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Two-on-One Pursuit when the Pursuers Have the Same Speed as the Evader ⋆

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Abstract: A two-on-one pursuit-evasion differential game is considered. The setup is akin to Isaacs' Two Cutters and Fugitive Ship differential game. In this paper it is however assumed that the three players have equal speeds and the two cutters/pursuers have a non-zero capture radius. The case where just one of the Pursuers is endowed with a circular capture set is also considered. The state space region where capture is guaranteed is delineated, thus providing the solution of the Game of Kind, and the players' optimal state feedback strategies and the attendant value function are synthesized, thus providing the solution of the Game of Degree.

1. INTRODUCTION

Isaacs' Two Cutters and Fugitive Ship differential game Isaacs (1999) is revisited – see Fig. 1 . In Isaacs' formulation, the cutters are faster than the fugitive ship and point capture is required. The solution of the game was obtained using a geometric method, sans a proof. In Pachter (2018) the validity of Isaacs' geometric approach was proven. It was shown that the geometric method–provided strategies are recovered from the solution of the HJI PDE. In Isaacs (1999), Pachter (2018), and Pachter (2019), point capture was considered and the case where the two pursuers are endowed with circular capture sets of radius $\ell > 0$ was addressed in Wasz (July 2019). In references Isaacs (1999), Pachter (2018), Pachter (2019),and Wasz (July 2019), the pursuers are faster than the evader.

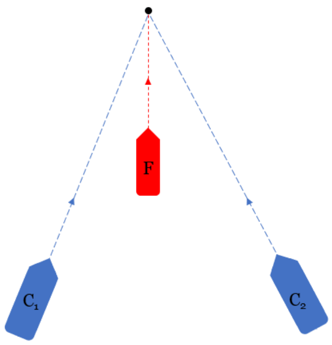

Fig. 1. Two Cutters and Fugitive Ship

In this paper it is assumed that all three players have the same speed. Many-on-one pursuit-evasion games where the players have simple motion and the pursuers and evader have the same speed require special consideration – see, e.g. the seminal paper Pshenichnyi (1976), where discriminating/stroboscopic pursuit strategies are employed. This has become a standard feature in the many-on-one pursuit-evasion literature as documented in the recent survey paper Kumkov (2016). Not so in this paper where the optimal strategies are state feedback strategies.

In this paper, the three players have equal speeds but, as in Wasz (July 2019), the cutters have a non-zero capture radius; when the two pursuers have the same speed as the evader, point capture is not possible, and this even if the Evader would be obliged to pre-announce his course; thus the need for finite capture sets. This game, where all the players have the same speed and the pursuers have a non-zero capture range, was considered in our preliminary work Wasz (Oct 2019). The state space region where capture is guaranteed was delineated thus providing the solution of the Game of Kind and the closed form solution of the Game of Degree which yields the players optimal state feedback strategies was outlined. In this paper, a streamlined derivation of the main result – the pursuers' and the evader's optimal state feedback strategies, and the Value function, is presented. The novel approach lends itself to the consideration of additional interesting scenarios, e.g., the case where the pursuers have different capture ranges.

The motivation behind this research is directly tied to air-to-air operations Kang (2010), Horie (2006). Previous research into this field has focused on games with fast pursuers where the objective is point capture, as in Breakwell (1979), but we are expanding this to include operationally relevant instances where both the blue and red sides have similar capabilities and the pursuers are endowed with finite capture sets, to reflect the range of aircraft weapon systems. This allows for the considerations of bounded capture regions, which was not the case when fast pursuers and point capture only is considered.

⋆ The views expressed in this paper are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the United States Government.

2. GEOMETRY

The Two Cutters and Fugitive Ship differential game is herein solved using a geometric method based on the solution of the max-min open-loop optimal control problem – as opposed to solving the HJI PDE – the validity of the geometric method in the case when the pursuers are faster than the evader having been proven in Pachter (2018). Now, when a pursuer and an evader; both with simple motion á la Isaacs, have the same speed and the pursuer is endowed with a circular capture set of radius ℓ, the locus of points in the Euclidean plane which they can reach at the same time is a hyperbola. Therefore, for any value of capture range $\ell > 0$ of the pursuer, what would have been a Cartesian oval had the pursuer been faster than the evader, as in Wasz (July 2019), will now become a hyperbola. The Boundary of the Safe Region of the Evader (BSR) will now be delineated by an arc of the hyperbola

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

with the parameters

$$a = \frac{\ell}{2}\ , \quad b = \frac{1}{2}\sqrt{d^2 - \ell^2}$$

where d is the $P - E$ separation. Since there are two pursuers, there are two hyperbolae at play. The asymptotes of those hyperbolae are used to solve the Game of Kind, and these are given by

$$Y = \pm\frac{b}{a}X$$

It will be useful to define the hyperbola's "eccentricity" $e \triangleq \frac{d}{\ell}$, and so the asymptotes' slope is

$$\frac{b}{a} = \sqrt{e^2 - 1} \tag{1}$$

The hyperbola locus, whose foci are the instantaneous positions of the pursuer P and the Evader E, and it's asymptotes, is shown in Figure 2. We use the hyperbola construct to designate the Safe Region (SR) of E in the two-on-one differential game when the Pursuers have the same speed as the Evader. Figure 2 shows the Boundary of the Evader's Safe Region (BSR) in the realistic plane (X,Y) when the pursuer is at $(-\frac{d}{2}, 0)$ and the evader at $(\frac{d}{2}, 0)$. Because the Pursuer is not faster than the Evader, this BSR is open; in other words, the Evader can escape. Hence, we need at least one other pursuer to obtain a closed SR so that the Evader might be isochronously captured by the two cooperating pursuers.

In the version of the Two Cutters and Fugitive Ship Differential Game investigated herein we have two pursers with capture radius ℓ and one evader, with all three having the same speed. We use a rotating reference frame (x, y), with the x-axis running through the instantaneous positions of the Pursuers P_1 and P_2 and the y-axis is the orthogonal bisector of the segment $\overline{P_1P_2}$. The state is specified by three variables: half of the separation of the pursuers, x_p, and the x and y position, (x_E, y_E), of the Evader in the rotation (x, y) frame. For example, the symmetric situation where E, P_1, and P_2 are collinear and the Evader is located halfway between P_1 and P_2 is illustrated in Figure 3. This Figure shows both the hyperbolae and their asymptotes, which intersect. The SR is therefore bounded and under optimal play the two pursuers will isochronously capture the Evader.

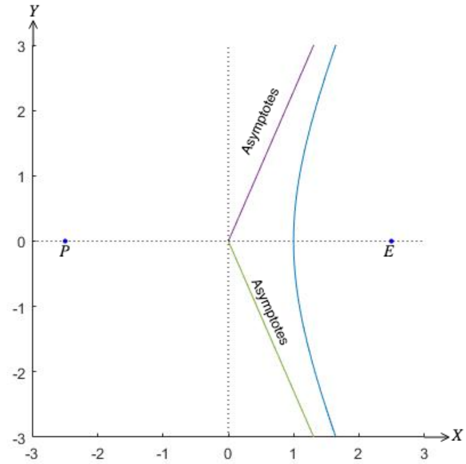

Fig. 2. The Hyperbola is the BSR of E

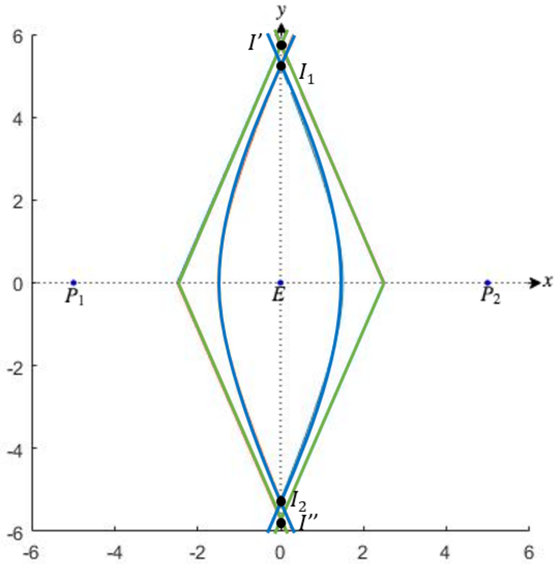

Fig. 3. Symmetric State – Two Pursuer Action

If the asymptotes don't intersect the evader can escape. But if the hyperbolae intersect and E is in the lens shaped region formed by the intersecting hyperbolae, if the pursuers play optimally, captures of the Evader is guaranteed. I_1 and I_2 are the points of intersection of the (P_1, E) and (P_2, E) based hyperbolae. Each of these points will be important in the sequel. Our immediate goal is to determine whether the SR is bounded, that is, the asymptotes intersect, which obviously is the case in the symmetric configuration illustrated in Figure 3 – when the evader is hemmed in by the pursuers, the asymptotes of the hyperbolae intersect at I' and I''.

3. GAME OF KIND

When the players are in general position, to find the solution to the Game of Kind, that is, whether under optimal pursuer play the Evader's capture is guaranteed, we need to determine whether the SR is bounded, which is the case if and only if the asymptotes of the hyperbolae intersect. Consider now the diagram in Figure 4. There are four points of interest, O_1,O_2, I', and I'' that are vertices of a quadrilateral. This quadrilateral contains the entirety of the evader's SR, so we can ensure capturability if we determine that this quadrilateral is indeed formed.

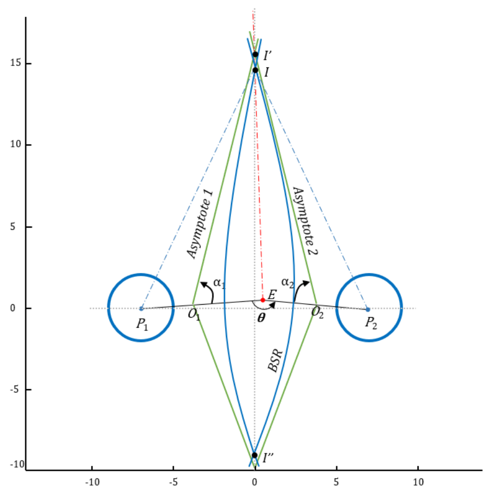

Fig. 4. Quadrilateral Formed by Intersecting Asymptotes

To this end, consider the angles θ, α_1, α_2 in Figure 4. Since a quadrilateral must have all internal angles sum to 360 degrees, we have the following

$$(360 - \theta) + \alpha_1 + \alpha_2 < 360$$

This yields the condition for a closed SR, and consequently the capturability condition is

$$\theta > \alpha_1 + \alpha_2$$

Since the slope of the asymptotes in the realistic plane (X, Y) are specified by Equation 1, we know that $\alpha_1 = \arctan(\sqrt{e_1^2 - 1})$ and $\alpha_2 = \arctan(\sqrt{e_2^2 - 1})$, with $e_1 = \frac{r_1}{\ell}$ and $e_2 = \frac{r_2}{\ell}$. The angles α_1, α_2, and θ are exclusively determined by the game's state (x_P, x_E, y_E). This is shown in Figure 5 where P_1, P_2, and E are in a general position. In Figure 5 the points

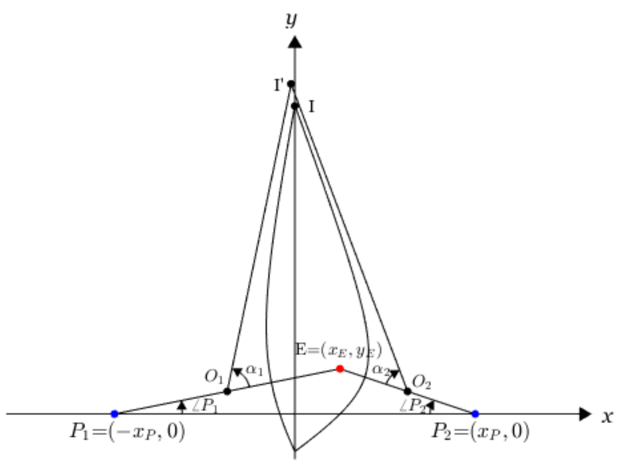

Fig. 5. The State (x_P, x_E, y_E)

$$O_1 = \frac{1}{2}(x_E - x_P, y_E) \ , \quad O_2 = \frac{1}{2}(x_E + x_P, y_E)$$

The angles

$$\tan \alpha_1 = \sqrt{e_1^2 - 1} \ , \quad \tan \alpha_2 = \sqrt{e_2^2 - 1}$$

and

$$\tan P_1 = \frac{y_E}{x_P + x_E} \ , \quad \tan P_2 = \frac{y_E}{x_P - x_E}$$

Therefore, summing those angles, we can characterize the captured zone in the reduced state space (x_P, x_E, y_E). Based on these arguments, in Ref. Wasz (Oct 2019) it was shown that the SR is closed and capturability is guaranteed if and only if in the realistic plane (x, y) the evader is located in the gray zone shown in Figure 6. If the y-coordinate is greater than ℓ, the evader can escape along a straight line path; he might even pre-announce his course and he'll still get away. The broken line in Figure 6 is not part of the gray zone where capturability is guaranteed.

The capture zone is limited. This is due to the fact that the pursuers are not faster than the evader – when both pursuers or just one pursuer, are/is faster than the evader, global capturability is guaranteed. Interestingly though, while the area of the Capture Zone is small, the pursuers can initially be far away from the evader and yet capturability is still guaranteed, provided the evader is in the narrow, gray, capturability zone.

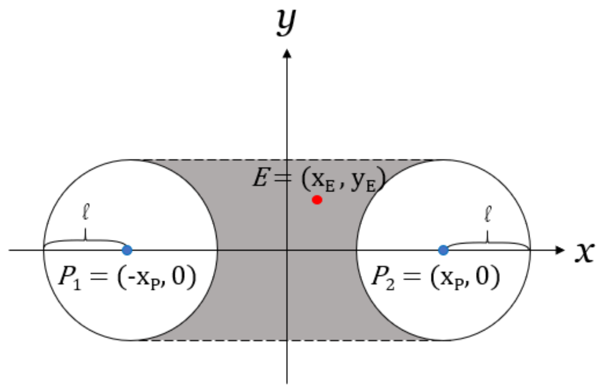

Fig. 6. Region of Capture

4. GAME OF DEGREE

Suppose the initial state is in the capture zone as shown in Figure 6. We focus on the Capture Zone area which is in the first quadrant of the (x,y) plane, that is, $x_E > 0$, $y_E > 0$.

Because both pursuers with equal speed and equal capture radii must travel the same distance in the same time, the interception ΔIP_1P_2 is isosceles, so the vertex I of the BSR must be on the orthogonal bisector of the segment $\overline{P_1P_2}$; therefore, the intercept point I is on the y-axis.

We now stipulate that the following must hold – see Fig. 7,

$$\sqrt{x_E^2 + (y - y_E)^2} = \sqrt{x_P^2 + y^2} - \ell,$$

as capture is only possible if $\overline{EI} = \overline{P_1I} - \ell = \overline{P_2I} - \ell$. Squaring both sides of the above equation, we obtain a quadratic equation in y

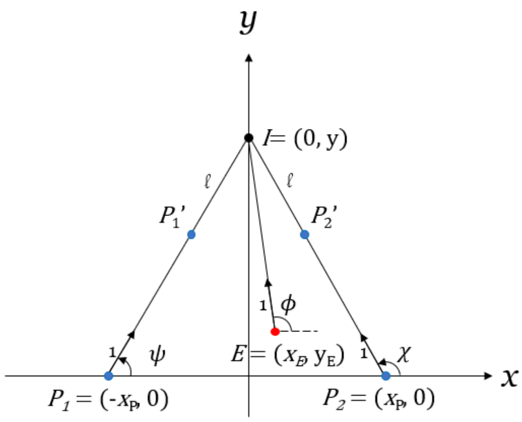

Fig. 7. Optimal Trajectories

$$4(\ell^2 - y_E^2)y^2 - 4y_E(\ell^2 - y_E^2 + x_P^2 - x_E^2)y \\ - (\ell^2 - y_E^2 + x_P^2 - x_E^2)^2 + 4\ell^2 x_P^2 = 0.$$

The discriminant must be non-negative. Thus, the following must be true:

$$4y_E^2(\ell^2 - y_E^2 + x_P^2 - x_E^2)^2 \\ + 4(\ell^2 - y_E^2)(\ell^2 - y_E^2 + x_P^2 - x_E^2)^2 - 16\ell^2(\ell^2 - y_E^2)x_P^2 > 0$$

$$(\ell^2 - y_E^2 + x_P^2 - x_E^2)^2 - 4(\ell^2 - y_E^2)x_P^2 > 0$$

We know $\ell^2 - y_E^2 > 0,\ x_P^2 - x_E^2 > 0$, thus

$$\ell^2 - y_E^2 + x_P^2 - x_E^2 > 2\sqrt{\ell^2 - y_E^2}x_P.$$

We need

$$(x_P - \sqrt{\ell^2 - y_E^2} > x_E^2),$$

but in the Capture Zone

$$x_P - \sqrt{\ell^2 - y_E^2} > x_E.$$

Thus, as long as the state $(x_P,\ x_E,\ y_E) \in$ Capture Zone, the quadratic equation has two real roots. If $y_E \geq 0$,

$$y = \frac{y_E(\ell^2 - y_E^2 + x_P^2 - x_E^2)}{2(\ell^2 - y_E^2)} \\ + \frac{\ell\sqrt{(\ell^2 - y_E^2 + x_P^2 - x_E^2)^2 - 4(\ell^2 - y_E^2)x_P^2}}{2(\ell^2 - y_E^2)} > 0 \quad (2)$$

The expression under the square root can be simplified so that eq. (2) can be written as

$$y(x_P, x_E, y_E) = \frac{\ell^2 - y_E^2 + x_P^2 - x_E^2}{2(\ell^2 - y_E^2)}y_E \\ + \frac{\ell\sqrt{(x_P + \sqrt{\ell^2 - y_E^2})^2 - x_E^2} \cdot \sqrt{(x_P - \sqrt{\ell^2 - y_E^2})^2 - x_E^2}}{2(\ell^2 - y_E^2)} \quad (3)$$

Eq. (3) can be applied $\forall\ x_P > 0,\ x_E > 0,\ y_E > 0$ to the Capture Zone part which is in the first quadrant of the realistic (x,y) plane. The players' optimal state feedback strategies are

$$sin\ \psi^* = \frac{y}{\sqrt{x_P^2 + y^2}}, \qquad cos\ \psi^* = \frac{x_P}{\sqrt{x_P^2 + y^2}}$$
$$sin\ \chi^* = \frac{y}{\sqrt{x_P^2 + y^2}}, \qquad cos\ \chi^* = -\frac{x_P}{\sqrt{x_P^2 + y^2}}$$
$$sin\ \phi^* = \frac{y - y_E}{\sqrt{x_E^2 + (y - y_E)^2}}, \quad cos\ \phi^* = -\frac{x_E}{\sqrt{x_E^2 + (y - y_E)^2}}$$

where the function $y(x_P, x_E, y_E)$ is given by eq. (3). The value function

$$V(x_P, x_E, y_E) = \sqrt{x_P^2 + y^2} - \ell.$$

When the state is symmetric $(x_E = 0)$

$$y(x_P, 0, y_E) = \frac{x_P^2 - (\ell - y_E)^2}{2(\ell - y_E)}$$

and the Value/time-to-capture

$$V(x_P, 0, 0) = \frac{x_P^2 - (\ell^2 - y_E^2)}{2(\ell - y_E)}$$

When $y_E = 0$,

$$y(x_P, x_E, 0) = \frac{\sqrt{(x_P + \ell)^2 - x_E^2} \cdot \sqrt{(x_P - \ell)^2 - x_E^2}}{2\ell}$$

5. CONTACT

Consider the case where the initial state is not in the interior of the gray capture zone and E is in contact with one of the pursuers, say P_2 – see Figure 8 – so

$$(x_P - x_E)^2 + y_E^2 = \ell^2$$

We insert this expression into eq. (3) and calculate the y-coordinate of the players' aim point,

$$y = \frac{x_P y_E}{x_P - x_E} \quad (4)$$

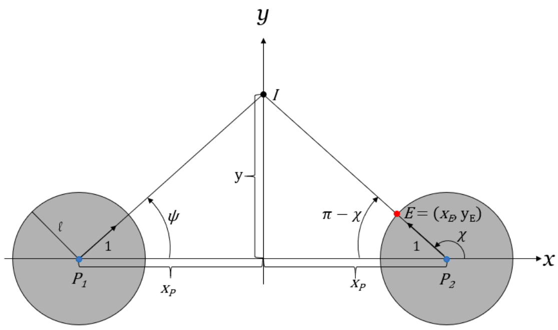

Fig. 8. E in Contact with P_2

But note:

$$tan(\pi - \chi) = \frac{y_E}{x_P - x_E}$$

and we calculate

$$\overline{y} = x_P\ tan(\pi - \chi) = \frac{x_P y_E}{x_P - x_E}$$

so,

$$\overline{y} = y$$

Thus, the optimal strategy of P_2 mandates that: Once in contact, P_2 pushes against E and does not let go of E. Once E reaches the y–axis which is the orthogonal bisector of the $\overline{P_1P_2}$ segment, the captive, but not yet captured, E will be met by P_1 and capture will be effected.

In general, if E does not play optimally by heading toward the interception point $I = (0, y)$, where y is specified by eq. (3), he will prematurely come into contact with one of the Pursuers, whereupon, as discussed above, he'll be pushed toward the y–axis where he'll be met by the companion pursuer and he'll be captured. Indeed, see Fig. 9 – the Evader's SR is closed and, trying to escape, he'll therefore run into one of the hyperbolae, say the P_2, E hyperbola. He will be met by P_2 who, by playing optimally, will push toward the point I' on the y-axis where he'll encounter P_1 and capture will be effected. This play is illustrated in Figure 9:

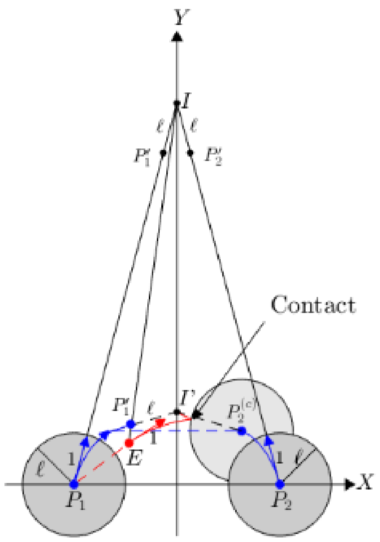

Fig. 9. Suboptimal Play of Evader. $x_{P_0} = 3$, $x_{E_0} = -1.4$, $y_{E_0} = 1.2$, $\ell = 1.5$

E erred by not running toward the aim point I and prematurely established contact with P_2. Consequently, once contact is established, P_2 relentlessly pushed E to the new aim point I' on the y–axis, where E will be met by P_1 and will be isochronously captured by P_1 and P_2. This is not good for E because $\overline{P_1I'} = \overline{P_2I'} < \overline{P_1I} = \overline{P_2I}$; E was captured prematurely. A simulation of the suboptimal trajectory depicted in Figure 9 resulted in a capture time of 4.9 seconds. A simulation using optimal feedback strategies subject to the same initial conditions as the suboptimal simulation resulted in a capture time of 20.1 seconds. The stark difference in capture time validates the notion that the optimal feedback strategies provide the evader with maximum longevity. A period of contact cannot arise in classical pursuit-evasion differential games where the pursuers are faster than the evader and this occurrence is unique to games where the pursuer's speed is the same, or even lower, than the evader's. When the speed ratio $\mu = \frac{v_E}{v_P}$, $0 < \mu < 1$, contact is immediately fatal for E.

6. DIFFERENT CAPTURE RANGES

We now consider the case where the Pursers are endowed with dissimilar capture ranges: $\ell_1 > \ell_2 \geq 0$. The intercept point I is no longer on the y-axis. Instead, the point of interception is defined by the intersections of three hyperbolae: the safe region-delimiting hyperbola whose foci are P_1 and E, the safe region-delimiting hyperbola whose foci are P_2 and E, plus a third hyperbola whose foci are the positions of the pursuers, P_1 and P_2. The parameters for the latter are

$$a = \frac{\ell_2 - \ell_1}{2}, \quad b = \sqrt{x_P^2 - a^2}.$$

The geometry is illustrated in Figure 10. The three hyperbolae are concurrent at the point I, which is the three players' aim point.

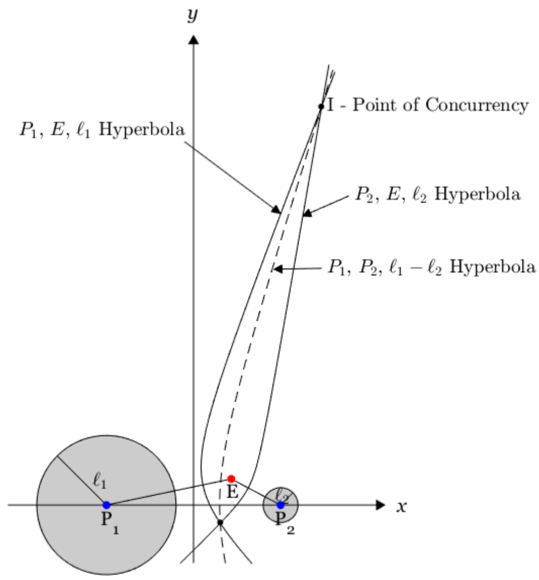

Fig. 10. Non-Equal Capture Disks. $\ell_1 = 1.2$, $\ell_2 = 0.3$.

We can now also consider the case where one of the pursuers, say P_2, is not endowed with a capture disk, thus point capture by P_2 is then necessary. The geometry when $\ell_2 = 0$, is depicted in Figure 11. The aim point I, where, under optimal play by the three players the evader will be captured, is defined by the intersection of the P_1, E hyperbola, the orthogonal bisector of the $\overline{EP_2}$ segment, and also the P_1, P_2 hyperbola; these three curves are concurrent at the aim point I.

7. CONCLUSION

A pursuit-evasion differential game in which two pursuers engage an equal-speed evader was analyzed. For capture to be effected, at least one of the pursuers must be endowed with a circular capture disk. A geometric approach based on the solution of the max-min open loop optimal control problem, whose validity was assumed in Isaacs (1999) and proven in Pachter (2018), is employed also when the pursuers have the same speed as the evader. In this paper, a streamlined derivation of the players' optimal state feedback strategies is provided. This, contingent on the evader being in the zone of capturability, as specified by the geometric solution of the Game of Kind. The zone of capturability is rather restricted due to the fact that the pursuers have the same speed as the evader. In extension, the optimal feedback strategies for pursuers with unequal capture radii were determined. Included was also the case in which only one pursuer is endowed with a capture disk.

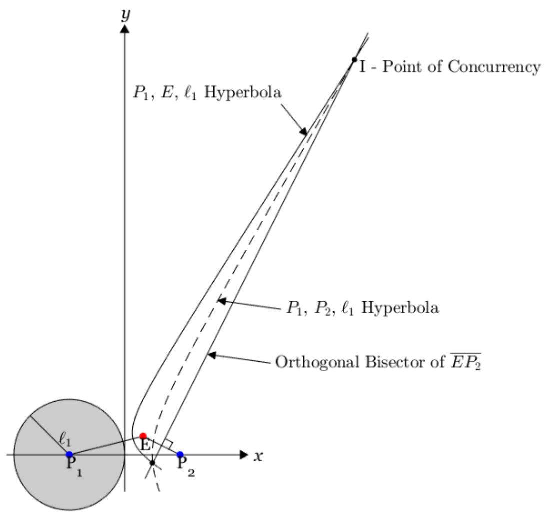

Fig. 11. Point Capture in Conjunction with Capture by a Pursuer Endowed with a Capture Disk of Radius. $\ell_1 = 1.5$.

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