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# An Alternative Method for Optimal Consensus Protocol Design for Scalar Single-integrators using Krotov Conditions

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**Abstract:** This article proposes a novel alternative approach for optimal consensus protocol design for scalar single-integrator multi-agent systems based on the Krotov methodology. The problem under consideration generally turns out to be non-convex due to the desired diffusive nature (i.e. using only relative information from the neighboring agents) of the control input. This work employs the Krotov framework which transforms the optimal control problem into another equivalent optimization problem via the selection of so-called Krotov function whose selection is *ad-hoc*. The formulation of this equivalent optimization problem provides sufficient conditions for the existence of globally optimal control law(s) and it is generally solved using iterative methods because of non-convex characteristics. In this work, these conditions are used to solve the optimal consensus protocol design problem for the single-integrator multi-agent systems by *choosing* the Krotov function in such a way the equivalent optimization problem can be solved non-iteratively and at the same time, the obtained optimal control law has the desired structure (as necessitated by the communication topology). The proposed method is demonstrated by a numerical example.

*Keywords:* Multi-agent Systems, consensus protocol, optimal control, Convex optimization, Krotov conditions,

## 1. INTRODUCTION

Recently, optimal control design for multi-agent systems has gained utmost popularity among control theorists and practitioners. This is due to the fact that the study of multi-agent systems plays a vital role in almost every application area (Mesbahi and Egerstedt (2010)). Furthermore, the consensus protocol design of multi-agent systems is one of the benchmark problems which serves as the background for developing control algorithms for more relevant involved problems like formation control, synchronization, etc. The consensus problem, also called as the *agreement* problem, requires the agents to arrive at a common agreement (value) via the exchange of *local information* with the neighbors (Mesbahi and Egerstedt (2010)). This problem is relevant in a number of disciplines including opinion dynamics, networked sensor design, nano-systems, social networking, energy networks, alignment of heading angles in UAVs (unmanned aerial vehicles), etc. It is well known that the convergence properties of a consensus protocol are highly dependent on the interaction topology of the agents. A variety of results have been derived which explore and exploit this dependency and interplay of the consensus characteristics with the network topology (Mesbahi and Egerstedt (2010); Ren et al. (2005); Chen et al. (2019)).

Optimal consensus protocol design is a more involved problem (than consensus problem) where an objective function also comes into the picture and, for the general case, it is unclear if the optimal solution to this problem exists (Jiao et al. (2019b)). This problem has been widely tackled using the *standard* tools of optimal control theory which include Hamilton-Jacobi-Bellman (HJB) based techniques, *Riccati equation-like* approaches, linear quadratic regulation (LQR) formalism based approaches, etc. See (Sun et al. (2017); Cao and Ren (2010); Zhang et al. (2015); Jovanovic (2005)) and the references therein for the details of the available results. Alongside, the technique of local optimal (with respect to a single agent) control design approaches has also been employed in the literature (Lewis et al. (2013); Jiao et al. (2019a)). The control law designed using this technique turns out to be sub-optimal for the original problem. Note that in (Cao and Ren (2010)) a solution of the considered problem has been provided based on the LQR framework and solution of the appropriate Riccati equation. In this work, we present an alternative method to reach the solution based on the Krotov framework. The following points serve as motivation for choosing this framework:

1. This framework provides the most general sufficient conditions for global optimality (Salmin (2017)).
2. This framework is barely explored for solving the multi-agent systems related problems, as per the best knowledge of the authors.
3. The exhaustive solution of the consensus design problem for single-integrators may serve as a background for further exploration of the framework specifically for multi-agent problems; for instance, the work can

 

serve as a guideline to make the selection of Krotov function for more involved consensus design problems.

To tackle the problem of obtaining a solution to the optimal consensus design problem, this work utilizes the global optimal control design framework introduced by V.F. Krotov in (Krotov (1995)). This framework remains highly unexplored in literature, to the best of author's knowledge. It is based upon the so-called extension principle which translates a constrained optimization problem into another equivalent optimization problem in which the constraints of the original problem are *excluded* but the formulation of the latter problem is done in a way that its solution still satisfies the excluded constraints of the original problem (Gurman et al. (2016)). The key idea is that the latter optimization problem could be easier to solve than the original optimization problem. Nevertheless, how to make the selection/formulation of the equivalent optimization problem remains an open question. Krotov proposed the equivalent optimization problem for an optimal control problem by utilizing the extension principle and then the sufficient conditions, known as Krotov conditions, for the existence of the solution to the original optimal control problem were derived (Krotov (1962, 1988)).

Consider the general form of an optimal control problem given as:

***GOCP (Generic Optimal Control Problem).*** Compute an optimal control law $\mathbf{u}^*(t)$ which minimizes the performance index/cost functional:

$$J(\mathbf{x}(t), \mathbf{u}(t)) = l_f(\mathbf{x}(t_f)) + \int_{t_0}^{t_f} l(\mathbf{x}(t), \mathbf{u}(t), t)\ dt \qquad (1)$$

subject to the system dynamics $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t)$ with $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n;\ t \in [t_0, t_f]$ to give the desired optimal trajectory $\mathbf{x}^*(t)$. Here, $l(\mathbf{x}(t), \mathbf{u}(t), t)$ is the running cost, $l_f(\mathbf{x}(t_f))$ is the terminal cost, $\mathbf{x}(t) \in \mathbb{X} \subset \mathbb{R}^n$ is the state vector and $\mathbf{u}(t) \in \mathbb{U} \subset \mathbb{R}^m$ is the control input vector to be designed. Also, $l_f : \mathbb{R}^n \to \mathbb{R}$ and $l : \mathbb{R}^n \times \mathbb{R}^m \times [t_0, t_f] \to \mathbb{R}$ are continuous. [1]

For this problem the Krotov conditions are given as per the following theorem (for details, refer Krotov (1995)): *Theorem: (Krotov Conditions).* For GOCP, let $q(\mathbf{x}(t), t)$ be a piecewise continuously differentiable function. Then, there is an equivalent representation of (1) given as:

$$\begin{aligned} J_{eq}(\mathbf{x}(t), \mathbf{u}(t)) =& s_f(\mathbf{x}(t_f)) + q(\mathbf{x}_0, t_0) \\ &+ \int_{t_0}^{t_f} s(\mathbf{x}(t), \mathbf{u}(t), t) dt \end{aligned}$$

where

$$\begin{aligned} s(\mathbf{x}(t), \mathbf{u}(t), t) &\triangleq \frac{\partial q}{\partial t} + l(\mathbf{x}(t), \mathbf{u}(t), t) \\ &+ \frac{\partial q}{\partial \mathbf{x}} f(\mathbf{x}(t), \mathbf{u}(t), t) \\ s_f(\mathbf{x}(t_f)) &\triangleq l_f(\mathbf{x}(t_f)) - q(\mathbf{x}(t_f), t_f) \end{aligned}$$

If $[\mathbf{x}^*(t), \mathbf{u}^*(t)]$ is an *admissible process* (i.e. a process which satisfies the dynamical equation $\dot{\mathbf{x}} = f(\mathbf{x}(t), \mathbf{u}(t), t)$ and the state and input constraints) such that

$$s(\mathbf{x}^*(t), \mathbf{u}^*(t), t) = \min_{\mathbf{x}\in\mathbb{X}, \mathbf{u}\in\mathbb{U}} s(\mathbf{x}(t), \mathbf{u}(t), t), \forall t \in [t_0, t_f) \qquad (2)$$

and

$$s_f(\mathbf{x}^*(t_f)) = \min_{\mathbf{x}\in\mathbb{X}_f} s_f(\mathbf{x}) \qquad (3)$$

then $[\mathbf{x}^*(t), \mathbf{u}^*(t)]$ is an optimal process. Here, $\mathbb{X}_f$ is the terminal set for admissible $\mathbf{x}(t)$ of $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t)$ i.e. if $\mathbf{x}(t)$ is admissible then $\mathbf{x}(t_f) \in \mathbb{X}_f$.

*Proof:* See (Krotov, 1995, Section 2.3) for the proof.

The function $q$ is called the Krotov function. Its selection is non-trivial and depends upon the optimal control problem under consideration. How to make the selection of this function for a generic optimal control problem still remains an open research problem. Furthermore, the previous work which has explored these conditions generally employs the iterative methods, one of them being the Krotov method, to compute the globally optimal control law(s). The reasons for choosing iterative methods are - (1) most contemporary optimal control problems are characterized by the lack of existence of globally optimal solutions (Krotov (1995)) and hence iterative methods are used to find the so-called *global bounds* and (2) the equivalent optimization problem in *Theorem: (Krotov Conditions)* turns out to be non-convex. To avoid the iterative computations, a novel technique based on the convexity imposition upon the optimization problems (2) and (3) has been reported for the standard linear quadratic problem in (Kumar and Jain (2019)). In this work, similar ideas are extended to optimal consensus protocol design. To summarize, this work proposes Krotov function which ensures that:

(1) the equivalent optimization problem (2)- (3) can be solved non-iteratively;
(2) the control law obtained upon solving the equivalent optimization problems is diffusive in nature.

## 2. PRELIMINARIES AND PROBLEM FORMULATION

Consider $N$ single-integrator agents interacting on a complete graph communication topology. Thus, each agent can communicate with every other agent. Furthermore, communication links are considered to be bi-directional. The dynamics of each agent is given as:

$$\dot{x}_i = u_i \text{ with } x_i,\, u_i \in \mathbb{R}$$

The optimal consensus protocol design problem for this set-up is as given below:

*Problem 1.* Design a diffusive control law i.e. a control law of the form

$$u_i = \sum_{j=1, j\neq i}^{N} a_{ij}\,(x_j - x_i)\,;\, i \in [1, 2, 3 \ldots N]$$

where $a_{ij}$'s are the design gain variables such that:

(1) the cost functional

$$J = \int_0^\infty \left[ \sum_{i,j=1; i\neq j}^{N} w_{ij}\, [x_i(t) - x_j(t)]^2 + \sum_{i=1}^{N} u_i^2 \right] dt$$

where $w_{ij}$'s are the weighting coefficients, is minimized;

[1] *Throughout this article the small alphabets represent scalar quantities, the small bold alphabets represent vector quantities and the capital alphabets represent matrices. The quantities with an "asterisk" represent the optimal counterparts.*

(2) consensus is achieved i.e. $e_{ij} \triangleq |x_i - x_j| \to 0$ as $t \to \infty$ $\forall\ i, j \in [1, 2, 3 \ldots N]$.

Since, *Problem* 1 is an infinite horizon optimal control problem, the terminal cost and hence the function $s_f$ in *Theorem :(Krotov Conditions)* shall not be present in the optimization problem obtained upon employing Krotov conditions. Furthermore, the control input is now constrained to be of the form

$$u_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij} \left(x_j - x_i\right); \, i \in [1, 2, 3 \ldots N]$$

Thus, only the control laws of this form are admissible. Furthermore, it is necessary to ensure that the consensus is achieved i.e. the error $e_{ij} = |x_i - x_j| \to 0$ as $t \to \infty$ $\forall\ i, j \in [1, 2, 3, \ldots N]$. Let $\mathbb{U}_a$ and $\mathbb{X}_a$ be the sets containing all the $\mathbf{x}$'s and $\mathbf{u}$'s for which both of these conditions are met. Finally, note that there may more than one control law which has the desired diffusive structure. In this case, the control law which results in the consensus is chosen. This point shall be made more clear when the proposed methodology is demonstrated by a numerical example. Employing *Theorem :(Krotov Conditions)*, the equivalent optimization problem for *Problem* 1 is as given below.

*Equivalent Problem 1.* Find an admissible $[\mathbf{x}^*, \mathbf{u}^*]$ pair such that

$$s(\mathbf{x}^*(t), \mathbf{u}^*(t), t) = \min_{\mathbf{x} \in \mathbb{X}_a, \mathbf{u} \in \mathbb{U}_a} s\left(\mathbf{x}(t), \mathbf{u}(t), t\right)$$

where

$$s = \sum_{k=1}^{N} \frac{\partial q}{\partial x_k} u_k + \left[ \sum_{i,j=1; i \neq j}^{N} w_{ij} \left[x_i(t) - x_j(t)\right]^2 + \sum_{i=1}^{N} u_i^2 \right]$$

and $q$ is the Krotov function, is minimized.

Recall that the objective is to make the selection of the Krotov function $q$ such that upon solving *Problem* 1, the control law of the desired structure is obtained. The selection of this function is done in the next section which helps in computing the desired control law.

## 3. MAIN RESULTS

In this section, the sufficient conditions for optimal consensus protocol design for *Problem* 1 are derived via suitable selection of the Krotov function $q$.

*Proposition 1.* (Optimal Consensus Protocol Design). For *Equivalent Problem* 1, let the Krotov function $q$ be chosen as :

$$q = \sum_{i=1}^{N} \left[ \frac{x_i^2}{2} \sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij} - x_i \sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij} x_j \right] \tag{4}$$

with $a_{ij} = a_{ji}$.

Then, if the control law $\mathbf{u} = (u_1 u_2 \ldots u_N)^T$:

$$u_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij} \left(x_j - x_i\right) \;\;, i \in [1, 2, 3 \ldots, N]$$

results in consensus then it is the globally optimal control law. Here, $a_{ij}$'s are computed by solving the following simultaneous $\dfrac{N(N+1)}{2}$ nonlinear algebraic equations:

$(i) N$ equations:

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} w_{ij} - 2 \left[ \sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij}^2 + \sum_{\substack{j,k=1 \\ j \neq k \neq i}}^{N} a_{ij} a_{ik} \right] = 0; \tag{5}$$

and $(ii)$ $\dfrac{N(N-1)}{2}$ equations:

$$4a_{ij}^2 - 2w_{ij} + 2a_{ij} \left( \sum_{\substack{k=1 \\ k \neq i}}^{N} a_{ik} + \sum_{\substack{k=1 \\ k \neq j}}^{N} a_{jk} \right) - 2 \left( \sum_{\substack{k=1 \\ k \neq i \neq j}}^{N} a_{ik} a_{jk} \right) = 0 \; ; \, i \neq j \tag{6}$$

with $i, j, k \in \{1, 2, \ldots, N\}$.

*Proof:* The function $2s$ for *Equivalent Problem* 1 can be rearranged as to give (9). Recall that the function $2s$ is to be minimized. Also, the first term in (9) attains the minimum value (zero) when

$$u_i = -\frac{\partial q}{\partial x_i} \;\; \forall i \;\in [1, 2, 3 \ldots, N] \tag{7}$$

Upon substituting $q$ (as in (4)) in (7), we obtain:

$$u_i = -\sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij} \left(x_j - x_i\right); i \in [1, 2, 3 \ldots, N] \tag{8}$$

which is the control law of the desired structure.

Furthermore, the remaining term in $2s$ which needs to be analyzed is given as (denoted as $\beta$).

$$\beta = -\sum_{i=1}^{N} \left( \frac{\partial q}{\partial x_i} \right)^2 + 2 \sum_{i,j=1; i \neq j}^{N} w_{ij} \left(x_i(t) - x_j(t)\right)^2$$

The substitution of $q$, rearrangement of the terms, and simplification give (10). Clearly, in order to minimize $s$, it suffices to minimize $\beta$. Furthermore, to have $\beta = 0$, it is sufficient that (5) and (6) hold. When $\beta = 0$, $s$ is independent of $\mathbf{x}$ and hence the optimization problem is solved. Hence, the control input (8) is a potential candidate for the optimal control. Finally, if this control law is admissible then it is optimal. This completes the proof. □

The admissibility of the control law (8) is to be ensured to conclude that it is the desired optimal control law for the problem under consideration. In the current work, the admissibility is checked by ensuring that the eigenvalues of the closed-loop error matrix are in the left half-plane. This point shall be more clear in the next section.

$$
\begin{aligned}
2s &= 2\left[\sum_{i=1}^{N}\frac{\partial q}{\partial x_i}u_i + \left(\sum_{i,j=1;i\neq j}^{N} w_{ij}\left(x_i(t)-x_j(t)\right)^2 + \sum_{i=1}^{N}u_i^2\right)\right] \\
&= -\sum_{i=1}^{N}\left(\frac{\partial q}{\partial x_i}\right)^2 + \sum_{i=1}^{N}\left(\frac{\partial q}{\partial x_i}\right)^2 + 2\left[\sum_{i=1}^{N}\frac{\partial q}{\partial x_i}u_i + \left(\sum_{i,j=1;i\neq j}^{N} w_{ij}\left(x_i(t)-x_j(t)\right)^2 + \sum_{i=1}^{N}u_i^2\right)\right] \\
&= \left(\sum_{i=1}^{N}\left(u_i + \frac{\partial q}{\partial x_i}\right)^2\right) - \sum_{i=1}^{N}\left(\frac{\partial q}{\partial x_i}\right)^2 + 2\sum_{i,j=1;i\neq j}^{N} w_{ij}\left(x_i(t)-x_j(t)\right)^2 \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
\beta =& \sum_{i=1}^{N}x_i^2\left(2\sum_{j=1}^{N}w_{ij} - 2\left[\sum_{j=1}^{N}a_{ij}^2 + \sum_{\substack{j,k=1\\ j\neq k}}^{N}a_{ij}a_{ik}\right]\right) \\
&+ \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}x_ix_j\left[4a_{ij}^2 + 2a_{ij}\left(\sum_{\substack{k=1\\ k\neq i}}^{N}a_{ik} + \sum_{\substack{k=1\\ k\neq j}}^{N}a_{jk}\right) - 2\left(\sum_{\substack{k=1\\ k\neq i,j}}^{N}a_{ik}a_{jk}\right) - 2w_{ij}\right] \qquad (10)
\end{aligned}
$$

## 4. NUMERICAL EXAMPLE

In this section, the proposed technique is used to solve the optimal consensus design problem for a four-agent system (*Example* 1). The communication links between the agents are assumed to be bi-directional.

*Example 1.* (Four agents). Compute a optimal diffusive control law for the single-integrators $\dot{x}_i = u_i,\ i \in [1, 2, 3, 4]$ interacting over the topology in Figure 1 such that:

(1)
$$
J = \frac{1}{2}\int_0^\infty \left[ (x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_4)^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2\right]dt
$$
is minimized;

(2) consensus is achieved i.e. $|x_1 - x_2| \to 0$, $|x_2 - x_3| \to 0$ and $|x_3 - x_4| \to 0$ as $t \to \infty$.

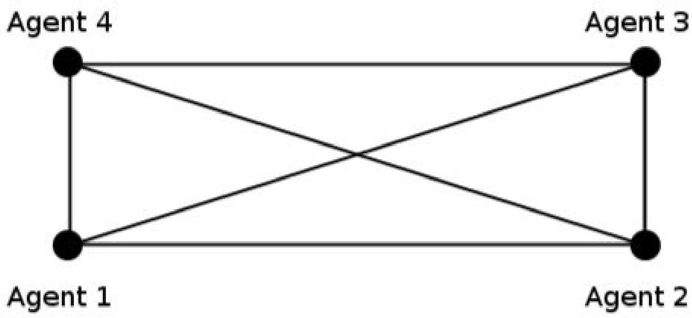


Fig. 1. Topology of agent interaction in *Example* 1

*Solution:*

(1) *Selection of the Krotov function*
Let the Krotov function, in accordance with *Proposition* 1, be chosen as:

$$
\begin{aligned}
q =& \frac{(a_{12}+a_{13}+a_{14})}{2}x_1^2 + \frac{(a_{12}+a_{23}+a_{24})}{2}x_2^2 + \\
& \frac{(a_{23}+a_{13}+a_{34})}{2}x_3^2 + \frac{(a_{14}+a_{24}+a_{34})}{2}x_4^2 \\
& - a_{12}x_1x_2 - a_{13}x_1x_3 - a_{14}x_1x_4 - a_{23}x_2x_3 \\
& - a_{24}x_2x_4 - a_{34}x_3x_4
\end{aligned}
$$

Then, the function $2s$ upon simplification becomes (11). The terms containing $u_1$, $u_2$. $u_3$ and $u_4$ are *convex* and attain their respective minimum values (zero) when

$$
\begin{aligned}
u_1 &= -\frac{\partial q}{\partial x_1} = a_{12}(x_2 - x_1) \\
&\qquad + a_{13}(x_3 - x_1) + a_{14}(x_4 - x_1) \\
u_2 &= -\frac{\partial q}{\partial x_2} = a_{12}(x_1 - x_2) \\
&\qquad + a_{23}(x_3 - x_2) + a_{24}(x_4 - x_2) \\
u_3 &= -\frac{\partial q}{\partial x_3} = a_{13}(x_1 - x_3) \\
&\qquad + a_{23}(x_2 - x_3) + a_{34}(x_4 - x_3) \\
u_4 &= -\frac{\partial q}{\partial x_4} = a_{14}(x_1 - x_4) \\
&\qquad + a_{24}(x_2 - x_4) + a_{34}(x_3 - x_4)
\end{aligned}
$$

Clearly, this control law is of the desired structure where the parameters $a_{12}$, $a_{13}$,$a_{14}$, $a_{23}$, $a_{24}$ and $a_{34}$ are yet to be determined. Furthermore, the remaining terms in $2s$ may be written as the quadratic form: $\mathbf{x}^T S\mathbf{x} (= \beta)$ with $\mathbf{x} = [x_1\ x_2\ x_3\ x_4]^T$ and $S = [\mathbf{s}_1\ \mathbf{s}_2\ \mathbf{s}_3\ \mathbf{s}_4]$ where $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$ and $\mathbf{s}_4$ are given in (12). Moreover, to ensure convexity of $2s$ requires that $S \succeq 0$. Finally, the minimization requires $S = 0$ which yields the ten nonlinear algebraic equations. These equations are then needed to be solved for the parameters $a_{12}$, $a_{13}$, $a_{14}$, $a_{23}$, $a_{24}$ and $a_{34}$. These equations yield **eight** solutions listed in Table 1 which are the potential candidates for the optimal control law as per *Proposition* 1.

$$
\begin{aligned}
2s =& x_1^2[1-\big(2a_{12}^2+2a_{13}^2+2a_{14}^2+2a_{12}a_{13}+2a_{13}a_{14}+2a_{12}a_{14}\big)]+x_2^2[2-\big(2a_{12}^2+2a_{23}^2+2a_{24}^2+2a_{12}a_{23}+2a_{23}a_{24}+2a_{12}a_{24}\big)]\\
&+x_3^2[2-\big(2a_{13}^2+2a_{23}^2+2a_{34}^2+2a_{13}a_{23}+2a_{13}a_{34}+2a_{23}a_{34}\big)]+x_4^2[1-\big(2a_{14}a_{24}+2a_{24}a_{34}+2a_{14}a_{34}+2a_{14}^2+2a_{24}^2+2_{34}^2\big)]\\
&-x_1x_4\left[-4a_{14}^2-2a_{13}a_{14}-2a_{12}a_{14}+2a_{12}a_{24}+2a_{13}a_{34}-2a_{14}a_{24}-2a_{14}a_{34}\right]+x_2x_3\Big[-2-(-4a_{23}^2+2a_{12}a_{13}-2a_{12}a_{23}\\
&-2a_{23}a_{24}-2a_{12}a_{23}-2a_{23}a_{34}+2a_{24}a_{34})\Big]+x_1x_2\Big[-2-\Big(-4a_{12}^2-2a_{12}a_{13}-2a_{12}a_{14}-2a_{12}a_{23}+2a_{14}a_{24}-2a_{12}a_{24}\\
&-2a_{12}a_{24}+2a_{13}a_{23}\Big)\Big]-x_1x_3\left[-4a_{13}^2-2a_{12}a_{13}-2a_{13}a_{14}+2a_{12}a_{23}-2a_{13}a_{23}-2a_{13}a_{34}+2a_{14}a_{34}\right]-x_2x_4\Big[-4a_{24}^2\\
&+2a_{12}a_{14}-2a_{23}a_{24}+2a_{23}a_{34}-2a_{14}a_{24}-2a_{24}a_{34}\Big]+x_3x_4\Big[-2-\Big(-4a_{34}^2+2a_{13}a_{14}+2a_{23}a_{24}-2a_{13}a_{34}\\
&-2a_{23}a_{34}-2a_{24}a_{34}-2a_{14}a_{34}\Big)\Big]+\left(u_1+\frac{\partial q}{\partial x_1}\right)^2+\left(u_2+\frac{\partial q}{\partial x_2}\right)^2+\left(u_3+\frac{\partial q}{\partial x_3}\right)^2+\left(u_4+\frac{\partial q}{\partial x_2}\right)^2
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\mathbf{s}_1 &= \begin{bmatrix} 1-\big(2a_{12}^2+2a_{13}^2+2a_{14}^2+2a_{12}a_{13}+2a_{13}a_{14}+2a_{12}a_{14}\big)\\ 0.5\left(-2-\left(-4a_{12}^2-2a_{12}a_{13}-2a_{12}a_{14}-2a_{12}a_{23}+2a_{14}a_{24}-2a_{12}a_{24}+2a_{13}a_{23}\right)\right)\\ 0.5\left(-\left(-4a_{13}^2-2a_{12}a_{13}-2a_{13}a_{14}+2a_{12}a_{23}-2a_{13}a_{23}-2a_{13}a_{34}+2a_{14}a_{34}\right)\right)\\ 0.5\left(-\left(-4a_{14}^2-2a_{13}a_{14}-2a_{12}a_{14}+2a_{12}a_{24}+2a_{13}a_{34}-2a_{14}a_{24}-2a_{14}a_{34}\right)\right) \end{bmatrix}\\
\mathbf{s}_2 &= \begin{bmatrix} 0.5\left(-2-\left(-4a_{12}^2-2a_{12}a_{13}-2a_{12}a_{14}-2a_{12}a_{23}+2a_{14}a_{24}-2a_{12}a_{24}+2a_{13}a_{23}\right)\right)\\ 2-\big(2a_{12}^2+2a_{23}^2+2a_{24}^2+2a_{12}a_{23}+2a_{23}a_{24}+2a_{12}a_{24}\big)\\ 0.5\left(-2-\left(-4a_{23}^2+2a_{12}a_{13}-2a_{12}a_{23}-2a_{23}a_{24}-2a_{13}a_{23}-2a_{23}a_{34}+2a_{24}a_{34}\right)\right)\\ 0.5\left(-\left(-4a_{24}^2+2a_{12}a_{14}-2a_{12}a_{24}-2a_{23}a_{24}+2a_{23}a_{34}-2a_{14}a_{24}-2a_{24}a_{34}\right)\right) \end{bmatrix}\\
\mathbf{s}_3 &= \begin{bmatrix} 0.5\left(-\left(-4a_{13}^2-2a_{12}a_{13}-2a_{13}a_{14}+2a_{12}a_{23}-2a_{13}a_{23}-2a_{23}a_{34}+2a_{14}a_{34}\right)\right)\\ 0.5\left(-2-\left(-4a_{23}^2+2a_{12}a_{13}-2a_{12}a_{23}-2a_{23}a_{24}-2a_{12}a_{23}-2a_{23}a_{34}+2a_{24}a_{34}\right)\right)\\ 2-\big(2a_{13}^2+2a_{23}^2+2a_{34}^2+2a_{13}a_{23}+2a_{13}a_{34}+2a_{23}a_{34}\big)\\ 0.5\left(-2-\left(-4a_{34}^2+2a_{13}a_{14}+2a_{23}a_{24}-2a_{13}a_{34}-2a_{23}a_{34}-2a_{24}a_{34}-2a_{14}a_{34}\right)\right) \end{bmatrix}\\
\mathbf{s}_4 &= \begin{bmatrix} 0.5\left(\left(-4a_{14}^2-2a_{13}a_{14}-2a_{12}a_{14}+2a_{12}a_{24}+2a_{13}a_{34}-2a_{14}a_{24}-2a_{14}a_{34}\right)\right)\\ 0.5\left(\left(-4a_{24}^2+2a_{12}a_{14}-2a_{12}a_{24}-2a_{23}a_{24}+2a_{23}a_{34}-2a_{14}a_{24}-2a_{24}a_{34}\right)\right)\\ 0.5\left(-2-\left(-4a_{34}^2+2a_{13}a_{14}+2a_{23}a_{24}-2a_{13}a_{34}-2a_{23}a_{34}-2a_{24}a_{34}-2a_{14}a_{34}\right)\right)\\ 1-\big(2a_{14}a_{24}+2a_{24}a_{34}+2a_{14}a_{34}+2a_{14}^2+2a_{24}^2+2a_{34}^2\big) \end{bmatrix}
\end{aligned} \tag{12}
$$

Table 1.
Potential values of parameters for optimal control law in *Example* 1

| No. | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{23}$ | $a_{24}$ | $a_{34}$ |
|---|---|---|---|---|---|---|
| 1. | -0.5449 | -0.1622 | -0.1084 | -0.4911 | -0.1622 | -0.5449 |
| 2. | -0.8155 | 0.1084 | 0.5449 | -0.379 | 0.1084 | -0.8155 |
| 3. | -0.1622 | -0.5449 | 0.8155 | 1.198 | -0.5449 | -0.1622 |
| 4. | 0.1084 | -0.8155 | 0.1622 | 1.086 | -0.8155 | 0.1084 |
| 5. | -0.1084 | 0.8155 | -0.1622 | -1.086 | 0.8155 | -0.1084 |
| 6. | 0.1622 | 0.5449 | -0.8155 | -1.198 | 0.5449 | 0.1622 |
| 7. | 0.8155 | -0.1084 | -0.5449 | 0.379 | -0.1084 | 0.8155 |
| 8. | **0.5449** | **0.1622** | **0.1084** | **0.4911** | **0.1622** | **0.5449** |

(2) *Choosing admissible solution*

The error vector is taken as $\mathbf{e} \triangleq \begin{bmatrix} e_1\\ e_2\\ e_3 \end{bmatrix} \triangleq \begin{bmatrix} x_1-x_2\\ x_2-x_3\\ x_3-x_4 \end{bmatrix}$.
Then, $\dot{\mathbf{e}} = E\mathbf{e}$ where $E$ is the error matrix as given in (13). Clearly, if the eigenvalues of $E$ are in the left half-plane then the obtained solution is indeed admissible and results in the consensus of the agents. It was found that for only the *eighth solution* i.e $a_{12} = 0.5449$ $a_{13} = 0.1622$, $a_{14} = 0.1084$, $a_{23} = 0.4911$, $a_{24} = 0.1622$, $a_{34} = 0.5449$, the matrix $E$ has all the eigenvalues in the left half of complex plane. Thus, this solution is admissible and hence the desired optimal control law is given as:

$$
\begin{aligned}
u_1^* &= 0.5449(x_2-x_1)+0.1622(x_3-x_1)\\
&\quad +0.1084(x_4-x_1)\\
u_2^* &= 0.5449(x_1-x_2)+0.4911(x_3-x_2)\\
&\quad +0.1622(x_4-x_2)\\
u_3^* &= 0.1622(x_1-x_3)+0.4911(x_2-x_3)\\
&\quad +0.5449(x_4-x_3)\\
u_4^* &= 0.1084(x_1-x_4)+0.1622(x_2-x_4)\\
&\quad +0.5449(x_3-x_4)
\end{aligned}
$$

The response obtained the initial condition $x_1(0) = 10$, $x_2(0) = -100$, $x_3(0) = 20$ and $x_4(0) = -90$ upon application of the designed control law is shown in Figure 2.

$$
E = \begin{bmatrix} -2a_{12}-a_{13}-a_{14} & -a_{13}-a_{14}+a_{23}+a_{24} & -a_{14}+a_{24}\\ a_{12}-a_{13} & -2a_{23}-a_{24}-a_{13} & -a_{24}+a_{34}\\ a_{13}-a_{14} & a_{13}+a_{23}-a_{14}-a_{24} & -2a_{34}-a_{14}-a_{24} \end{bmatrix} \tag{13}
$$

***Subsidiary results:*** Following a similar procedure, the following optimal consensus control laws were obtained for the two-agent and three-agent systems respectively.

(1) For the two agents with dynamics $\dot{x}_1 = u_1$ and $\dot{x}_2 = u_2$ respectively, the optimal consensus control law was found to be $u_1^* = \frac{1}{\sqrt{2}}(x_2-x_1)$ and $u_2 = \frac{1}{\sqrt{2}}(x_1-x_2)$.

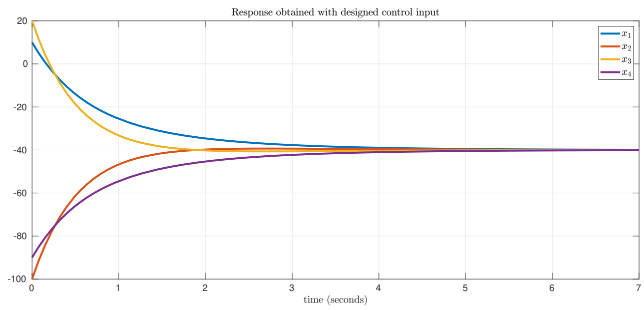


Fig. 2. Response obtained with the designed control law

(2) For the three agents with dynamics $\dot{x}_1 = u_1$, $\dot{x}_2 = u_2$ and $\dot{x}_3 = u_3$ respectively, the optimal consensus control law was found to be $u_1^* = 0.577\,(x_2 - x_1) + 0.211\,(x_3 - x_1)$, $u_2^* = 0.577\,(x_1 - x_2) + 0.577\,(x_3 - x_1)$ and $u_3^* = 0.211\,(x_2 - x_1) + 0.577\,(x_3 - x_1)$.

## 5. CONCLUSION

In this work, the application of Krotov conditions to the problem of the optimal consensus protocol design is considered. The solution to this problem is difficult to compute due to the desired diffusive nature of the control law which results in the non-convex nature of the problem. A rather unexplored tool, namely Krotov conditions is utilized to tackle the problem. These conditions translate the optimal control design problem into an optimization problem via the so-called Krotov function and consequently provide sufficient conditions for the existence of optimal control law(s). Since this latter optimization problem is usually not convex, the iterative methods are used to compute the optimal control laws. At the same time, the Krotov function could be any continuously differentiable function. This fact is exploited in this work to make the selection so as to compute the required optimal control laws non-iteratively. The future work includes the extension of similar ideas to the higher-order multi-agent systems.

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