m}$ be the sets of (nonnegative) real numbers, n-dimensional (nonnegative) vectors and $n \times m$ matrices, respectively. The Euclidean norm $|x|$ and ∞-norm of a vector $x =$

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$(x_1, \ldots, x_n)^T \in \Re^n$ are defined as $|x| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ and $|x|_\infty = \max_{1\le i\le n} |x_i|$, respectively. If it is not stated explicitly, $|x|$ represents one of the norms introduced before for $x \in \Re^n$, or an absolute value for $x \in \Re$. For a continuous function $\phi : [a, b] \to \Re^n$, $a, b \in \Re$, its uniform norm is defined as $\|\phi\| = \sup_{t\in[a,b]} |\phi(t)|_2$; the space of such functions we will denote as $C_{[a,b]}$. For a Lebesgue measurable function of time $u : \Re_+ \to \Re^m$ define the norm $||u||_\infty = \text{ess}\sup_{t\ge0}|u(t)|_2$ and the space of u with $||u||_\infty < +\infty$ we further denote as $\mathcal{L}_\infty^m$. A continuous function $\alpha : \Re_+ \to \Re_+$ belongs to class $\mathcal{K}$ if it is increasing and $\alpha(0) = 0$, and $\alpha \in \mathcal{K}_\infty$ if it belongs to the class $\mathcal{K}$ and is also unbounded. A continuous function $\beta : \Re_+ \times \Re_+ \to \Re_+$ belongs to $\mathcal{KL}$ class if $\beta(\cdot, x)$ is a $\mathcal{K}$ class function and $\beta(x, \cdot)$ is decreasing to zero for any fixed $x \in \Re_+$. For a locally Lipschitz continuous function $V : \Re^n \to \Re_+$, its upper directional Dini derivative is defined as: $D^+V(x)v = \limsup_{h\to0^+} \frac{V(x+hv)-V(x)}{h}$ for any $x \in \Re^n$ and $v \in \Re^n$.

2. PRELIMINARIES

Consider a class of nonlinear time-delay systems (Gu et al., 2003):

$$\frac{dx(t)}{dt} = f(x_t, u(t)),\ t \ge 0, \tag{1}$$

where $x(t) \in \Re^n$ and $x_t \in C_{[-d,0]}$ is the state function, $x_t(s) = x(t + s)$, $-d \le s \le 0$ and $d > 0$ is a finite delay; $u(t) \in \Re^m$ is the external input, $u \in \mathcal{L}_\infty^m$; $f : C_{[-d,0]} \times \Re^m \to \Re^n$ is a continuous function, $f(0, 0) = 0$, and such that existence and uniqueness of solutions in forward time for the system (1) is ensured. Denote such a unique solution satisfying the initial condition $x_0 \in C_{[-d,0]}$ with the input $u \in \mathcal{L}_\infty^m$ by $x(t, x_0, u)$, and $x_t(s, x_0, u) = x(t + s, x_0, u)$ for $-d \le s \le 0$, which is defined on some interval $[-d, T)$.

Definition 1. (Teel, 1998; Pepe and Jiang, 2006) The system (1) is called practical ISS, if there exist $q \ge 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|x(t, x_0, u)| \le \max\{\beta(\|x_0\|, t), \gamma(||u||_\infty), q\} \quad \forall t \ge 0$$

for all $x_0 \in C_{[-d,0]}$ and all $u \in \mathcal{L}_\infty^m$. If $q = 0$ then (1) is called ISS.

There exist two methods evaluating ISS property of the system (1) based on a Lyapunov-Razumikhin function or a Lyapunov-Krasovskii functional. The former approach can be formulated as follows:

Theorem 2. (Teel, 1998) If there exists a locally Lipschitz continuous Lyapunov-Razumikhin function $V : \Re^n \to \Re_+$ such that

(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and all $x \in \Re^n$:

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|);$$

(ii) for some $\alpha, \gamma_x, \gamma_u \in \mathcal{K}$ and $r \ge 0$, with $\gamma_x(s) < s$ for all $s > 0$:

$$\max\{\gamma_x\left(\max_{\theta\in[-d,0]} V(\varphi(\theta))\right), \gamma_u(|u|), r\} < V(\varphi(0)) \Rightarrow$$
$$D^+V(\varphi(0)) f(\varphi, u) \le -\alpha(|\varphi(0)|)$$

for all $\varphi \in C_{[-d,0]}$ and all $u \in \Re^m$, then the system (1) is practically ISS, and it is just ISS if $r = 0$.

In the following, we introduce the notion of average dwell time switching.

Definition 3. (Hespanha and Morse, 1999) For a right-continuous signal $\sigma : \Re_+ \to \Re_+$ and $t_1 \ge t_0 \ge 0$, let $N_\sigma(t_0, t_1)$ denote the number of discontinuities of σ in the interval (t_0, t_1). If

$$N_\sigma(t_0, t_1) \le N_0 + \frac{t_1 - t_0}{\tau}$$

holds with $\tau > 0$ and $N_0 \ge 1$, then τ is called the average dwell time of the signal σ. If $N_0 = 1$ then τ is called the dwell time of σ.

3. ISS WITH GIVEN DECADE RATE

The drawback of Theorem 2 is that the form of the functions β and γ, which appear in Definition 1, is not evaluated. In order to use the dwell-time stability conditions (Hespanha and Morse, 1999; Efimov et al., 2008), we need to establish the shape of the function β, and for this purpose the following extension of Theorem 2 is developed:

Lemma 4. If there exists a locally Lipschitz continuous Lyapunov-Razumikhin function $V : \Re^n \to \Re_+$ such that

(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \Re^n$:

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|);$$

(ii) for some $\gamma' < 1$, $\alpha' \ge -\frac{\ln\gamma'}{d}$, $\gamma_u \in \mathcal{K}$ and $r \ge 0$:

$$\max\{\gamma' \max_{\theta\in[-d,0]} V(\varphi(\theta)), \gamma_u(|u|), r\} < V(\varphi(0)) \Rightarrow$$
$$D^+V(\varphi(0)) f(\varphi, u) \le -\alpha' V(\varphi(0))$$

for all $\varphi \in C_{[-d,0]}$ and all $u \in \Re^m$, then the system (1) is practically ISS, and for all $x_0 \in C_{[-d,0]}$, all $u \in \mathcal{L}_\infty^m$ and all $t \ge 0$:

$$|x(t, x_0, u)|$$
$$\le \alpha_1^{-1} \circ \max\{\exp\left(\frac{\ln\gamma'}{d}t\right) \alpha_2(\|x_0\|), \gamma_u(||u||_\infty), r\}.$$

Proof. Since all conditions of Theorem 2 are satisfied with $\alpha(s) = \alpha'\alpha_1(s)$ and $\gamma_x(s) = \gamma's$, then the system (1) is practically ISS, and we have only to demonstrate that the proposed upper bound on $|x(t, x_0, u)|$ is valid.

For any $x_0 \in C_{[-d,0]}$ and any $u \in \mathcal{L}_\infty^m$, the corresponding solution $x(t, x_0, u)$ is defined for all $t \ge 0$. Then the time domain of the solution existence $\Re_+ = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, where

$$\max\{\gamma' \max_{\theta\in[-d,0]} V(x(t + \theta)), \gamma_u(|u(t)|), r\} \ge V(x(t))$$

for all $t \in \mathcal{T}_1 \cup \mathcal{T}_2$, with

$$\gamma' \max_{\theta\in[-d,0]} V(x(t + \theta)) > \max\{\gamma_u(|u(t)|), r\}$$

for all $t \in \mathcal{T}_1$ and otherwise for all $t \in \mathcal{T}_2$:

$$\gamma' \max_{\theta\in[-d,0]} V(x(t + \theta)) \le \max\{\gamma_u(|u(t)|), r\}.$$

Consequently,

$$\max\{\gamma' \max_{\theta\in[-d,0]} V(x(t + \theta)), \gamma_u(|u(t)|), r\} < V(x(t))$$

for all $t \in \mathcal{T}_3$. Since $x(t)$ is a continuous function of time, and the same property is satisfied for $V(x(t))$, the sets $\mathcal{T}_1$ and $\mathcal{T}_3$ are concatenated by disjoint open intervals of the time, *i.e.* $\mathcal{T}_i = \bigcup_{j=1}^{J_i}(t_{i,s}^j, t_{i,e}^j)$ where $J_i > 0$ is (possibly infinite) quantity of these intervals in $\mathcal{T}_i$, $i = 1, 3$.

By construction $V(x(t)) \leq \max\{\gamma_u(||u||_\infty), r\}$ for all $t \in \mathcal{T}_2$.

Next, for any $j = 1, \ldots, J_1$ and the corresponding interval $(t^j_{1,s}, t^j_{1,e}) \subset \mathcal{T}_1$, for all $t \in (t^j_{1,s}, t^j_{1,e})$ there exists

$$\theta_t = \min\{\vartheta \in [-d, 0] : V(x_t(\vartheta)) = \max_{\theta\in[-d,0]} V(x_t(\theta))\},$$

where we introduce the minimum over $\vartheta \in [-d, 0]$ to resolve the non-uniqueness issue. Note that the inequality $\theta_t \leq -\varepsilon_{x_t}$ is satisfied for some $\varepsilon_{x_t} \in (0, d]$ dependent on x_t, since the maximum is calculated under the restriction that $\gamma' \max_{\theta\in[-d,0]} V(x(t+\theta)) \geq V(x(t))$ with $\gamma' < 1$. Thus,

$$V(x(t)) < \gamma' V(x_t(\theta_t)) = \exp(\ln \gamma') V(x_t(\theta_t)) \\ \leq \exp\left(-\ln\gamma' \frac{\theta_t}{d}\right) V(x_t(\theta_t)).$$

Recursively applying this estimate, *i.e.*,

$$V(x_t(\theta_t)) < \exp\left(-\ln\gamma' \frac{\theta_{t+\theta_t}}{d}\right) V(x_{t+\theta_t}(\theta_{t+\theta_t})),$$

we obtain

$$V(x(t)) \leq \exp\left(-\ln\gamma' \frac{\theta_t + \theta_{t+\theta_t}}{d}\right) V(x_{t+\theta_t}(\theta_{t+\theta_t})),$$

and by induction,

$$V(x(t)) \leq \exp\left(\frac{\ln\gamma'}{d}(t - t^j_{1,s})\right) \max_{\theta\in[-d,0]} V\left(x_{t^j_{1,s}}(\theta)\right) \quad (2)$$

for all $t \in (t^j_{1,s}, t^j_{,1e})$ (*i.e.*, for t sufficiently close to $t^j_{1,s}$ it could be $t + \theta_t < t^j_{1,s}$ and $\max_{\theta\in[-d,0]} V\left(x_{t^j_{1,s}}(\theta)\right)$ has to be used).

Finally, for any $j = 1, \ldots, J_3$ and the corresponding interval $(t^j_{3,s}, t^j_{3,e}) \subset \mathcal{T}_3$, for all $t \in (t^j_{3,s}, t^j_{3,e})$, we have

$$D^+V(x(t)) f(x_t, u(t)) \leq -\alpha' V(x(t))$$

by the conditions of the lemma and, consequently,

$$V(x(t)) \leq \exp\left(-\alpha'(t - t^j_{3,s})\right) V(x(t^j_{3,s})).$$

Since $\alpha' \geq -\frac{\ln\gamma'}{d}$ we obtain that the estimate (2) is satisfied for all $t \in (t^j_{i,s}, t^j_{,i,e})$ with $i = 1, 3$.

By definition $V(x(0)) \leq \max_{\theta\in[-d,0]} V(x_0(\theta))$, then combining the estimates derived for $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ we get that the relation

$$V(x(t)) \leq \\ \max\{\exp\left(\frac{\ln\gamma'}{d}t\right) \max_{\theta\in[-d,0]} V(x_0(\theta)), \gamma_u(||u||_\infty), r\}, \quad (3)$$

for any $x_0 \in C_{[-d,0]}$ and any $u \in \mathcal{L}^m_\infty$, is satisfied for $t = 0$, and let $t' > 0$ be a time instant such that (3) is true for all $t \in [0, t')$ and at $t = t'$ it is violated for the first time, *i.e.*

$$V(x(t')) > \\ \max\{\exp\left(\frac{\ln\gamma'}{d}t'\right) \max_{\theta\in[-d,0]} V(x_0(\theta)), \gamma_u(||u||_\infty), r\}.$$

Obviously, $\exp\left(\frac{\ln\gamma'}{d}t\right) \max_{\theta\in[-d,0]} V(x_0(\theta))$ is a strictly decreasing function of time $t \geq 0$, and always there is $T_{x_0,u} \geq 0$ such that

$$\exp\left(\frac{\ln\gamma'}{d}t\right) \max_{\theta\in[-d,0]} V(x_0(\theta)) \leq \max\{\gamma_u(||u||_\infty), r\}$$

for all $t \geq T_{x_0,u}$. Therefore, for $t' \in [0, T_{x_0,u})$

$$\max\{\gamma_u(||u||_\infty), r\} < \exp\left(\frac{\ln\gamma'}{d}t'\right) \max_{\theta\in[-d,0]} V(x_0(\theta)) \\ < V(x(t'))$$

and $V(x(t)) \leq \exp\left(\frac{\ln\gamma'}{d}t\right) \max_{\theta\in[-d,0]} V(x_0(\theta))$ for all $t \in [\max\{0, t' - d\}, t')$, then latter condition leads to $V(x(t')) > \gamma' \max_{\theta\in[-d,0]} V(x(t' + \theta))$. Hence,

$$\max\{\gamma' \max_{\theta\in[-d,0]} V(x(t'+\theta)), \gamma_u(|u(t)|), r\} < V(x(t')) \quad (4)$$

and, consequently,

$$D^+V(x(t')) f(x_{t'}, u(t')) \leq -\alpha' V(x(t')), \quad (5)$$

which means that (3) cannot be violated due to $\alpha' \geq -\frac{\ln\gamma'}{d}$ and the estimate (3) has to be also preserved at the instant t'. If $t' > T_{x_0,u} + d$, then

$$V(x(t)) \leq \max\{\gamma_u(||u||_\infty), r\},$$

$$\exp\left(\frac{\ln\gamma'}{d}t\right) \max_{\theta\in[-d,0]} V(x_0(\theta)) \leq \max\{\gamma_u(||u||_\infty), r\}$$

for all $t \in [t' - d, t')$, these facts imply that the relation (4) holds, then again (5) is valid and (3) cannot be violated at t'. Finally, let $T_{x_0,u} \leq t' \leq T_{x_0,u} + d$, then

$$\exp\left(\frac{\ln\gamma'}{d}t'\right) \max_{\theta\in[-d,0]} V(x_0(\theta)) < \max\{\gamma_u(||u||_\infty), r\} \\ < V(x(t'))$$

and $V(x(t')) > \gamma' \max_{\theta\in[-d,0]} V(x(t' + \theta))$, which again leads to (4) and impossibility of violation of (3) at the instant t' due to (5). The same arguments can be applied further for all $t \geq t'$, and the required upper estimate on the solutions of (1) follows.

Let $\alpha_1(|x|) = a_1|x|^b$ and $\alpha_2(|x|) = a_2|x|^b$ for some positive constants a_1, a_2, b in Lemma 4, then the exponential ISS of the system (1) is covered. The only additional restriction is imposed in Lemma 4 with respect to Theorem 2, which allows us to evaluate an exponential convergence rate, is $\alpha' \geq -\frac{\ln\gamma'}{\tau}$ (the chosen form of functions $\alpha(s)$ and $\gamma_x(s)$ is common in Gu et al. (2003)). Such a modification is not restrictive. Similar Lyapunov-Razumikhin conditions for exponential stability of nonlinear time-delay systems (Myshkis, 1995) and stochastic nonlinear time-delay systems (Mao, 1996) were established, respectively.

Remark 5. It is worth to highlight an important aspect of the Lyapunov-Razumikhin approach. Consider the formulation of Lemma 4. The rate of convergence, which is achievable in the system if the condition

$$\max\{\gamma' \max_{\theta\in[-d,0]} V(\varphi(\theta)), \gamma_u(|u|), r\} < V(\varphi(0))$$

is satisfied, is straightforward to derive under the Lyapunov-Razumikhin framework. It is given by the formulation of the method, since in such a case

$$D^+V(\varphi(0)) f(\varphi, u) \leq -\alpha' V(\varphi(0)).$$

However, it is also necessary to substantiate that a similar rate of convergence is valid when this condition is not satisfied, *i.e.* when $t \in \mathcal{T}_1 \cup \mathcal{T}_2$, and

$$\max\{\gamma' \max_{\theta\in[-d,0]} V(\varphi(\theta)), \gamma_u(|u|), r\} \geq V(\varphi(0)).$$

This constitutes the core of the proof of Lemma 4.

For the system (1) with $u(t) = 0$ and $r = 0$, the exponential practical ISS presented in Lemma 4 is reduced to the exponential stability:

Corollary 6. If there exists a locally Lipschitz continuous Lyapunov-Razumikhin function $V : \Re^n \to \Re_+$ such that

(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \Re^n$:
$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|);$$
(ii) for some $\gamma' > 1$ and $\alpha' \ge \frac{\ln \gamma'}{d}$:
$$\max_{\theta\in[-d,0]} V(\varphi(\theta)) < \gamma' V(\varphi(0)) \Rightarrow$$
$$D^+V(\varphi(0)) f(\varphi, 0) \le -\alpha' V(\varphi(0))$$
for all $\varphi \in C_{[-d,0]}$, then for all $x_0 \in C_{[-d,0]}$ and all $t \ge 0$ the solutions of the system (1) with $u(t) = 0$ admit the estimate: $|x(t, x_0, 0)| \le \alpha_1^{-1}\left(\exp\left(-\frac{\ln\gamma'}{d}t\right)\alpha_2(\|x_0\|)\right)$.

In this work we will also be interested in the unstable case. We have to restrict ourselves by studying only forward complete systems (*i.e.*, the systems as in (1) that for any $x_0 \in C_{[-d,0]}$ and any $u \in \mathcal{L}_\infty^m$ admit solutions $x(t, x_0, u)$ defined for all $t \ge 0$). A Razumikhin-type condition for forward completeness (1) is given below:

Lemma 7. If there exists a locally Lipschitz continuous Lyapunov-Razumikhin function $V : \Re^n \to \Re_+$ such that

(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \Re^n$:
$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|);$$
(ii) for some $\alpha_{00} > 0$, $\gamma_u \in \mathcal{K}$, and $r \ge 0$:
$$\max\{\max_{\theta\in[-d,0]} V(\varphi(\theta)), \gamma_u(|u|), r\} < V(\varphi(0))$$
$$\Rightarrow D^+V(\varphi(0)) f(\varphi, u) \le \alpha_{00}V(\varphi(0)),$$
for all $\varphi \in C_{[-d,0]}$ and all $u \in \Re^m$, then solutions of the system (1) for all $x_0 \in C_{[-d,0]}$, all $u \in \mathcal{L}_\infty^m$ and all $t \ge 0$ admit an upper estimate:
$|x(t, x_0, u)| \le \alpha_1^{-1}\circ\max\{\exp(\alpha_{00}t)\alpha_2(\|x_0\|), \gamma_u(||u||_\infty), r\}$.

In Lemma 4, the Lyapunov-Razumikhin approach is presented to check ISS property for the system (1). Lemma 7 is a counterpart of Lemma 4, which provides an upper estimate on increasing rate of the states.

Remark 8. Note that if the condition (ii) of Lemma 7 is replaced with
$$\max\{\max_{\theta\in[-d,0]} V(\varphi(\theta)), \gamma_u(|u|), r\} > V(\varphi(0)) \Rightarrow D^+V(\varphi(0)) f(\varphi, u) \ge \alpha_{00}V(\varphi(0)), \tag{6}$$
then we obtain a Razumikhin-type condition on instability of (1) with at least exponential growth of $V(x(t))$.

4. MAIN RESULTS

Consider the following switched time-delay systems:
$$dx(t)/dt = f_{\sigma(t)}(x_t, u(t)),\ t \ge 0, \tag{7}$$
where $x(t) \in \Re^n$ and $x_t \in C_{[-d,0]}$ is the state, $x_t(s) = x(t+s)$ for $s \in [-d, 0]$ with a delay $d > 0$; $x_0 \in C_{[-d,0]}$ is the initial condition; $u(t) \in \Re^m$ is the input, $u \in \mathcal{L}_\infty^m$; $\sigma(t)$ is a right-continuous switching signal taking values in a finite set $S = \{1, 2, \dots, M\}$ for some integer $M > 1$; for each $i \in S$, $f_i : C_{[-d,0]} \times \Re^m \to \Re^n$ is a nonlinear function that guarantees the unique existence of solutions of (7) in forward time; $f_i(0, 0) = 0$.

Our objective is to formulate Razumikhin-type conditions of ISS in (7) for a class of switching signals admitting an average dwell time property:

Theorem 9. If there exist locally Lipschitz continuous Lyapunov-Razumikhin functions $V_i : \Re^n \to \Re_+$ for all $i \in S$ such that

(i) for some $\alpha_{1i}, \alpha_{2i} \in \mathcal{K}_\infty$ and all $x \in \Re^n$:
$$\alpha_{1i}(|x|) \le V_i(x) \le \alpha_{2i}(|x|);$$
(ii) for some $\gamma_1 \in \mathcal{K}$, $\gamma_0 < 1, \alpha_0 > -\frac{\ln\gamma_0}{d}$, all $\varphi \in C_{[-d,0]}$ and all $u \in \Re^m$:
$$\max\{\gamma_0 \max_{\theta\in[-d,0]} V_i(\varphi(\theta)), \gamma_1(|u|)\} \le V_i(\varphi(0))$$
$$\Rightarrow D^+V_i(\varphi(0))f_i(\varphi, u) \le -\alpha_0 V_i(\varphi(0));$$
(iii) for some $\lambda \ge \gamma_0^{-1}$, all $j \ne i \in S$ and all $x \in \Re^n$:
$$V_i(x) \le \lambda V_j(x), \tag{8}$$
then the system (7) is exponentially ISS provided that the switching signal σ yields a dwell time bigger than d and the average dwell time $\tau > \left(1 - \frac{\ln\lambda}{\ln\gamma_0}\right)d$.

Proof. Consider a time sequence $0 = t_0 \le t_1 \le \dots \le t_m \le t < t_{m+1}$, where t_i, $i = 1, \dots, m+1$ is the switching time instant, with $\sigma(t_i) \in S$ being active in the interval $[t_i, t_{i+1})$. Since all conditions of Lemma 4 are satisfied for $r = 0$, then for all $t \in [t_m, t_{m+1})$:
$$V_{\sigma(t_m)}(x(t)) \le \max\left(e^{\frac{\ln\gamma_0}{d}(t-t_m)} \max_{\theta\in[-d,0]} V_{\sigma(t_m)}(x(t_m+\theta)), \gamma_1(||u||_\infty)\right).$$
Taking into account the relation (8) between $V_{\sigma(t_m)}$ and $V_{\sigma(t_{m-1})}$ we obtain:
$$V_{\sigma(t_m)}(x(t)) \le \max\Big\{\lambda e^{\frac{\ln\gamma_0}{d}(t-t_m)} \times \max_{\theta\in[-d,0]} V_{\sigma(t_{m-1})}(x(t_m+\theta)), \gamma_1(||u||_\infty)\Big\}.$$
In addition,
$$\max_{s\in[-d,0]} V_{\sigma(t_m)}(x(t+s)) \le \max\Big\{\lambda e^{\frac{\ln\gamma_0}{d}\max\{0, t-d-t_m\}} \times \max_{\theta\in[-d,0]} V_{\sigma(t_{m-1})}(x(t_m+\theta)), \gamma_1(||u||_\infty)\Big\}.$$
Iterative repeating these steps gives (9). By definition, $m = N_\sigma(t_0, t) \le N_0 + \frac{t-t_0}{\tau}$, note also that
$$\lambda e^{\frac{\ln\gamma_0}{d}(t-t_m)} = \lambda^{N(t_m,t)} e^{\frac{\ln\gamma_0}{d}(t-t_m)}, \dots,$$
$$\lambda^{m-1} e^{\frac{\ln\gamma_0}{d}(t-t_m+\sum_{i=m}^{2}\max\{0, t_i-d-t_{i-1}\})}$$
$$\le \lambda^{N(t_1,t)} e^{\frac{\ln\gamma_0}{d}(t-t_m+\sum_{i=m}^{3}\max\{0, t_i-d-t_{i-1}\})},$$
and if $t_i > t_{i-1} + d$ (the dwell time is bigger than d), then
$$\lambda^m e^{\frac{\ln\gamma_0}{d}(t-t_m+\sum_{i=m}^{1}\max\{0, t_i-d-t_{i-1}\})}$$
$$= \lambda^{N_\sigma(t_0,t)} e^{\frac{\ln\gamma_0}{d}(t-t_0-N_\sigma(t_0,t)d)}$$
and
$$\lambda e^{\frac{\ln\gamma_0}{d}(t-t_m)} = \lambda^{N(t_m,t)} e^{\frac{\ln\gamma_0}{d}(t-t_m-(N(t_m,t)-1)d)}, \dots,$$
$$\lambda^{m-1} e^{\frac{\ln\gamma_0}{d}(t-t_m+\sum_{i=m}^{3}\max\{0, t_i-d-t_{i-1}\})}$$
$$= \lambda^{N(t_1,t)} e^{\frac{\ln\gamma_0}{d}(t-t_1-(N_\sigma(t_1,t)-1)d)}.$$

$$
\begin{aligned}
V_{\sigma(t_m)}(x(t)) &\le \max\Big\{\lambda e^{\frac{\ln\gamma_0}{d}(t-t_m)}\max_{\theta\in[-d,0]}V_{\sigma(t_{m-1})}(x(t_m+\theta)),\gamma_1(||u||_\infty)\Big\}\\
&\le \max\Big\{\lambda^3 e^{\frac{\ln\gamma_0}{d}(t-t_m+\sum_{i=m}^{m-1}\max\{0,t_i-d-t_{i-1}\})}\max_{\theta\in[-d,0]}V_{\sigma(t_{m-3})}(x(t_{m-2}+\theta)),\\
&\quad \max\{1,\lambda e^{\frac{\ln\gamma_0}{d}(t-t_m)},\lambda^2 e^{\frac{\ln\gamma_0}{d}(t-t_m+\max\{0,t_m-d-t_{m-1}\})}\}\gamma_1(||u||_\infty)\Big\}\\
&\le \cdots\\
&\le \max\Big\{\lambda^m e^{\frac{\ln\gamma_0}{d}(t-t_m+\sum_{i=m}^{2}\max\{0,t_i-d-t_{i-1}\})}\max_{\theta\in[-d,0]}V_{\sigma(t_0)}(x(t_1+\theta)),\\
&\quad \max\{1,\lambda e^{\frac{\ln\gamma_0}{d}(t-t_m)},\dots,\lambda^{m-1}e^{\frac{\ln\gamma_0}{d}(t-t_m+\sum_{i=m}^{3}\max\{0,t_i-d-t_{i-1}\})}\}\gamma_1(||u||_\infty)\Big\}\\
&\le \max\Big\{\lambda^m e^{\frac{\ln\gamma_0}{d}(t-t_m+\sum_{i=m}^{1}\max\{0,t_i-d-t_{i-1}\})}\max_{\theta\in[-d,0]}V_{\sigma(t_0)}(x(t_0+\theta)),\\
&\quad \max\{1,\lambda e^{\frac{\ln\gamma_0}{d}(t-t_m)},\dots,\lambda^{m-1}e^{\frac{\ln\gamma_0}{d}(t-t_m+\sum_{i=m}^{2}\max\{0,t_i-d-t_{i-1}\})}\}\gamma_1(||u||_\infty)\Big\}.
\end{aligned} \tag{9}
$$

Due to restrictions imposed on the dwell time there is $\varrho > 0$ such that $\frac{\ln\gamma_0}{d}(1-\frac{d}{\tau})+\frac{\ln\lambda}{\tau}\le -\varrho$, then for all $i=1,\dots,m$:

$$
\begin{aligned}
&\lambda^{N(t_i,t)}e^{\frac{\ln\gamma_0}{d}(t-t_i-(N(t_i,t)-1)d)}\\
&\le \lambda^{N_0}e^{-\ln\gamma_0(N_0-1)}e^{-\varrho(t-t_i)}\le \lambda^{N_0}e^{-\ln\gamma_0(N_0-1)},
\end{aligned}
$$

hence,

$$
\begin{aligned}
&V_{\sigma(t_m)}(x(t))\le \lambda^{N_0}e^{-\ln\gamma_0(N_0-1)}\times\\
&\max\Big\{e^{-\varrho(t-t_0)}\max_{\theta\in[-d,0]}V_{\sigma(t_0)}(x(t_0+\theta)),\gamma_1(||u||_\infty)\Big\}.
\end{aligned}
$$

Noting that $\alpha_1(|x|)\le V_{\sigma(t_m)}(x)$ and $\max_{\theta\in[-d,0]}V_{\sigma(t_0)}(x(t_0+\theta))\le\alpha_2(\|\varphi_0\|)$, where $\alpha_1(s)=\min_{1\le i\le M}\{\alpha_{1i}(s)\}$ and $\alpha_2(s)=\max_{1\le i\le M}\{\alpha_{2i}(s)\}$, then

$$
\begin{aligned}
&|x(t)|\le\alpha_1^{-1}\Big(\lambda^{N_0}e^{-\ln\gamma_0(N_0-1)}\times\\
&\max\Big\{e^{-\varrho(t-t_0)}\alpha_2(\|x_0\|),\gamma_1(||u||_\infty)\Big\}\Big),
\end{aligned}
$$

which implies the desired ISS property for (7).

Remark 10. In Jiang et al. (2013, 2016), the asymptotic stability and ISS of switched time-delay systems were addressed by virtue of the average dwell-time switching method and some additional small-gain conditions. In Theorem 9, an additional exponential-kind estimate on the rate of convergence of solutions of the switched system is calculated.

Assume that $u(t)=0$, then Theorem 9 reduces to the exponential stability of the system (7):

Corollary 11. If there exist locally Lipschitz continuous Lyapunov-Razumikhin functions $V_i:\Re^n\to\Re_+$ for all $i\in S$ such that

(i) for some $\alpha_{1i},\alpha_{2i}\in\mathcal{K}_\infty$ and all $x\in\Re^n$:

$$\alpha_{1i}(|x|)\le V_i(x)\le\alpha_{2i}(|x|);$$

(ii) for some $\gamma_0>1,\alpha_0>\frac{\ln\gamma_0}{d}$, all $\varphi\in C_{[-d,0]}$:

$$
\begin{aligned}
&\max_{\theta\in[-d,0]}V_i(\varphi(\theta))\le\gamma_0V_i(\varphi(0))\\
&\Rightarrow D^+V_i(\varphi(0))f_i(\varphi,0)\le-\alpha_0V_i(\varphi(0));
\end{aligned}
$$

(iii) (8) holds for some $\lambda\ge\gamma_0$, all $j\ne i\in S$ and all $x\in\Re^n$,

then solutions of the system (7) with $u(t)=0$ admit an estimate $|x(t)|\le\alpha_1^{-1}\left(\lambda^{N_0}e^{\ln\gamma_0(N_0-1)}e^{-\varrho(t-t_0)}\alpha_2(\|x_0\|)\right)$ provided that the switching signal σ yields a dwell time bigger than d and the average dwell time with $\tau>\big(1+\frac{\ln\lambda}{\ln\gamma_0}\big)d$.

Remark 12. In Theorem 9 and Corollary 12, it is required that all subsystems are ISS and exponentially stable, respectively. In Lemma 7, an upper bound estimate of the state of the system is provided. Combining Theorem 9, Corollary 12, and Lemma 7, the ISS and exponentially stability of the system (7) with part stable subsystems and part unstable (or, bounded) subsystems can be obtained.

5. ILLUSTRATIVE EXAMPLE

This section provides one example to illustrate the obtained results. Consider the system (7) with two stable subsystems:

$$
\begin{aligned}
\dot{x}_1(t)&=-3x_1(t)+\int_{-0.5}^{0}x_1(t+s)ds,\\
\dot{x}_2(t)&=2x_1(t)-3x_2(t)+\int_{-0.5}^{0}x_2(t+s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{x}_1(t)&=-4x_1(t)+\int_{-0.5}^{0}x_1(t+s)ds,\\
\dot{x}_2(t)&=x_1(t)-4x_2(t)+\int_{-0.5}^{0}x_2(t+s)ds.
\end{aligned}
$$

Choose $V_1(x(t))=\frac{1}{2}(x_1^2(t)+x_2^2(t))$ and $V_2(x(t))=\frac{1}{4}(x_1^2(t)+x_2^2(t))$, then $\lambda=2$ and $\tau\ge1$. We have

$$\dot{V}_1(x(t))\le-\frac{7}{4}x_1^2(t)-\frac{7}{4}x_2^2(t)+\frac{1}{4}x_1^2(t+s)+\frac{1}{4}x_2^2(t+s),$$

and

$$\dot{V}_2(x(t))\le-\frac{13}{8}x_1^2(t)-\frac{13}{8}x_2^2(t)+\frac{1}{8}x_1^2(t+s)+\frac{1}{8}x_2^2(t+s).$$

Supposing that $V_i(x(t+s))\le 2V_i(x(t))$ for $\gamma_0=2$, that is, $x_1^2(t+s)+x_2^2(t+s)\le2(x_1^2(t)+x_2^2(t))$. Choose $\alpha_0=\frac{3}{2}\ge\frac{\ln\gamma_0}{d}=1.3863$. It follows that

$$
\begin{aligned}
\dot{V}_1(x(t))&\le-\frac{5}{4}(x_1^2(t)+x_2^2(t))\le-\frac{3}{2}V_1(x(t)),\\
\dot{V}_2(x(t))&\le-\frac{11}{8}(x_1^2(t)+x_2^2(t))\le-\frac{3}{2}V_2(x(t)).
\end{aligned}
$$

From the above derivation, the conditions of Theorem 9 hold. Figs. 1 and 2 show the simulations of the states of the subsystems without switching and the state of the system with average dwell time switching.

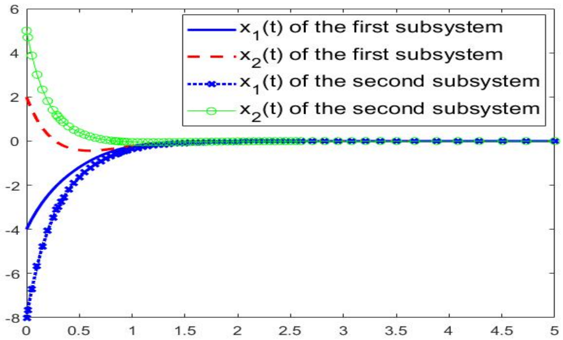

Fig. 1. The simulations of the states of the subsystems

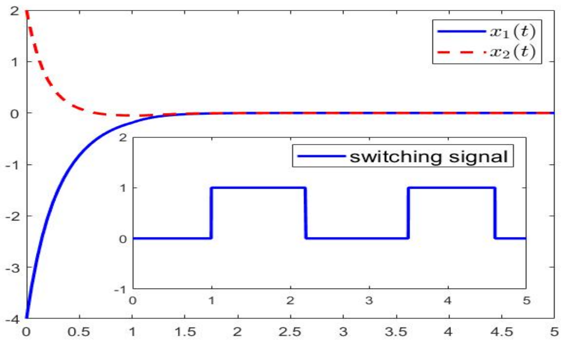

Fig. 2. The simulations of the states under the average dwell time switching

6. CONCLUSION

An estimate approach is presented for the decay rate of ISS nonlinear time-delay systems by virtue of Lyapunov-Razumikhin method. The exponential ISS of switched nonlinear systems is explored for average dwell-time switching laws without additional restrictions on used Lyapunov functions. Further relaxation of the requirements imposed on the switched signal can be considered in future research.

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