On the Selection of Lambda in Lambda Tuning for PI(D) Controllers

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Abstract: Lambda tuning is frequently employed for PI(D) controllers in the process industry because of its simplicity and intuitiveness for the user. In this paper we analyze the fundamental choice of the tuning parameter λ, that is, of the desired closed-loop time constant, by considering different trade-offs that can be posed by the user in the controller design. In particular, we consider the trade-off between bandwidth and gain or phase margin and that between performance and maximum sensitivity. The achievement of a specified maximum sensitivity is also addressed. Simple analytical formulas are determined so that they can be used in order to provide an optimized tuning in different contexts.

Keywords: PID controllers, Tuning, Optimization.

1. INTRODUCTION

Proportional-Integral-Derivative (PID) controllers are widely used in the process industry because of their ability to provide a satisfactory performance in spite of their relative simplicity (Åström and Hägglund, 2006). Indeed, the cost/benefit ratio they ensure is difficult to be improved by other more sophisticated control algorithms. Actually, the great majority of the controllers employed in practice is of PI type, as the derivative action is often switched off in order to avoid problems related to the amplification of noise and, in general, to the increased complexity in the overall controller design (Visioli, 2006).

Many tuning rules have been devised in the last century in order to simplify the task required by the user to determine the most suitable PI(D) parameters for a given application (O’Dwyer, 2006). Between the many different available techniques, one of the most employed in industry is surely the so-called λ-tuning (see, for example, (McMillan, 1999; Rice, 2010)). In fact, despite the method is based on pole-zero cancellation and therefore it is not suitable for the load disturbance rejection task in lag-dominant processes (Shinskey, 1994), its ease of use and its capability to set the speed of response of the control loop in a very intuitive way makes it a preferred choice for the user in many situations (Olsen and Bielkowsi, 2002).

Basically, λ-tuning, as in the Internal Model Control (IMC) approach (Morari and Zafiriou, 1989; Rivera et al., 1986), consists in cancelling the dominant pole(s) of the process dynamics so that the closed-loop time constant λ can be arbitrarily specified. In other words, starting from a simple first-order-plus-dead-time (FOPDT) process model, the PI parameters are automatically determined by using very simple formulas (see Section 2) once the desired closed-loop time constant λ is selected by the user. It clearly appears that the overall design effort is therefore restricted to the suitable choice of λ, by taking into account that its value handles the trade-off between aggressiveness and robustness (and control effort) of the control system. This can be indeed a critical issue and methods currently used in industry links the appropriate value of λ to the process parameters. For example, in (Coughran, 2013) λ has to be selected as three times the dead time of the process, while in (Smuts, 2011), it is suggested that its value is less than three times the value of the dominant time constant of the process. The well-known SIMC tuning rules proposed in (Skogestad, 2003) are based on selecting a value of λ equal to the dead time of the process. Software tools are also available to help the user to effectively accomplish this task (see, for example, Figure 1 where a practical probe is available to help the user to select the most suitable value of λ based on an estimated process model).

In this paper we provide different rules for the choice of λ by considering different requirements made by the user. In fact, it has to be taken into account that a (possibly skilled) user can be more familiar with some control concept (for example: bandwidth, phase margin, maximum sensitivity, etc.) than other ones and he/she may want to exploit these concepts in selecting the optimized value of λ for a given application. In particular, we consider different trade-offs between performance and robustness measures that typically arise in industrial applications. The devised rules are mostly based on the exploitation of an analytical approach than can be applied thanks to the simplicity of the mathematical expressions obtained by means of the pole-zero cancellation.

The paper is organized as follows. The problem is formulated in Section 2, where the λ-tuning approach is also briefly reviewed mainly for the purpose of introducing the notation. The different rules for the selection of λ are presented in Section 3. Worked examples are given in Section 4 and conclusions are drawn in Section 5.
2. PROBLEM FORMULATION

We consider the standard unity-feedback control scheme of Figure 2. As it is typical for industrial plants, the process dynamics is described by a FOPDT model as

\[ P(s) = \frac{\mu}{Ts + 1} e^{-\theta s} \]  \hspace{1cm} (1)

where \( \mu \) is the gain, \( T \) is the (dominant) time constant and \( \theta \) is the (apparent) dead time of the process. The controller is chosen as a PI controller whose transfer function is

\[ C(s) = K_p \left( 1 + \frac{1}{T_s} \right) \]  \hspace{1cm} (2)

where \( K_p \) is the proportional gain and \( T_i \) is the integral time constant. The \( \lambda \) tuning rules consists in selecting the PI parameters as

\[ K_p = \frac{\tau}{\mu(\theta + \lambda)} \]

\[ T_i = \tau \]  \hspace{1cm} (3)

In this way, the loop transfer function \( L(s) := C(s)P(s) \) results in

\[ L(s) = \frac{e^{-\theta s}}{(\theta + \lambda)s} \]  \hspace{1cm} (4)

This implies that the closed-loop transfer function is

\[ F(s) = \frac{L(s)}{1 + L(s)} = \frac{e^{-\theta s}}{(\theta + \lambda)s + e^{-\theta s}} \]  \hspace{1cm} (5)

By considering the approximation (to be used in the denominator of (5))

\[ e^{-\theta s} \approx 1 - \theta s \]  \hspace{1cm} (6)

we obtain

\[ F(s) \approx \frac{e^{-\theta s}}{\lambda s + 1} \]  \hspace{1cm} (7)

It turns out that the only design parameter \( \lambda \) represents the desired closed-loop time constant, which has to be suitably selected depending on the given control requirements and on the required robustness. In fact, the process has unavoidable uncertainties and, in any case, the achieved performance is never exactly as expected because of the approximation (6). In the next section we give some methodologies for the sound selection of \( \lambda \), where the previous approximation is not used.

3. TUNING RULES

3.1 Tuning to avoid overshoot and for minimum IAE

Although, in principle, the set-point step response of system (7) has no overshoot, it has to be taken into account that the approximation (6) has been employed so that in practical cases there is an overshoot for small values of \( \lambda \). In (Guzman et al., 2015) it has been determined that, by approximating \( e^{-\theta s} \) with the first-order Padé approximation \( (1 - s \frac{\lambda}{2})/(1 + s \frac{\lambda}{2}) \), the denominator of the closed-loop transfer function does not exhibit complex poles if \( \lambda \geq \left( \frac{1}{2} + \sqrt{2} \right) \theta \). The absence of overshoot implies also that the integrated absolute error (IAE) is equal to the integrated error (IE), that is
By imposing different values of the ratio \( \frac{a}{\theta} \), the values of IAE and IE can be evaluated numerically for acceptable values for the gain crossover frequency and for the phase margin, the Nash point can be determined as the optimal trade-off between the gain crossover frequency and gain margin (we denote them as \( \omega_c^x \) and \( \kappa_m^x \), respectively, as they represent the utopia point), then we have
\[
\omega_c = \omega_c^x := \frac{1}{2} \sqrt{\frac{2\pi \omega_c^d}{\theta \kappa_m^d}} \quad \kappa_m = \kappa_m^x := \frac{1}{2} \sqrt{\frac{2\pi \kappa_m^d}{\theta \omega_c^d}} \quad \lambda = \lambda_o := \sqrt{\frac{2\theta \kappa_m^d}{\pi \omega_c^d}} \quad \theta
\]

If, on the contrary, the user prefers to specify the desired gain crossover frequency and gain margin (we denote them as \( \omega_c^v \) and \( \kappa_m^v \), respectively), then we have
\[
\omega_c = \omega_c^v := \frac{1}{2} \sqrt{\frac{2\pi \omega_c^v}{\theta \kappa_m^v}} \quad \kappa_m = \kappa_m^v := \frac{1}{2} \sqrt{\frac{2\pi \kappa_m^v}{\theta \omega_c^v}} \quad \lambda = \lambda_o := \sqrt{\frac{2\theta \kappa_m^v}{\pi \omega_c^v}} \quad \theta
\]

Thus, by replacing (16) in (15), we have
\[
\omega_c = \omega_c^v := \frac{1}{2} \sqrt{\frac{2\pi \omega_c^v}{\theta \kappa_m^v}} \quad \kappa_m = \kappa_m^v := \frac{1}{2} \sqrt{\frac{2\pi \kappa_m^v}{\theta \omega_c^v}} \quad \lambda = \lambda_o := \sqrt{\frac{2\theta \kappa_m^v}{\pi \omega_c^v}} \quad \theta
\]

A similar reasoning can be applied by considering the trade-off between the gain crossover frequency and the phase margin. In this case the relationship is linear and expressed by (10).

By denoting as \( \varphi_m^{\varphi_m} \) the minimum acceptable value of the phase margin, the Nash point is determined as:
\[
\omega_c = \omega_c^\varphi_m := \frac{\pi - 2\varphi_m^{\varphi_m} + 2\theta \omega_c^d}{4\theta} \quad \varphi_m = \varphi_m^{\varphi_m} := \frac{\pi + 2\varphi_m^{\varphi_m} - 2\theta \omega_c^d}{4} \quad \lambda = \lambda_o := \frac{4\theta}{\pi + 2\varphi_m^{\varphi_m} - 2\theta \omega_c^d - \theta}
\]

If the user may specifies his desired phase margin \( \varphi_m^{\varphi_m} \) together with the desired value of \( \omega_c^\varphi_m \), then we have

\[
\omega_c^\varphi_m = \frac{\varphi_m - \varphi_m^{\varphi_m}}{\theta} \quad \varphi_m^\varphi_m = \frac{\pi}{2} - \theta \omega_c^\varphi_m
\]

and, by replacing (19) in (18), we have

\[
A = \left( \omega_c - \omega_c^d \right) \left( \kappa_m - \kappa_m^d \right) = \left( \omega_c - \omega_c^v \right) \left( \frac{\pi}{2\theta \omega_c} - \kappa_m^d \right)
\]

and, by solving
\[
\frac{dA}{d\omega_c} = \frac{\pi}{2\theta \omega_c} - \kappa_m^d = \frac{\left( \omega_c - \omega_c^d \right)}{2\theta \omega_c} \quad 0
\]
A Nash point is selected considering the trade-off between the gain crossover frequency and the phase margin.

\[
\omega_c = \omega^\nu := \frac{\pi - 2\varphi_m^\mu + 2\theta \omega^\nu}{4\theta}
\]

\[
\varphi_m = \varphi_m^\mu := \frac{\pi + 2\varphi_m^\mu - 2\theta \omega^\mu}{4}
\]

\[
\lambda = \lambda_0 := \frac{4\theta}{\pi + 2\varphi_m^\mu - 2\theta \omega^\mu} - \theta
\]

### 3.3 Tuning for specified maximum sensitivity

A typical approach for the tuning of a PID controller is to consider the resulting maximum sensitivity as a measure of the system robustness (see, for example, (Padula and Visioli, 2011; Åström et al., 2015; Alfaro and Vilanova, 2016)). In \(\lambda\)-tuning this can be relevant, for example, in order to compensate for an imperfect pole-zero cancellation or an imperfect dead time estimation (as uncertainties in the process parameters cannot be avoided in practice). In this context, defining the maximum sensitivity as

\[
M_s := \max_w \left| \frac{1}{1 + L(j\omega)} \right|
\]

its value can be determined numerically for different values of \(\alpha\) (see Figure 5). Then, a fitting function can be determined in order to obtain a relationship between \(\alpha\) and \(M_s\). It results:

\[
M_s = \frac{\alpha + 1.2256}{\alpha + 0.3943}
\]

Therefore, once a desired value of \(M_s\) has been selected by the user, then the corresponding value of \(\lambda\) can be obtained as:

\[
\alpha = \frac{1.2256 - 0.3943M_s}{M_s - 1}
\]

### 3.4 Trade-off between performance and maximum sensitivity

In Veronesi and Visioli (2009, 2010), the set-point following performance of a PID controller is assessed by comparing the obtained integrated absolute error with a benchmark one calculated as \(2A_s\theta\), that is, the one given by the response of (7), where \(\lambda = \theta\) as suggested in (Skogestad, 2003), to a set-point signal of amplitude \(A_s\).

![Fig. 4. Illustrative picture regarding the selection of the Nash point given the disagreement or the utopia point considering the trade-off between the gain crossover frequency and the phase margin.](image)

![Fig. 5. Values of \(M_s\) for different values of \(\alpha\) (black stars) and the determined fitting function (solid line).](image)

![Fig. 6. Minimization of the cost function \(J\). The dashed straight lines represent the points corresponding to constant values of \(J\) (the solid line corresponds to the optimal value). Values of \((\text{IAE}, M_s)\) for different values of \(\alpha\) are shown with the symbol ‘*’.](image)

![Fig. 7. Values of \(\alpha\) (black stars) for different values of \(w\) and the determined fitting function (solid line).](image)
It is therefore sensible to consider the trade-off between the performance assessed in this way and the achieved robustness, which can be considered as the ratio between the value of $M_I$ obtained with a given value of $\lambda$ and a target one equal to 1.2. Thus, the following new performance index can be minimized:

$$J = \frac{w IAE}{2 \lambda \theta} + (1 - w) \frac{M_I}{1.2}$$  \hspace{1cm} (24)

where the user can select a value of $w$ between 0 and 1 in order to suitably weight the two terms. It is worth noting that, in the plane $M_I-IAE$, the considered cost function is represented by straight lines corresponding to constant values of $J$ and, of course, the bigger is the value of $J$ the worse is the overall performance. On the other hand, for a given value of $w$, the trade-off between $IAE$ and $M_I$ can be obtained by simulation for different values of $\alpha$ and drawn as a curve in the same plane. Since the overall cost function has to be minimized, the best achievable point achievable by $\lambda$-tuning is the one where that line is tangent to the constraint represented by the trade-off: the situation is shown in Figure 6, where the slope of the straight lines depends on the value of $w$.

By fitting the optimal values of $\alpha$ obtained for different values of $w$ (see Figure 7), we obtain:

$$\alpha = \frac{0.7375}{(w - 0.0673)^{0.4903}}$$ \hspace{1cm} (25)

### 4. ILLUSTRATIVE EXAMPLES

Some results are shown hereafter only with the aim to illustrate the use of the devised formulas, rather than showing the effectiveness of the $\lambda$-tuning and the role of $\alpha$ in the performance, as these are already well known. For this reason, we focus only of the set-point step response, even if load disturbance rejection is also relevant in general and has not to be neglected. We consider the process with $\mu = 1$, $\tau = 4$, and $\theta = 1$, that is,

$$P(s) = \frac{1}{4s + 1}e^{-s}$$ \hspace{1cm} (26)

Note that, in the process model, only the the dead time is relevant in the set-point step response. However, the PI controller, and therefore also the control variable, depends also on the process time constant and on the process gain.

#### 4.1 Example 1

As a first case we consider the trade-off between the bandwidth (that is, the gain crossover frequency), and the gain margin. By setting the minimum acceptable values (that is, the disagreement point) as $\omega_m^c = 0.1/\theta$ and $\delta_m^c = 1.1$, by applying (15) we obtain $\omega_m^c = 0.3779$, $\delta_m^c = 4.1568$, $\lambda_o = 1.6463$. This gives $K_p = 1.5115$ and $T_i = 4$. On the other hand, if we set the desirable values of the gain crossover frequency and of the gain margin (utopia point) as $\omega_m^c = 1/\theta$ and $\delta_m^c = 4$, it results (see (17)) $\omega_m^c = 0.6267$, $\delta_m^c = 2.5066$, and $\lambda_o = 0.5958$, which gives $K_p = 2.5066$ and $T_i = 4$. The set-point unit step responses with these two tunings are shown for the sake of completeness in Figure 8 where also the minimum $IAE$ case ($\lambda = 0.7$, which gives $K_p = 2.3529$ and $T_i = 4$) is plotted. Of course, as it is well known, results confirm that a smaller value of $\lambda$ implies a smaller rise time but also a bigger control effort and a larger overshoot.

#### 4.2 Example 2

In the second example we consider the trade-off between the gain crossover frequency, and the phase margin. By setting the minimum acceptable values as $\omega_m^c = 0.1/\theta$ and $\phi_m^c = \pi/6$, by applying (18) we obtain $\omega_m^c = 0.5736$, $\phi_m^c = 0.9972$, and $\lambda_o = 0.7434$. This gives $K_p = 2.2943$ and $T_i = 4$. Alternatively, if we set the desirable values of the gain crossover frequency and of the phase margin as $\omega_m^c = 0.75/\theta$ and $\phi_m^c = \pi/3$, it results (see (20)) $\omega_m^c = 0.6368$, $\phi_m^c = 0.934$, and $\lambda_o = 0.570$, which gives a value of the proportional gain $K_p = 2.547$. The set-point unit step responses in the two cases are shown in Figure 9.

#### 4.3 Example 3

In the third example we consider the tuning based on a specification of a desired maximum sensitivity. By setting the maximum sensitivity as $M_I = 1.2$, $M_t = 1.6$, and $M_o = 2$ (which are typical values), we obtain through (23) the values $\lambda = 3.7622$, $\lambda = 0.9912$, and $\lambda = 0.437$ respectively. The corresponding values of the proportional gain of the PI controller are, respec-
the devised formulas are very useful because they can be used by users with different skills and different knowledge of the given industrial process. It is therefore believed that they can be exploited to optimize the performance and to obtain a fast commissioning of the control loop, which is one of the main features to improve the cost/benefit ratio of PID controllers.

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REFERENCES


