Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ State-Feedback Control of Continuous-Time Markov Jump Systems with Partial Observations of the Markov Chain

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Abstract: We study the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback control of continuous-time Markov jump linear systems considering that the Markov chain is not observable, with the only information available to the controller coming from the output of a fault-detection and isolation device. We present sufficient design conditions given in terms of linear matrix inequalities so that the closed-loop system is stable and its $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are bounded. We present an illustrative example in which we investigate the behavior of the proposed algorithm.

Keywords: Markov jump linear systems, hidden Markov models, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control, linear matrix inequalities

1. INTRODUCTION

The study of systems subject to abrupt changes in their structure has taken a lot of effort in the control literature in the past decades. The situations in which those changes arise due to *failures* in real-time applications are of special concern, since that could lead to performance degradation and, in a critical scenario, instability. One approach to deal with those situations relies in the so-called Markov jump linear systems (MJLS), that are a class of switched systems whose switching rule is a Markov chain (see Costa et al. (2013), Dragan et al. (2013), and the references therein). An interesting application of MJLS is found in Active Fault-Tolerant Control Systems (AFTCS), see for instance, Aberkane et al. (2008), Mahmoud et al. (2003), and the references therein, in which the failure process is modeled by a Markov chain and the controller switches according to the estimates provided by a fault-detection and isolation (FDI) device.

An usual assumption employed in MJLS control design is that the Markov chain θ (or mode of operation) can be perfectly measured, which yields to the so-called mode-dependent case and arguably leads to "simpler" problems to be solved, see, for instance, Cardelíquio et al. (2014). However there are applications in which θ is hard to obtain, e.g., in Networked Control Systems (NCS), see, for instance, Gonçalves et al. (2010), or in AFTCS, in which that assumption would be equivalent to consider that we know perfectly when and where the fault occurred in the system. Considering the NCS, some aproaches have been proposed such as the mode-independent and cluster cases, see, for instance, Val et al. (2002) and Morais et al. (2016), in which there is only one controller for all modes of operation, or a smaller set of controllers for some distinguishable modes of operation. In the context of AFTCS though, the aforementioned problem of using the information coming from a FDI device to actively switch the controller is appealing.

In this work, we study the design of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback controllers considering that the failure process θ cannot be measured, but instead there is a *detector* $\hat{\theta}$ that provides to the controller the only information on the underlying process. For that we rely on the model presented in Stadtmann and Costa (2017) and Stadtmann and Costa (2018) dubbed *exponential hidden Markov model* that emulates the behavior of a FDI device. Our contributions are as follows:

- We propose new linear matrix inequalities (LMIs) design conditions for a stabilizing state-feedback controller that switches according to $\hat{\theta}$ so that the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the closed-loop system are bounded by given constants.
- We briefly discuss some of the aforementioned observation cases, that is, the mode-dependent and -independent formulations. We show in the illustrative example that our conditions can get very close to the optimal $\mathcal{H}_2$ control in the case where the failure process can be perfectly measured.

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It is worth noting that there are alternative observation models employed in the literature such as the one used in Aberkane et al. (2008) and Mahmoud et al. (2003) that does not consider simultaneous jumps of the Markov chain and the detector as done in the exponential hidden Markov model of Stadtmann and Costa (2017); the $\epsilon-$model of Rodrigues et al. (2019) that requires the study of rate of jumps of $\hat{\theta}$; and the continuous-time formulation formulation of the detector approach of Costa and Fragoso (1995) presented in Fragoso and Costa (2004). Besides that this paper can also be viewed as the partial information counterpart of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback control presented in (Costa et al., 2013, Chapter 9) for the case in which the Markov chain can be perfectly measured, as well as a continuous-time version of Oliveira and Costa (2017) and Oliveira and Costa (2018). It is also worth noting that a similar formulation for discrete-time MJLS, known in the literature as *asynchronous control*, was studied in, for instance, in Shen et al. (2019), Wu et al. (2017), and the references therein.

This work is organized as follows. We introduce the notation in Section 2 and the problem formulation in Section 3, where the system and the controller structure are introduced, as well as the basic definitions such as the mean-square stability and the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. The main results are presented in Section 4 in which we study a sufficient condition for obtaining the desired bounds on the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for a given controller by means of an extended bounded-real lemma, and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ LMI design condition. We present an Illustrative Example in Section 5 in which we are able to control an MJLS with two unstable subsystems, and briefly discuss the behavior of the costs by changing the detector rates. We present our final remarks in Section 6.

2. NOTATION

The real Euclidean space of dimension n is represented by $\mathbb{R}^n$, and the space of real matrices of dimension $n \times m$, by $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$. We represent by $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ the space of $m \times n$ real matrices with $\mathbb{B}(\mathbb{R}^n) \triangleq \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$. The identity matrix of size $n \times n$ is given by I_n (or simply I), $(\cdots)'$ is the transpose operator and, for a square matrix G, we set $Her(G) \triangleq G + G'$, and $Tr(\cdot)$ is the trace operator. Given positive integers N and M, we set $\mathscr{N} \triangleq \{1, \ldots, N\}$, $\mathscr{M} \triangleq \{1, \ldots, M\}$, and $\mathscr{V} \subseteq \mathscr{N} \times \mathscr{M}$. The linear space composed by all sequence of matrices $\mathbf{V} = (V_{ik} \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m); (i,k) \in \mathscr{V})$ is represented by $\mathbb{H}^{n,m}$, and for ease of notation we set $\mathbb{H}^n \triangleq \mathbb{H}^{n,n}$ and $\mathbb{H}^{n+} \triangleq \{\mathbf{V} \in \mathbb{H}^n : V_{ik} \geq 0, (i,k) \in \mathscr{V}\}$. Given $\mathbf{V}, \mathbf{S} \in \mathbb{H}^{n+}$, we write that $\mathbf{V} \geq \mathbf{S}$ if $V_{ik} - S_{ik} \geq 0$ for all $(i,k) \in \mathscr{V}$. Similarly we define the set $\mathbb{M}^{n,m} \triangleq \{M_k \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m), k \in \mathscr{M}\}$, $\mathbb{M}^n \triangleq \mathbb{M}^{n,n}$, and $\mathbb{M}^{n+}$ accordingly. We fix the probability space as $(\Omega, \mathscr{F}, I\!\!P)$ and represent by $L_2^r(\Omega, \mathscr{F}, I\!\!P)$ (or simply L_2^r) the set of square integrable stochastic processes $x = \{x(t) \in \mathbb{R}^n, t \in \mathbb{R}^+\}$ with $x(t)$ $\mathscr{F}_t$-measurable for each $t \in \mathbb{R}^+$.

3. PROBLEM FORMULATION

We consider the following MJLS in the probability space $(\Omega, \mathscr{F}, I\!\!P)$ equipped with the filtration $\{\mathscr{F}_t\}$,

$$\mathcal{G} : \begin{cases} \dot{x}(t) = A_{\theta(t)} x(t) + B_{\theta(t)} u(t) + J_{\theta(t)} w(t) \\ z(t) = C_{\theta(t)} x(t) + D_{\theta(t)} u(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^q$, and $w(t) \in \mathbb{R}^r$. $\theta(t)$ is a homogeneous Markov chain taking values in $\mathscr{N}$ with transition rate matrix $\Lambda \triangleq [\lambda_{ij}]$ and $\theta(0)$ is a random variable in $\mathscr{N}$. We also consider that $x(0) = 0$.

The main goal is to design the following controller

$$K : u(t) = K_{\hat{\theta}(t)} x(t) \quad (2)$$

that depends only on an observed variable $\hat{\theta}(t)$ taking values in $\mathscr{M}$. The closed-loop system is obtained by plugging (1) and (2) yielding to

$$\mathcal{G}_K : \begin{cases} \dot{x}(t) = A_{\theta(t)\hat{\theta}(t)} x(t) + J_{\theta(t)} w(t) \\ z(t) = C_{\theta(t)\hat{\theta}(t)} x(t) \end{cases} \quad (3)$$

where

$$A_{\theta(t)\hat{\theta}(t)} = A_{\theta(t)} + B_{\theta(t)} K_{\hat{\theta}(t)} \quad (4)$$

$$C_{\theta(t)\hat{\theta}(t)} = C_{\theta(t)} + D_{\theta(t)} K_{\hat{\theta}(t)} \quad (5)$$

By setting the stochastic process as $Z(t) \triangleq (\theta(t), \hat{\theta}(t))$, we consider that

$$I\!\!P(Z(t+h) = (j,l) \mid Z(t) = (i,k)) = \begin{cases} \nu_{(i,k)(j,l)} h + o(h), \ (j,l) \neq (i,k) \\ 1 + \nu_{(i,k)(i,k)} h + o(h), \ (j,l) = (i,k) \end{cases}$$

where

$$\nu_{(i,k)(j,l)} = \begin{cases} \alpha_{jl}^k \lambda_{ij}, \ j \neq i, l \in \mathscr{M}, \\ q_{kl}^i, \ j = i, l \neq k, \\ \lambda_{ii} + q_{kk}^i, \ j = i, l = k, \end{cases} \quad (6)$$

where $\sum_{l \in \mathscr{M}} \alpha_{jl} = 1$, $\lambda_{ij} \geq 0$ for all $i \neq j$, $q_{kl}^i \geq 0$, $l \neq k$, $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$, $q_{kk}^i = -\sum_{l \neq k} q_{kl}^i$. As studied in Stadtmann and Costa (2017), we get that λ_{ij} represents the transition rate of $\theta(t)$, and α_{jl}^k and q_{kl}^i models simultaneous and spontaneous jumps of $\hat{\theta}(t)$. We also define the invariant set for $Z(t)$ by $\mathscr{V} \subseteq \mathscr{N} \times \mathscr{M}$ so that $I\!\!P(Z(t) \in \mathscr{V}) = 1$ whenever $Z(0) \in \mathscr{V}$.

Remark 1. (Detector rates). For the case of simultaneous jump of $\theta(t)$ and $\hat{\theta}(t)$, we get that α_{jl}^k is the probability of $\hat{\theta}(t^+) = l$ given that $\theta(t^+) = j$, $\theta(t^-) = i$, $\hat{\theta}(t^-) = k$. That is, by setting $I\!\!P(\hat{\theta}(t+h) = l \mid \theta(t+h) = j, Z(t) = (i,k)) = \alpha_{jl}^k + r(h)$ with $\lim_{h \to 0} r(h) = 0$, we get for $j \neq i$ that

$$\begin{aligned} & I\!\!P(Z(t+h) = (j,l) \mid Z(t) = (i,k)) \\ & = (\alpha_{jl}^k + r(h)) I\!\!P(\theta(t+h) = j \mid \theta(t) = i) \\ & = (\alpha_{jl}^k + r(h))(\lambda_{ij} h + o(h)) = \alpha_{jl}^k \lambda_{ij} h + o(h). \end{aligned}$$

Conversely q_{kl}^i can be viewed as the transition rate of spontaneous jumps of $\hat{\theta}(t)$ whenever $\theta(t) = i$. We have for $l \neq k$ that

$$\begin{aligned} & I\!\!P(Z(t+h) = (i,l) \mid Z(t) = (i,k)) \\ & = (q_{kl}^i h + o(h))(I\!\!P(\theta(t+h) = i \mid \theta(t) = i)) \\ & = (q_{kl}^i h + o(h))(1 - \lambda_{ii} h + o(h)) = q_{kl}^i h + o(h). \end{aligned}$$

□

Remark 2. According to Stadtmann and Costa (2018) we can retrieve some interesting cases presented in the literature such as

- The *mode-dependent case* ($\mathscr{M} = \mathscr{N}$, $q^i_{kl} = 0$, $\alpha^k_{jj} = 1$, and $\alpha^k_{jl} = 0$ for $j \neq l$, with invariant set $\mathscr{V} = \{(i,i) \in \mathscr{N} \times \mathscr{N}\}$);
- The *mode-independent case* ($\mathscr{M} = \{1\}$, $q^i_{kl} = 0$, and $\alpha^1_{j1} = 1$);
- *No mutual jumps*: $\alpha^k_{jk} = 1$ and $\alpha^k_{jl} = 0$ for $k \neq l$. □

In the following we set $\mathcal{Q}_i \triangleq [q^i_{kl}]$, and $\mathcal{A}_k \triangleq [\alpha^k_{jl}]$. We also introduce the next technical assumption on the transition rates, used in Section 4.

Assumption 1. $\nu_{(i,k)(i,k)} \neq 0$ for all $(i,k) \in \mathscr{V}$. □

Assumption 1 imposes that there exists at least one element in $\nu_{(i,k)(j,l)} > 0$ for $(i,k) \neq (j,l) \in \mathscr{V}$. Effectively it avoids the so-called absorbing states in the Markov chain, see, for instance Leon-Garcia (2008).

We introduce next the concept of internal mean-square stability (iMSS) adapted from Costa et al. (2013).

Definition 1. The system (3) is said to be iMSS with $w = 0$ if $\lim_{t\to\infty} \mathbf{E}(\|x(t)\|^2) = 0$ for arbitrary $x(0)$ and $(\theta(0), \hat{\theta}(0))$. □

We get that if (3) is iMSS and $w \in L^r_2$, then $x \triangleq \{x(t), t \geq 0\} \in L^n_2$, see, for instance, (Costa et al., 2013, Theorem 3.27).

We now present conditions for accessing the iMSS of (3). For that we define the operator $\mathcal{T} \in \mathbb{H}^n$ such that

$$\mathcal{T}_{ik}(\mathbf{P}) \triangleq Her(A'_{ik}P_{ik}) + \sum_{(j,l)\in\mathscr{V}} \nu_{(i,k)(j,l)} P_{jl} \tag{7}$$

for $\mathbf{P} \in \mathbb{H}^n$. We have the following lemma adapted from Costa et al. (2013).

Lemma 3. The system (3) is iMSS if and only if there exists $\mathbf{P} \in \mathbb{H}^{n+}$ such that

$$\mathbf{P} > 0, \mathcal{T}(\mathbf{P}) < 0. \tag{8}$$

□

The set of admissible controllers is described as follows.

$$\mathfrak{K} \triangleq \{K \text{ as in (2)}: \text{ such that (8) holds}\}. \tag{9}$$

We now introduce the concept of $\mathcal{H}_2$ norm of (3) for a fixed $K \in \mathfrak{K}$, adapted from Stadtmann and Costa (2017).

Definition 2. Given that $K \in \mathfrak{K}$ and $x(0) = 0$, the $\mathcal{H}_2$ norm of (3) is given by

$$\|\mathcal{G}_K\|_2^2 \triangleq \sum_{s=1}^{r} \sum_{(i,k)\in\mathscr{V}} \mu_{ik} \|z_{s,(i,k)}\|_2^2,$$

where $z_{s,(i,k)}(k)$ is the output of (3) for $w(t) = v_s\delta(t)$, v_s is the $s-$th element of the standard basis of $\mathbb{R}^r$; and $I\!P((\theta(0), \hat{\theta}(0)) = (i,k)) = \mu_{ik}$, for all $(i,k) \in \mathscr{V}$. □

Following Costa et al. (2013) and Stadtmann and Costa (2017), the $\mathcal{H}_2$ norm of (3) for a fixed $K \in \mathfrak{K}$ is given by

$$\|\mathcal{G}_K\|_2^2 = \sum_{(i,k)\in\mathscr{V}} \mu_{ik} Tr(J'_i \bar{P}_{ik} J_i), \tag{10}$$

where $\bar{\mathbf{P}} \in \mathbb{H}^{n+}$ is the unique solution of the following coupled Lyapunov equations,

$$\mathcal{T}(\bar{\mathbf{P}}) + \mathbf{C} = 0, \tag{11}$$

for $\mathbf{C} \triangleq (\mathbf{C_{ik}}, (i,k) \in \mathscr{V})$, $\mathbf{C} \in \mathbb{H}^{n+}$, $\mathbf{C_{ik}} \triangleq C'_{ik}C_{ik}$.

We introduce next the concept of $\mathcal{H}_\infty$ norm for (3), taken from De Farias et al. (2000).

Definition 3. Given that $K \in \mathfrak{K}$ and $x(0) = 0$, the $\mathcal{H}_\infty$ norm of (3), $\|\mathcal{G}_K\|_\infty$, is defined as the smallest γ_∞ such that $\|z\|_2 < \gamma_\infty \|w\|_2$, for all $(\theta(0), \hat{\theta}(0)) = (i,k) \in \mathscr{V}$ and $0 \neq w \in L^r_2$. □

For achieving a given attenuation degree γ of the $\mathcal{H}_\infty$ norm of (3), we introduce the following extended bounded-real lemma adapted from Stadtmann and Costa (2018).

Lemma 4. Given K, we get that $K \in \mathfrak{K}$ with $\|\mathcal{G}_K\|_\infty < \gamma_\infty$ for $x(0) = 0$ if there exists $\mathbf{P} \in \mathbb{H}^{n+}$, $P_{ik} > 0$, $(i,k) \in \mathscr{V}$, such that

$$Her(P_{ik}A_{ik}) + \sum_{(j,l)\in\mathscr{V}} \nu_{(i,k)(j,l)} P_{jl} + C'_{ik}C_{ik} + \frac{1}{\gamma_\infty^2} P_{ik} J_i J'_i P_{ik} < 0 \tag{12}$$

for all $(i,k) \in \mathscr{V}$. □

Considering the previous definitions, we are now able to state our main goal, that is, finding controllers $K \in \mathfrak{K}$ such that a suitable upper bound for the $\mathcal{H}_2$ norm of (3) is minimized for a given bound on the $\mathcal{H}_\infty$ norm, that is,

$$\inf_{K\in\mathfrak{K}} \gamma_2 \ : \ \text{ such that } \|\mathcal{G}_K\|_2 < \gamma_2 \text{ and } \|\mathcal{G}_K\|_\infty < \gamma_\infty. \tag{13}$$

To achieve our goal we will discuss in the next section a property of (12) that echoes the studied in Costa and Marques (1998) and Oliveira and Costa (2017) for the discrete-time MJLS with hidden observations. For that, we recall the next auxiliary result presented in (Costa et al., 2013, Theorem 3.25).

Lemma 5. Given that $K \in \mathfrak{K}$ in (3), we have that if $\mathbf{S} \geq \mathbf{T} \geq 0$ ($\mathbf{S} > \mathbf{T} \geq 0$), as well as $\mathcal{T}(\mathbf{P}) + \mathbf{S} = 0$ and $\mathcal{T}(\bar{\mathbf{P}}) + \mathbf{T} = 0$, then $\mathbf{P} \geq \bar{\mathbf{P}} \geq 0$ ($\mathbf{P} > \bar{\mathbf{P}} \geq 0$). □

4. MAIN RESULTS

In this section we present an auxiliary result that provides a bound on both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of (3) by exploiting the structure of (12). Furthermore by applying LMI techniques on this auxiliary tool, we are able to obtain the main design result of this paper shown in Theorem 7.

Lemma 6. Given K, γ_2, and γ_∞, if there exists $P \in \mathbb{H}^{n+}$, $P > 0$ such that (12) and

$$\sum_{(i,k)\in\mathscr{V}} \mu_{ik} Tr(J'_i P_{ik} J_i) < \gamma_2^2 \tag{14}$$

jointly hold, then $K \in \mathfrak{K}$, $\|\mathcal{G}_K\|_2 < \gamma_2$, and $\|\mathcal{G}_K\|_\infty < \gamma_\infty$. □

Proof. Note that if (12) and (14) jointly hold, we get from (12) that $\mathcal{T}(\mathbf{P}) + \mathbf{C} + \mathbf{J} + \mathbf{V} = 0$, for $\mathbf{V} > 0$, where $\mathbf{J} \triangleq (\mathbf{J_{ik}}, (i,k) \in \mathscr{V})$, $\mathbf{J} \in \mathbb{H}^{n+}$, $\mathbf{J_{ik}} \triangleq P_{ik}J_iJ'_iP_{ik} \geq 0$ for all $(i,k) \in \mathscr{V}$, since $J_iJ'_i \geq 0$. Thus it is direct that $\mathbf{C} + \mathbf{J} + \mathbf{V} > \mathbf{C} \geq 0$, and so from Lemma 5, we get that $\mathbf{P} \geq \bar{\mathbf{P}}$, where $\bar{\mathbf{P}}$ is the solution of (11). It follows from simple manipulations along with (10) and (14) that $\|\mathcal{G}_K\|_2^2 = \sum_{(i,k)\in\mathscr{V}} \mu_{ik} Tr(J'_i \bar{P}_{ik} J_i) \leq \sum_{(i,k)\in\mathscr{V}} \mu_{ik} Tr(J'_i P_{ik} J_i) < \gamma_2^2$. □

It is clear that if K is considered as a variable in (12) and (14), we would get a non-linear problem in the decision variables due to the products between K and $\mathbf{P}$. Instead we resort to the use of slack variables as employed in Morais et al. (2016) and classical LMI techniques for obtaining a convex formulation. Consider the following inequalities for $(i,k) \in \mathscr{V}$,

$$\sum_{(i,k)\in\mathscr{V}} \mu_{ik} Tr(W_{ik}) < \varsigma, \tag{15}$$

$$\begin{bmatrix} W_{ik} & \bullet \\ J_i & X_{ik} \end{bmatrix} > 0, \tag{16}$$

$$\mathcal{H}_{ik} + Her(\Psi_{ik}\Phi_{ik}) < 0, \tag{17}$$

$$\begin{bmatrix} Z_{(ik),(jl)} & \bullet \\ H_{ik} & X_{jl} \end{bmatrix} > 0 \tag{18}$$

along with $X_{ik} > 0$, where

$$\mathcal{H}_{ik} \triangleq \begin{bmatrix} \nu_{(i,k)(i,k)}X_{ik} & \bullet & \bullet & \bullet & \bullet \\ X_{ik} & 0_{n\times n} & \bullet & \bullet & \bullet \\ X_{ik} & 0_{n\times n} & \Xi_{ik} & \bullet & \bullet \\ J_i^T & 0_{r\times n} & 0_{r\times n} & -\gamma_\infty^2 I_r & \bullet \\ 0_{q\times n} & 0_{q\times n} & 0_{q\times n} & 0_{q\times r} & -I_q \end{bmatrix},$$

$$\Xi_{ik} \triangleq -Her(H_{ik}) + \sum_{(j,l)\in\mathscr{V}^{(i,k)}} \nu_{(i,k)(j,l)} Z_{(i,k)(j,l)}$$

$$\Psi'_{ik} \triangleq [I_n\zeta \;\; I_n \;\; 0_{n\times n} \;\; 0_{n\times r} \;\; 0_{n\times q}],$$

$$\Phi_{ik} \triangleq \left[(A_iG_k + B_iY_k)' \;\; -G'_k \;\; 0_{n\times n} \;\; 0_{n\times r} \;\; (C_iG_k + D_iY_k)'\right],$$

and $\mathscr{V}^{(i,k)} \triangleq \{(j,l) \in \mathscr{V} : (j,l) \neq (i,k)\}$.

Theorem 7. Given $\gamma_\infty \in \mathbb{R}_+$ and $\zeta \in \mathbb{R}$, if there exist $\varsigma > 0$, $0 < W_{ik} \in \mathbb{B}(\mathbb{R}^r)$, $0 < Z_{(i,k)(j,l)} \in \mathbb{B}(\mathbb{R}^n)$, $H_{ik} \in \mathbb{B}(\mathbb{R}^n)$, $0 < X_{ik} \in \mathbb{B}(\mathbb{R}^n)$, $G_k \in \mathbb{B}(\mathbb{R}^n)$, $Y_k \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, such that (15)-(18) hold, then by setting $K_k = Y_k G_k^{-1}$, we get that $K \in \mathfrak{K}$, $\|\mathcal{G}_K\|_2 < \varsigma^{1/2}$ and $\|\mathcal{G}_K\|_\infty < \gamma_\infty$. □

Proof. Given that (15)-(18) holds, we note that $Her(G_k) < 0$, and thus G_k is non-singular. We get, by setting $\gamma_2^2 = \varsigma$ and $Y_k = K_k G_k$, that

$$\mathcal{H}_{ik} + Her(\Psi_{ik} G'_k \bar{\Phi}_{ik}) < 0 \tag{19}$$

holds, where

$$\bar{\Phi}_{ik} \triangleq \left[A'_{ik} \;\; -I_n \;\; 0_{n\times n} \;\; 0_{n\times r} \;\; C'_{ik}\right].$$

By defining

$$\mathcal{N} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ A'_{ik} & 0 & 0 & C'_{ik} \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & I_q \end{bmatrix}$$

so that $Rank(N) = 2n + r + q$, we get by multiplying (19) to the left-hand side by $\mathcal{N}'$ and to the right-hand side by $\mathcal{N}$, that

$$\mathcal{N}'\mathcal{H}_{ik}\mathcal{N} = \begin{bmatrix} \nu_{(i,k)(i,k)}X_{ik} + Her(A_{ik}X_{ik}) & \bullet & \bullet & \bullet \\ X_{ik} & \Xi_{ik} & \bullet & \bullet \\ J_i^T & 0_{r\times n} & -\gamma_\infty^2 I_r & \bullet \\ C_{ik}X_{ik} & 0_{q\times n} & 0_{q\times r} & -I_q \end{bmatrix} < 0 \tag{20}$$

holds. Then, by considering the reasoning presented in Cardelíquio et al. (2014) and Geromel et al. (2009), we get that (17) yields to $Z_{(i,k)(j,l)} > H'_{ik} X_{jl}^{-1} H_{ik}$. Then it follows by multiplying the last inequality by $\nu_{(i,k)(j,l)}$ and summing everything up by $(j,l) \in \mathscr{V}^{(i,k)}$ that $\sum_{(j,l)\in\mathscr{V}^{(ik)}} \nu_{(i,k)(j,l)} Z_{(i,k)(j,l)} \geq H'_{ik}\mathfrak{P}_{ik}H_{ik}$, where $\mathfrak{P}_{ik} \triangleq \sum_{(j,l)\in\mathscr{V}^{(i,k)}} \nu_{(i,k)(j,l)} X_{jl}^{-1}$. Therefore,

$$Her(H_{ik}) - H'_{ik}\left(\sum_{(j,l)\in\mathscr{V}^{(i,k)}} \nu_{(i,k)(j,l)} X_{jl}^{-1}\right) H_{ik} \geq Her(H_{ik}) - \sum_{(j,l)\in\mathscr{V}^{(i,k)}} \nu_{(i,k)(j,l)} Z_{(i,k)(j,l)} \tag{21}$$

and then, considering Assumption 1, (21), and that $(\mathfrak{G}'_{ik} - \mathfrak{P}_{ik}^{-1})\mathfrak{P}_{ik}(\mathfrak{G}_{ik} - \mathfrak{P}_{ik}^{-1}) \geq 0$ for $\mathfrak{G}_{ik} \triangleq H_{ik}$, we get from the same reasoning as presented in Cardelíquio et al. (2014) that $-\Xi_{ik} = Her(H_i) - \sum_{(j,l)\in\mathscr{V}^{(i,k)}} \nu_{(i,k)(j,l)} Z_{(i,k)(j,l)} \leq Her(\mathfrak{G}_{ik}) - \mathfrak{G}'_{ik}\mathfrak{P}_{ik}\mathfrak{G}_{ik} = \mathfrak{P}_{ik}^{-1} - (\mathfrak{G}'_{ik} - \mathfrak{P}_{ik}^{-1})\mathfrak{P}_{ik}(\mathfrak{G}_{ik} - \mathfrak{P}_{ik}^{-1}) \leq \mathfrak{P}_{ik}^{-1}$. Thus, we can substitute Ξ_{ik} by $-\mathfrak{P}_{ik}^{-1}$ in (20) and the inequality still holds, that is,

$$\begin{bmatrix} \nu_{(i,k)(i,k)}X_{ik} + Her(A_{ik}X_{ik}) & \bullet & \bullet & \bullet \\ X_{ik} & -\mathfrak{P}_{ik}^{-1} & \bullet & \bullet \\ J_i^T & 0_{r\times n} & -\gamma_\infty^2 I_r & \bullet \\ C_{ik}X_{ik} & 0_{q\times n} & 0_{q\times n} & -I_q \end{bmatrix} < 0.$$

By applying the congruence transformation $diag(X_{ik}^{-1}, I_n, I_r, I_q)$ and the Schur complement with respect to $-\mathfrak{P}_{ik}^{-1}$ in the previous inequality, we get that (12) holds for $P_{ik} = X_{ik}^{-1}$, and then $K \in \mathfrak{K}$ and $\|\mathcal{G}_K\|_\infty < \gamma_\infty$. It remains to show that (15)-(16) leads to the bound on the $\mathcal{H}_2$ norm. By applying the Schur complement to (16), we get that $W_{ik} > J'_i X_{ik}^{-1} J_i = J'_i P_{ik} J_i$ holds. By taking the trace operator on both sides of this inequality, multiplying them by μ_{ik}, summing everything up for all $(i,k) \in \mathscr{V}$, and considering (15), we get that (14) holds. By Lemma 6, the claim follows. □

Bearing in mind the result of Theorem 7, we can rewrite the main goal posed in (13) as follows

$$\inf_{\psi\in\Psi(\gamma_\infty,\zeta)} \varsigma : \text{ subject to } (15) - (18) \tag{22}$$

where $\psi \triangleq (\varsigma, W_{ik}, Z_{(i,k)(j,l)}, X_{ik}, H_{ik}, G_k, Y_k, (i,k) \times (j,l) \in \mathscr{V} \times \mathscr{V}, k \in \mathscr{M})$, and $\Psi(\gamma_\infty, \zeta)$ is the set of all solutions of (15)-(18) for given $\gamma_\infty \in \mathbb{R}^+$ and $\zeta \in \mathbb{R}$, which adds a degree of freedom to the problem. This parameter is discussed in Section 5.

Remark 8. ("Pure" $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control). From the result of Theorem 7, we can obtain alternative design conditions for the "pure" $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control of exponential hidden Markov models presented in Stadtmann and Costa (2017) and Stadtmann and Costa (2018). By taking $\varsigma \to \infty$ and minimizing γ_∞, we note that (15) is fulfilled for arbitrary big values of W_{ik}, and thus by the Schur complement in (16), we get that $X_{ik} > J_i W_{ik}^{-1} J'_i \approx 0$, retrieving the "pure" $\mathcal{H}_\infty$ control. On the other hand, by taking the Schur complement of (17) with respect to the block $-\gamma_\infty I_r$, $\gamma_\infty \to \infty$, and minimizing ς, we get that the resulting conditions will yield to $Her(X_{ik}^{-1}A_{ik}) + C'_{ik}C_{ik} < 0$, that by Lemma 5, provides a "tighter bound" on the solution $\bar{P}_{ik}$ of (11). □

5. ILLUSTRATIVE EXAMPLE

In this section we study problem (22) and the effects of the rates q and α on the problem. Consider the following

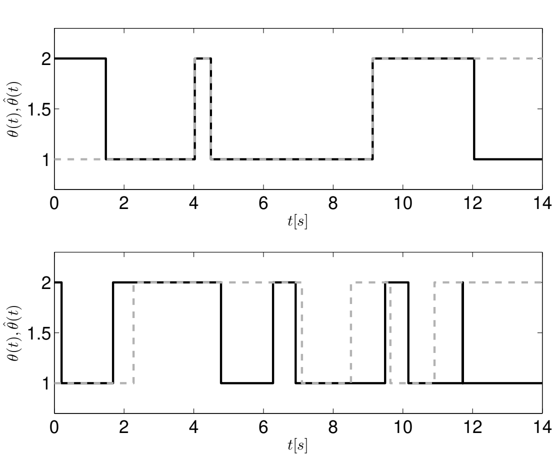

Fig. 1. $\theta(t)$ (full black line) and $\hat{\theta}(t)$ (dashed grey line) against t for Case (i) (upper figure) and Case (ii) (lower figure).

unstable MJLS with two modes of operation ($N = 2$) and system matrices given by

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, J_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, J_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix},$$

and rate transition matrix Λ given by

$$\Lambda = \begin{bmatrix} -0.3 & 0.3 \\ 0.8 & -0.8 \end{bmatrix}.$$

Note that both subsystems are unstable. We set the controlled output matrices as follows

$$C_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, D_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for $i \in \mathscr{N}$. Regarding the detector $\hat{\theta}(t)$, we consider that $M = 2$ and investigate the effects of the rates α_{jl}^k and q_{kl}^i presented in (6) and Remark 1 in the behavior of the joint process $(\theta, \hat{\theta})$:

- (i) (Simultaneous Jumps only). We set $q_{kl}^i = 0$ for all $i \in \mathscr{N}$, $k, l \in \mathscr{M}$, and $\alpha_{jj}^k = 0.7$ for all $j \in \mathscr{N}$, $k \in \mathscr{M}$ in (6), that is, the process $\hat{\theta}(k)$ can jump only when $\theta(k)$ changes. The probability that $\theta(k+1) = \hat{\theta}(k+1)$ in this case is given by 0.7.
- (ii) (No mutual jumps). We set $q_{kk}^i = -0.5$ for all $i \in \mathscr{N}$, $k \in \mathscr{M}$, and $\alpha_{jk}^k = 1$ for $j \in \mathscr{N}$, $k \in \mathscr{M}$. In this case, the detector will change according to its own conditioned rates as in (24). Note that at the times when θ changes, the detector $\hat{\theta}(t)$ will not jump.

The trajectories of θ and $\hat{\theta}$ for one realization of Cases (i) and (ii) are shown in Figure 1. We note in Figure 1 at the top of the page that for Case (i) (upper figure), the expected behavior of the detector is that $\hat{\theta}$ is able to jump only when the Markov chain θ switches. By looking at the interval $t \in [1.5, 12)$, we get that the detector is able to follow the Markov chain, even in the jump times $t \in \{4, 4.5, 9.1\}$. However in the transition instant $t = 12$ s, we note that both trajectories take different paths, due to the probability $\alpha_{jj}^k = 0.7$. On the other hand, considering

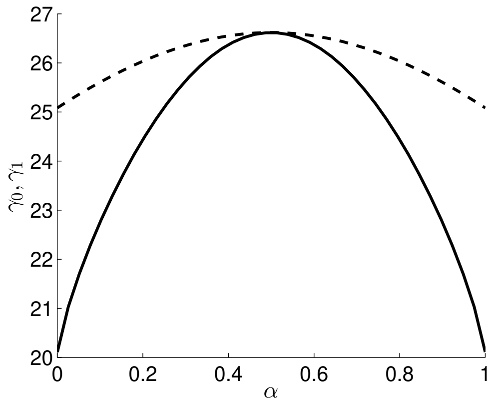

Fig. 2. ς_1^* (dashed black line) and ς_0^* (full black line) against $\alpha \in (0, 1)$ for $\zeta = 1$ for $\gamma_\infty = 12$.

Case (ii), since $\alpha_{jk}^k = 1$, we get that the detector will always stay in the same state whenever the Markov chain $\theta(t)$ changes. This can be noted in Figure 1 (lower figure), that is, the two processes θ and $\hat{\theta}$ switch at different time instants, as expected.

We now set the detector rates as follows,

$$\mathscr{A}_k = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix}, \forall k \in \mathscr{M} \tag{23}$$

$$\mathscr{Q}_i = \begin{bmatrix} -\beta & \beta \\ \beta & -\beta \end{bmatrix}, \forall i \in \mathscr{N}, \tag{24}$$

for $\alpha \in [0, 1]$ and $\beta \in [0, 2]$, and investigate the effects of α and β in (22) by setting $\gamma_\infty = 12$ and $\zeta = 1$, and solving (22) for $\alpha \in (0, 1)$, for $\beta \in \{0, 1\}$. The initial distribution of the Markov chain is taken as the stationary one for each case. The solution ς_β^* for $\beta \in \{0, 1\}$ is shown in Figure 2 at the top of the page. First we note in Figure 2 that the worst-case scenario is given by $\alpha = 0.5$ for both $\beta \in \{0, 1\}$. That is to say that the detector will switch with equal probability to 1 or 2 whenever the Markov chain jumps, thus we do not have any clear information being provided by $\hat{\theta}$. Specially for the case $\beta = 1$, we note that due to $q_{11}^i = q_{22}^i$ in (24), we get that the detector behavior is not distinguishable between the two possible outputs, for fixed values of θ. In this situation, we get the mode-independent case (see Remark 2 and Stadtmann and Costa (2017)), and therefore the controller becomes

$$K_1 = K_2 = [-1.66 \;\; -2.47]$$

Besides, we note a degradation on the upper bounds in Figure 2 for the case $\beta = 1$ with respect to $\beta = 0$ for $\zeta = 1$, that is, we increase the uncertainty of the detector in this case.

Finally, we briefly investigate the effects of $\zeta \in \{1, 2, 10\}$ of (17) in (22) in the case $\beta = 0$ through Figure 3 at the top of page 6. We note that as we increase ζ, we get smaller upper bounds ς^* in a similar effect as the one obtained in Stadtmann and Costa (2017) and Morais et al. (2016). That seems to be consistent behavior that leads to the optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control for the mode-dependent case. Indeed considering Remark 8, by solving (22) with $\zeta = 10$ and $\gamma_\infty = 1e3$ with $\beta = 0$, $\alpha = 1$, and invariant set $\mathscr{V} = \{(11), (22)\}$, we get $\varsigma^* = \|\mathcal{G}_K\|_2^2 = 11.28$. The controller in this case is given by

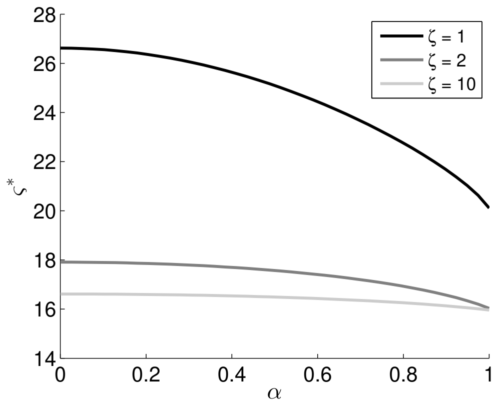

Fig. 3. ς^* against $\alpha \in (0,1)$ for $\zeta \in \{1,2,10\}$ and $\beta = 0$ for $\gamma_\infty = 12$.

$$K_1 = [-1.6086 \ \ -0.5676]$$
$$K_2 = [-0.2703 \ \ -1.6324]$$

6. CONCLUSION

In this work we proposed new LMI conditions for the design of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback controllers for continuous-time MJLS considering that we have access only to the output of a detector of the Markov chain. We investigate some interesting behaviors that the detector can exhibit, such as the cases of simultaneous and spontaneous jumps and noticed through in a numerical example that we are able to obtain optimal $\mathcal{H}_2$ state-feedback controllers for the case in which the detector can perfectly observe the Markov chain.

For future steps, the study of more general forms of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control is desirable, since the equalized case tackled in this paper is very restrictive due to the use of a common weight for both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control. Besides, as we noted in the Illustrative Example, we are able to obtain optimal $\mathcal{H}_2$ controllers in the case of perfect observation of the Markov chain, and thus a formal proof is required for properly accessing this property. Finally some robustness aspects of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control should also be exploited.

REFERENCES

Aberkane, S., Ponsart, J.C., R., M., and Sauter, D. (2008). Output feedback control of a class of stochastic hybrid systems. *Automatica*, 44(5), 1325–1332.

Cardelíquio, C.B., Fioravanti, A.R., and Gonçalves, A.P.C. (2014). H_2 and H_∞ state-feedback control of continuous-time MJLS with uncertain transition rates? In *2014 European Control Conference (ECC)*, 2237–2241.

Costa, O.L.V. and Fragoso, M.D. (1995). Discrete-time LQ-optimal control problems for infinite Markov jump parameter systems. *IEEE Trans. Autom. Control*, 40(12), 2076–2088.

Costa, O.L.V., Fragoso, M.D., and Todorov, M.G. (2013). *Continuous-Time Markov Jump Linear Systems*. Springer-Verlag Berlin Heidelberg, New York City.

Costa, O.L.V. and Marques, R.P. (1998). Mixed H_2/H_∞-control of discrete-time Markovian jump linear systems. *IEEE Trans. Autom. Control*, 43(1), 95–100.

De Farias, D.P., Geromel, J.C., Do Val, J.B.R., and Costa, O.L.V. (2000). Output feedback control of Markov jump linear systems in continuous-time. *IEEE Trans. Autom. Control*, 45(5), 944–949.

Dragan, V., Morozan, T., and Stoica, A.M. (2013). *Mathematical Methods in Robust Control of Linear Stochastic Systems*. Springer, New York City.

Fragoso, M.D. and Costa, O.L.V. (2004). Mean square stabilizability of continuous-time linear systems with partial information on the Markovian jumping parameters. *Stochastic Analysis and Applications*, 22(1), 99–111.

Geromel, J.C., Gonçalves, A.P.C., and Fioravanti, A.R. (2009). Dynamic output feedback control of discrete-time Markov jump linear systems through linear matrix inequalities. *SIAM Journal on Control and Optimization*, 48(2), 573–593.

Gonçalves, A.P., Fioravanti, A.R., and Geromel, J.C. (2010). Markov jump linear systems and filtering through network transmitted measurements. *Signal Processing*, 90(10), 2842–2850.

Leon-Garcia, A. (2008). *Probability, Statistics, and Random Processes for Electrical Engineering*. Pearson/Prentice Hall, Upper Saddle River, NJ, 3rd edition.

Mahmoud, M., Jiang, J., and Zhang, Y. (2003). *Active Fault Tolerant Control Systems - Stochastic Analysis and Synthesis*. Springer Verlag, Germany.

Morais, C.F., Braga, M.F., Oliveira, R.C.L.F., and Peres, P.L.D. (2016). $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control design for polytopic continuous-time Markov jump linear systems with uncertain transition rates. *Int. J. of Robust Nonlinear Control*, 26(3), 599–612.

Oliveira, A.M. and Costa, O.L.V. (2017). Mixed H_2/H_∞ state feedback control for Markov jump linear systems with hidden observations. *IFAC-PapersOnLine*, 50(1), 3800–3805. 20th IFAC World Congress.

Oliveira, A.M. and Costa, O.L.V. (2018). Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of hidden Markov jump systems. *Int. J. of Robust Nonlinear Control*, 28(4), 1261–1280.

Rodrigues, C.C.G., Todorov, M.G., and Fragoso, M.D. (2019). Detector-based approach for $\mathcal{H}_\infty$ filtering of markov jump linear systems with partial mode information. *IET Control Theory & App.*, 13, 1298–1308.

Shen, Y., Wu, Z.G., Shi, P., Shu, Z., and Karimi, H.R. (2019). H_∞ control of Markov jump time-delay systems under asynchronous controller and quantizer. *Automatica*, 99, 352 – 360.

Stadtmann, F. and Costa, O.L.V. (2017). H_2-control of continuous-time hidden Markov jump linear systems. *IEEE Trans. Autom. Control*, 62(8), 4031–4037.

Stadtmann, F. and Costa, O.L.V. (2018). Exponential hidden Markov models for H_∞ control of jumping systems. *IEEE Control Systems Letters*, 2(4), 845–850.

Val, J.B., Geromel, J.C., and Gonçalves, A.P.C. (2002). The H_2-control for jump linear systems: cluster observations of the Markov state. *Automatica*, 38(2), 335–340.

Wu, Z., Shi, P., Shu, Z., Su, H., and Lu, R. (2017). Passivity-based asynchronous control for Markov jump systems. *IEEE Trans. Autom. Control*, 62(4), 2020–2025.