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Quadratic Stabilization of Discrete-Time Bilinear Control Systems Subjected to Exogenous Disturbances ⋆

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Abstract: We consider the design problem for discrete-time bilinear control systems subjected to arbitrary bounded exogenous disturbances. A procedure for the construction of the stabilizability ellipsoids and stabilizability domain for discrete-time bilinear control systems is proposed and its efficiency is proved. The main tools are the linear matrix inequality technique and the apparatus of quadratic Lyapunov functions. This simple yet general approach is of great potential; for instance, it can be generalized to the various robust statements of the problem.

Keywords: Discrete-time bilinear systems, bounded exogenous disturbances, quadratic Lyapunov functions, linear feedback, stabilizability ellipsoid, stabilizability domain, linear matrix inequalities.

1. INTRODUCTION

Since the appearance of the monograph Mohler (1973), also see Ryan and Buckingham (1983); Isidori (ed.) (1990); Chen et al. (1991), problems related to stability, stabilization, and control design for bilinear systems are in the focus of numerous publications. There exists a variety of problem statements as well as approaches to the solution; e.g., see Ryan and Buckingham (1983); Chen et al. (1991); Čelikovský (1990, 1993); Tibken et al. (1996); Belozyorov (2002, 2005); Andrieu and Tarbouriech (2013); Coutinho and de Souza (2012); Kung et al. (2012); Omran et al. (2014). In particular, an ellipsoidal approach to such problems was the subject of discussion in Tibken et al. (1996).

In Khlebnikov (2015, 2016), an approach to the description of a stabilizability domain was proposed for disturbance-free continuous time bilinear systems. Using the linear matrix inequality technique and quadratic Lyapunov functions, a so-called stabilizability ellipsoid was constructed. This enabled efficient construction of nonconvex approximations to the stabilizability domains of disturbance-free bilinear control systems. In Khlebnikov (2018) this results were generalized to the discrete-time case. Among the most close publications we mention Amato et al. (2009); Tarbouriech et al. (2009); they are also targeted at the construction of quadratic Lyapunov functions in the problem of stabilizability of bilinear systems, and use the apparatus of linear matrix inequalities.

A number of recent publications are devoted to discrete-time bilinear control systems; e.g., see Goka et al. (1973); Tie and Lin (2015); Athanasopoulos and Bitsoris (2008, 2011), etc.; however the considerations in most of them are limited to the issue of controllability. The present paper differs essentially from all these works, since it deals with discrete-time bilinear systems subjected to bounded exogenous disturbances. In the present paper, an efficient procedure is proposed for the construction of stabilizability ellipsoids for discrete-time bilinear control systems subjected to exogenous disturbances; moreover, a new problem is formulated and solved, which is construction the stabilizability domains.

The paper is organized as follows. Section 2 presents an important auxiliary result that represents a generalization of the so-called Petersen's lemma; Section 3 contains the statement of the problem. Section 4 is devoted to the analysis problem and to construction of stabilizability ellipsoids, whereas Section 5 contains the main result of the paper. In Section 6 we present a procedure for the construction of stabilizability domains for discrete-time bilinear control systems subjected to exogenous disturbances, and Section 7 presents comments and conclusions.

In all formulations throughout the paper, scalar control input is considered; however, the proposed approach can be fully generalized to the case of many-dimensional controls; the derivations become somewhat more involved technically, whereas the ideological part remains nearly the same.

We stress that the proposed approach is based on the solution of *convex* optimization problems, however it leads to *nonconvex* approximations of stabilizability domains of discrete-time bilinear systems.

In the sequel, we use the following notation: $\|\cdot\|$ is the Euclidean norm of a vector and the spectral matrix norm,

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$^\top$ is the transposition sign, I is the identity matrix of appropriate size, and all matrix inequalities are understood in the sense of sign-definiteness.

2. PRELIMINARY RESULT

The so-called *Petersen's lemma*, see Petersen (1987), is an effective tool widely used in various statements of the problems of stabilization and control.

Lemma 1. (Petersen's lemma). *Let $G = G^\top \in \mathbb{R}^{n\times n}$, $0 \neq M \in \mathbb{R}^{n\times p}$, and $0 \neq N \in \mathbb{R}^{q\times n}$ be given matrices. The inequality*

$$G + M\Delta N + N^\top \Delta^\top M^\top \preceq 0$$

is valid for all $\Delta \in \mathbb{R}^{p\times q}$: $\|\Delta\| \leq 1$ if and only if there exists a number $\varepsilon > 0$ such that

$$\begin{pmatrix} G + \varepsilon M M^\top & N^\top \\ N & -\varepsilon I \end{pmatrix} \preceq 0.$$

We introduce the following modification of Petersen's lemma. Namely, instead of norm-bounded matrix uncertainty we consider vector uncertainty subjected to the ellipsoidal constraint.

Lemma 2. Let $G = G^\top \in \mathbb{R}^{n\times n}$, $0 \neq M \in \mathbb{R}^{n\times q}$, $0 \neq N \in \mathbb{R}^{1\times n}$, and $0 \prec P = P^\top \in \mathbb{R}^{q\times q}$ be given matrices. The inequality

$$G + M\delta N + N^\top \delta^\top M^\top \preceq 0$$

is valid for all

$$\delta \in \mathbb{R}^q: \ \delta^\top P^{-1}\delta \leq 1$$

if and only if there exists a number $\varepsilon > 0$ such that

$$\begin{pmatrix} G & MP & N^\top \\ PM^\top & -\varepsilon P & 0 \\ N & 0 & -\varepsilon^{-1} I \end{pmatrix} \preceq 0.$$

The proof of the Lemma 2 follows from Petersen's lemma for the vector uncertainty $\Delta = P^{-1/2}\delta$ such that $\|\Delta\| \leq 1$. This result will be essentially used in the exposition to follow.

3. STATEMENT OF THE PROBLEM

Consider the discrete-time bilinear control system

$$x_{\ell+1} = Ax_\ell + Bx_\ell u_\ell + bu_\ell + Dw_\ell, \qquad (1)$$

with state variable $x_\ell \in \mathbb{R}^n$, scalar control input $u_\ell \in \mathbb{R}$, initial state x_0 and exogenous disturbance $w_\ell \in \mathbb{R}^m$, which is bounded at every time instant:

$$\|w_\ell\| \leq \gamma \quad \text{for all } \ell = 0, 1, 2, \dots \qquad (2)$$

Here $A, B \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$, and $b \in \mathbb{R}^n$ are given matrices and vectors.

Definition 1. The ellipsoid

$$\mathcal{E} = \left\{x \in \mathbb{R}^n: \quad x^\top P^{-1} x \leq 1\right\}, \quad P \succ 0, \qquad (3)$$

is said to be *stabilizability ellipsoid* associated with the static linear state feedback

$$u_\ell = k^\top x_\ell, \quad k \in \mathbb{R}^n, \qquad (4)$$

if trajectory of the closed-loop system (1) embraced with feedback (4), emanating from any point in this ellipsoid remains to stay inside this ellipsoid for all admissible disturbances (2).

We are interested to construct the stabilizability ellipsoid for the considered system. In what follows, this ellipsoid will be made as large (in a certain sense) as possible.

Embracing the bilinear system (1), (2) by a static linear feedback (4) we arrive at the *quadratic* discrete-time dynamical system

$$x_{\ell+1} = (A_c + Bx_\ell k^\top)x_\ell + Dw_\ell,$$

where

$$A_c = A + bk^\top.$$

The dynamical systems of this form are the subject of our interest in the next section of this paper.

4. ANALYSIS

Consider the quadratic discrete-time dynamical system of the form

$$x_{\ell+1} = (A + Bx_\ell k^\top)x_\ell + Dw_\ell, \qquad (5)$$

where $A, B \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$, $k \in \mathbb{R}^n$, $x_\ell \in \mathbb{R}^n$ is the state variable, and w_ℓ is the exogenous disturbance (2). Note that no other constraints are imposed on the disturbance w_ℓ; e.g., it is not assumed to be stochastic or harmonic. Hence, we consider l_∞-bounded exogenous disturbances. Assume that the matrix A is Schur stable.

System (5) in the absence of exogenous disturbances ($D = 0$) was the subject of analysis in Khlebnikov (2015, 2016). Using the linear matrix inequality technique and the apparatus of quadratic Lyapunov functions, a regular approach was proposed in these papers for the construction of stabilizability ellipsoids for such bilinear system. The goal of this section is to construct a stabilizability ellipsoid for system (5) in the presence of disturbances (2).

In contrast to disturbance-free systems, trajectories enter the reachability set of the closed-loop system (or approach a point on its boundary), whereas the trajectories have to converge towards the origin in the disturbance-free case. An important common feature of these two cases is that the trajectory of system (5) that emanates from any point x_0 inside the stabilizability ellipsoid, remains to stay inside this ellipsoid for all admissible disturbances (5).

The theorem below establishes a sufficient condition for the ellipsoid (3) to be a stabilizability ellipsoid for system under consideration.

Theorem 1. Ellipsoid (3) *is a stabilizability ellipsoid for system* (5), (2) *for given bound γ, if its matrix P satisfies the matrix inequalities*

$$\begin{pmatrix} -\alpha P & 0 & 0 & 0 & Pk & PA^\top \\ 0 & -P & 0 & BP & 0 & 0 \\ 0 & 0 & -(1-\alpha)I & 0 & 0 & \gamma D^\top \\ 0 & PB^\top & 0 & -\varepsilon P & 0 & PB^\top \\ k^\top P & 0 & 0 & 0 & -\varepsilon^{-1} I & 0 \\ AP & 0 & \gamma D & BP & 0 & -P \end{pmatrix} \preceq 0,$$

$$P \succ 0,$$

for some α and $\varepsilon > 0$.

Proof. Introduce the quadratic form

$$V(x) = x^\top Q x, \quad 0 \prec Q = P^{-1} \in \mathbb{R}^{n\times n}.$$

To force the trajectories x_ℓ of system (5) remain in the ellipsoid

$$\mathcal{E} = \left\{x \in \mathbb{R}^n: \quad V(x) \leq 1\right\},$$

it is sufficient to require the following condition to hold:

$$V(x_{\ell+1}) \le 1 \quad \text{for} \quad V(x_\ell) \le 1 \quad \text{and all} \quad \|w_\ell\| \le \gamma.$$

Keeping in mind that

$$\begin{aligned} V(x_{\ell+1}) &= x_{\ell+1}^\top Q x_{\ell+1} \\ &= x_\ell^\top A^\top Q A x_\ell + x_\ell^\top k x_\ell^\top B^\top Q A x_\ell + w_\ell^\top D^\top Q A x_\ell \\ &\quad + x_\ell^\top A^\top Q B x_\ell k^\top x_\ell + x_\ell^\top k x_\ell^\top B^\top Q B x_\ell k^\top x_\ell \\ &\quad + w_\ell^\top D^\top Q B x_\ell k^\top x_\ell + x_\ell^\top A^\top Q D w_\ell \\ &\quad + x_\ell^\top k x_\ell^\top B^\top Q D w_\ell + w_\ell^\top D^\top Q D w_\ell \\ &= x_\ell^\top (A^\top Q A + A^\top Q B x_\ell k^\top + k x_\ell^\top B^\top Q A \\ &\quad + k x_\ell^\top B^\top Q B x_\ell k^\top) x_\ell \\ &\quad + w_\ell^\top (D^\top Q A + D^\top Q B x_\ell k^\top) x_\ell \\ &\quad + x_\ell^\top (A^\top Q D + k x_\ell^\top B^\top Q D) w_\ell + w_\ell^\top D^\top Q D w_\ell, \end{aligned}$$

this condition can be rewritten as

$$\begin{aligned} & x_\ell^\top (A^\top Q A + A^\top Q B x_\ell k^\top + k x_\ell^\top B^\top Q A \\ & + k x_\ell^\top B^\top Q B x_\ell k^\top) x_\ell + w_\ell^\top (D^\top Q A + D^\top Q B x_\ell k^\top) x_\ell \\ & + x_\ell^\top (A^\top Q D + k x_\ell^\top B^\top Q D) w_\ell + w_\ell^\top D^\top Q D w_\ell \le 1 \\ & \qquad \text{for} \quad x_\ell^\top Q x_\ell \le 1 \quad \text{and} \quad w_\ell^\top w_\ell \le \gamma^2, \end{aligned}$$

and, after introducing the compound vector

$$s_\ell = \begin{pmatrix} x_\ell \\ w_\ell \end{pmatrix} \in \mathbb{R}^{n+m},$$

it takes the form

$$s^\top \begin{pmatrix} \begin{matrix} A^\top Q A + A^\top Q B x_\ell k^\top \\ + k x_\ell^\top B^\top Q A \\ + k x_\ell^\top B^\top Q B x_\ell k^\top \end{matrix} & \begin{matrix} A^\top Q D \\ + k x_\ell^\top B^\top Q D \end{matrix} \\ D^\top Q A + D^\top Q B x_\ell k^\top & D^\top Q D \end{pmatrix} s \le 1$$

$$\text{for} \quad s^\top \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} s \le 1 \quad \text{and} \quad s^\top \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{-2} I \end{pmatrix} s \le 1.$$

Use of the sufficiency part of the S-procedure implies the condition

$$\begin{pmatrix} \begin{matrix} A^\top Q A + A^\top Q B x_\ell k^\top \\ + k x_\ell^\top B^\top Q A \\ + k x_\ell^\top B^\top Q B x_\ell k^\top \end{matrix} & A^\top Q D + k x_\ell^\top B^\top Q D \\ D^\top Q A + D^\top Q B x_\ell k^\top & D^\top Q D \end{pmatrix} - \alpha \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} - \beta \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{-2} I \end{pmatrix} \preceq 0$$

or

$$\begin{pmatrix} \begin{matrix} A^\top Q A - \alpha Q \\ + A^\top Q B x_\ell k^\top \\ + k x_\ell^\top B^\top Q A \\ + k x_\ell^\top B^\top Q B x_\ell k^\top \end{matrix} & A^\top Q D + k x_\ell^\top B^\top Q D \\ D^\top Q A + D^\top Q B x_\ell k^\top & D^\top Q D - \beta \gamma^{-2} I \end{pmatrix} \preceq 0$$

for some $\alpha, \beta \ge 0$ such that $\alpha + \beta \le 1$.

The obtained condition can be rewritten as

$$\begin{pmatrix} \begin{matrix} A^\top Q A - \alpha Q \\ + A^\top Q B x_\ell k^\top \\ + k x_\ell^\top B^\top Q A \end{matrix} & k x_\ell^\top B^\top Q & \begin{matrix} A^\top Q D \\ + k x_\ell^\top B^\top Q D \end{matrix} \\ Q B x_\ell k^\top & -Q & 0 \\ \begin{matrix} D^\top Q A \\ + D^\top Q B x_\ell k^\top \end{matrix} & 0 & D^\top Q D - \beta \gamma^{-2} I \end{pmatrix} \preceq 0$$

or in the equivalent form

$$\begin{aligned} & \begin{pmatrix} A^\top Q A - \alpha Q & 0 & A^\top Q D \\ 0 & -Q & 0 \\ D^\top Q A & 0 & D^\top Q D - \beta \gamma^{-2} I \end{pmatrix} \\ & \qquad + \begin{pmatrix} A^\top Q B \\ Q B \\ D^\top Q B \end{pmatrix} x_\ell \begin{pmatrix} k^\top & 0 & 0 \end{pmatrix} \\ & \qquad + \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix} x_\ell^\top \begin{pmatrix} B^\top Q A & B^\top Q & B^\top Q D \end{pmatrix} \preceq 0. \end{aligned} \tag{6}$$

Require now that the matrix inequality (6) holds for all x_ℓ in the ellipsoid

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n : \ V(x) \le 1 \right\} = \left\{ x \in \mathbb{R}^n : \ x^\top P^{-1} x \le 1 \right\}.$$

Then Lemma 2 implies the equivalent matrix inequality

$$\begin{pmatrix} A^\top Q A - \alpha Q & * & * & * & * \\ 0 & -Q & * & * & * \\ D^\top Q A & 0 & D^\top Q D - \beta \gamma^{-2} I & * & * \\ B^\top Q A & B^\top Q & B^\top Q D & -\varepsilon Q & * \\ k^\top & 0 & 0 & 0 & -\varepsilon^{-1} I \end{pmatrix} \preceq 0$$

or

$$\begin{pmatrix} -\alpha Q & 0 & 0 & 0 & k \\ 0 & -Q & 0 & Q B & 0 \\ 0 & 0 & -\beta \gamma^{-2} I & 0 & 0 \\ 0 & B^\top Q & 0 & -\varepsilon Q & 0 \\ k^\top & 0 & 0 & 0 & -\varepsilon^{-1} I \end{pmatrix} + \begin{pmatrix} A^\top \\ 0 \\ D^\top \\ B^\top \\ 0 \end{pmatrix} Q \begin{pmatrix} A & 0 & D & B & 0 \end{pmatrix} \preceq 0.$$

Using the Schur complement, see Horn and Johnson (1985) we obtain

$$\begin{pmatrix} -\alpha Q & 0 & 0 & 0 & k & A^\top \\ 0 & -Q & 0 & Q B & 0 & 0 \\ 0 & 0 & -\beta \gamma^{-2} I & 0 & 0 & D^\top \\ 0 & B^\top Q & 0 & -\varepsilon Q & 0 & B^\top \\ k^\top & 0 & 0 & 0 & -\varepsilon^{-1} I & 0 \\ A & 0 & D & B & 0 & -Q^{-1} \end{pmatrix} \preceq 0. \tag{7}$$

We transform inequality (7) to another form, more convenient for design problem. Namely, defining

$$P = Q^{-1} \succ 0,$$

and pre- and post-multiplying inequality (7) by the matrix

$$\begin{pmatrix} P & 0 & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma I & 0 & 0 & 0 \\ 0 & 0 & 0 & P & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix},$$

we arrive at

$$\begin{pmatrix} -\alpha P & 0 & 0 & 0 & Pk & PA^\top \\ 0 & -P & 0 & BP & 0 & 0 \\ 0 & 0 & -\beta I & 0 & 0 & \gamma D^\top \\ 0 & PB^\top & 0 & -\varepsilon P & 0 & PB^\top \\ k^\top P & 0 & 0 & 0 & -\varepsilon^{-1} I & 0 \\ AP & 0 & \gamma D & BP & 0 & -P \end{pmatrix} \preceq 0.$$

Finally, for simplicity we can eliminate parameter β by letting

$$\beta = \beta_{\max} = 1 - \alpha,$$

see Boyd et al. (1994) for the details. Theorem is proved.

It is natural to maximize the stabilizability ellipsoid via one or another criterion. Specifically, the maximization of its volume leads to the following corollary of Theorem 1.

Corollary 2. Let $\widehat{P}$ be a solution of the convex optimization problem

$$\max \log \det P$$

subject to the constraints

$$\begin{pmatrix} -\alpha P & 0 & 0 & 0 & Pk & PA^\top \\ 0 & -P & 0 & BP & 0 & 0 \\ 0 & 0 & -(1-\alpha) I & 0 & 0 & \gamma D^\top \\ 0 & PB^\top & 0 & -\varepsilon P & 0 & PB^\top \\ k^\top P & 0 & 0 & 0 & -\varepsilon^{-1} I & 0 \\ AP & 0 & \gamma D & BP & 0 & -P \end{pmatrix} \preceq 0,$$

$$P \succ 0,$$

where the maximization is performed in the matrix variable $P = P^\top \in \mathbb{R}^{n\times n}$ and the scalar parameters ε and α.

Then

$$\widehat{\mathcal{E}} = \left\{x \in \mathbb{R}^n : \quad x^\top \widehat{P}^{-1} x \le 1\right\}$$

is a stabilizability ellipsoid for system (5), (2).

5. DESIGN PROBLEM

We turn back to the bilinear control system (1), (2). With linear state feedback (4), the bilinear system takes the form

$$x_{\ell+1} = (A_c + Bx_\ell k^\top)x_\ell + Dw_\ell,$$

where $A_c = A + bk^\top$.

By Theorem 1, we arrive at the matrix inequality

$$\begin{pmatrix} -\alpha P & 0 & 0 & 0 & Pk & PA_c^\top \\ 0 & -P & 0 & BP & 0 & 0 \\ 0 & 0 & -(1-\alpha) I & 0 & 0 & \gamma D^\top \\ 0 & PB^\top & 0 & -\varepsilon P & 0 & PB^\top \\ k^\top P & 0 & 0 & 0 & -\varepsilon^{-1} I & 0 \\ A_c P & 0 & \gamma D & BP & 0 & -P \end{pmatrix} \preceq 0$$

or

$$\begin{pmatrix} -\alpha P & * & * & * & * & * \\ 0 & -P & * & * & * & * \\ 0 & 0 & -(1-\alpha) I & * & * & * \\ 0 & PB^\top & 0 & -\varepsilon P & * & * \\ k^\top P & 0 & 0 & 0 & -\varepsilon^{-1} I & * \\ (A + bk^\top)P & 0 & \gamma D & BP & 0 & -P \end{pmatrix} \preceq 0.$$

Now introduce an auxiliary vector variable

$$y = Pk \in \mathbb{R}^n,$$

to get rid of the variable k. Since $P \succ 0$, the vector k can be uniquely recovered as

$$k = P^{-1} y.$$

As a result, we arrive at the linear matrix inequality

$$\begin{pmatrix} -\alpha P & * & * & * & * & * \\ 0 & -P & * & * & * & * \\ 0 & 0 & -(1-\alpha) I & * & * & * \\ 0 & PB^\top & 0 & -\varepsilon P & * & * \\ y^\top & 0 & 0 & 0 & -\varepsilon^{-1} I & * \\ AP + by^\top & 0 & \gamma D & BP & 0 & -P \end{pmatrix} \preceq 0$$

in the matrix variable P, vector variable y, with the scalar parameters ε and α.

Hence, we obtained the following result.

Theorem 2. Let a matrix P and a vector y satisfy the matrix inequalities

$$\begin{pmatrix} -\alpha P & * & * & * & * & * \\ 0 & -P & * & * & * & * \\ 0 & 0 & -(1-\alpha) I & * & * & * \\ 0 & PB^\top & 0 & -\varepsilon P & * & * \\ y^\top & 0 & 0 & 0 & -\varepsilon^{-1} I & * \\ AP + by^\top & 0 & \gamma D & BP & 0 & -P \end{pmatrix} \preceq 0,$$

$$P \succ 0,$$

for a certain values of the scalar parameters ε and α, and given bound γ.

Then the linear feedback (4) with gain matrix

$$k = P^{-1} y$$

stabilizes system (1) inside the ellipsoid

$$\mathcal{E} = \{x \in \mathbb{R}^n : \quad x^\top P^{-1} x \le 1\}$$

for all admissible disturbances (2).

It is natural to maximize the stabilizability ellipsoid via one or another performance index, say by maximizing its volume. This leads to the following corollary.

Corollary 3. Let $\widehat{P}$, $\widehat{y}$ be the solution of the convex optimization problem

$$\max \log \det P$$

subject to the constraints

$$\begin{pmatrix} -\alpha P & * & * & * & * & * \\ 0 & -P & * & * & * & * \\ 0 & 0 & -(1-\alpha) I & * & * & * \\ 0 & PB^\top & 0 & -\varepsilon P & * & * \\ y^\top & 0 & 0 & 0 & -\varepsilon^{-1} I & * \\ AP + by^\top & 0 & \gamma D & BP & 0 & -P \end{pmatrix} \preceq 0, \tag{8}$$

$$P \succ 0,$$

with respect to the matrix variable $P = P^\top \in \mathbb{R}^{n\times n}$, vector variable $y \in \mathbb{R}^n$, and the scalar parameters ε and α.

Then the set

$$\widehat{\mathcal{E}} = \left\{x \in \mathbb{R}^n : \quad x^\top \widehat{P}^{-1} x \le 1\right\}$$

is a stabilizability ellipsoid for system (1) embraced with the linear feedback (4) with gain matrix

$$\widehat{k} = \widehat{P}^{-1} \widehat{y}$$

for all admissible disturbances (2).

6. STABILIZABILITY DOMAIN

In the previous section, the volume-maximizing stabilizability ellipsoid $\mathcal{E}$ was found for system (1), (2). Obviously, there exist other stabilizability ellipsoids, in particular, those optimal with respect to other criteria. Let us consider the set composed as the union of stabilizability ellipsoids; we will refer to it as the *stabilizability domain* of the system (1), (2). Clearly, by construction, this stabilizability domain possesses the same property as each of the individual constituent stabilizability ellipsoids; i.e., at every time instant, any trajectory emanating from any point x_0 in this domain remains inside this domain for all admissible exogenous disturbances.

Given an arbitrary vector c, an efficient procedure for finding a point on the boundary of the stabilizability domain of the considered system in the direction c can be proposed. Indeed, let us pick a direction defined by a unit-length vector c and require that a point ρc belongs to a stabilizability ellipsoid. By Schur complement, the condition that the point ρc belongs to the ellipsoid with matrix P has the form

$$\begin{pmatrix} 1 & \rho c^\top \\ \rho c & P \end{pmatrix} \succeq 0.$$

Appending this matrix inequality to the constraint (8), and maximizing over them the parameter ρ we provide a simple characterization of stabilizability domain of the considered discrete-time bilinear system subjected to exogenous disturbances.

7. CONCLUSIONS AND FURTHER RESEARCH

The approach proposed in this paper is easily implementable from the computational point of view; it allows for efficient construction of stabilizability ellipsoids and stabilizability domains for the discrete-time bilinear systems subjected to bounded exogenous disturbances via use of linear static state feedback.

This approach can be extended to the various robust statements of the problem, and to the case of multidimensional control input.

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