An Auto-tuning LQR based on Correlation Analysis

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Abstract: In this paper, we present an auto-tuning method for Linear Quadratic Regulator (LQR) based on correlation analysis. Unlike previous studies which focused on LQR tuning strategies exclusively by evaluating the control performance, we propose to explore the explicit relationship between the model and weighting parameters in LQR. The objective of this paper is twofold: (1) we introduce an approach to the identification and quantification of the correlation between a model parameter and a weighting parameter in LQR; (2) an auto-tuning method is worked out which is explicitly related to the variation of the model parameter. As a result, an optimal value of the weighting parameter can be effectively determined and, in the meantime, the parameter variation estimated. Through the numerical example, we demonstrate the effectiveness of the proposed auto-tuning method in restoring the control performance under unknown parameter variations.

Keywords: LQR, optimal control, auto-tuning, correlation analysis

1. INTRODUCTION

The primary purpose of the Linear Quadratic Regulator (LQR) is to derive an optimal feedback controller that manipulates the system at a minimum cost according to user preferences. The feedback gain matrix of LQR is easily determined if an accurate dynamic model can be available. The feedback gain is considered to be suboptimal in the case of parametric uncertainties, such as payload variation or component aging (Clarke and Gawthrop (1979)). The existing tuning methods to address the variation of the model parameter are all based on the controller performance analysis. However, there is no available approach to tuning LQR by explicitly considering the relationship between the model parameters and the weighting parameters. Our purpose is to investigate these relationships by correlation analysis, such that the parameter tuning can be made explicitly related to parameter variations.

In general, the tuning algorithms for the weighting parameters in LQR can be categorized into two groups: indirect and direct approaches (Astrom and Wittenmark (1994)). The indirect approach consists of two stages (Clarke et al. (1985); Grimble (1984)), i.e., the model parameters are estimated at first and then the control parameters in the feedback gain matrix updated using a tuning algorithm. The direct approach in LQR mostly refers to the selection of the weighting parameters (Johnson and Grimble (1987)). A simple heuristic scheme is widely applied based on the inverse-square method (Bryson and Ho (1969)). These techniques offer considerable flexibility in system design but require intensive numerical or practical experiments to achieve the expected control performance.

In Trimpe et al. (2014), the authors proposed a gradient-based auto-tuning approach with a simultaneous perturbation stochastic approximation (Spall (2003)). The search for the weighting parameters is based on the symmetric Bernoulli distribution around their nominal value. This work was extended in Marco et al. (2016) by applying an entropy search (Hennig and Schuler (2012)) to reduce the experiment times. However, the search space grows dramatically when the number of weighting parameters increases.

The conventional correlation analysis is to identify the dependence between different data sets (Cohen et al. (2014)). In general, it refers to the degree of the linear relationship between two data sets. Recently, correlation analysis was applied to identify the relationships among model parameters of differential equation systems to determine their identifiability (Li and Vu (2013, 2015)). If there is a parameter correlation, the model is non-identifiable. In Lazutkin et al. (2015), the authors proposed to evaluate the degree of difficulty in solving dynamic optimization problems by analyzing correlations among control variables in the model. It is shown that the presence of correlated controls negatively affects the convergence rate of the numerical solution algorithm.

Inspired by these studies, in this paper, we intend to explore the correlations between the model and weighting parameters for developing an auto-tuning method for LQR. The purpose of our auto-tuning method is to restore the control performance of the system when the values of model parameters vary, by adjusting the weighting parameters based on the identified relationship.

In short, our approach can restore the control performance effectively by extracting two pieces of the essential information from the correlation analysis. First, a correlation coefficient is introduced to describe the correlation level

between a model parameter and a weighting parameter. In this way, if multiple weighting parameters are correlated with one model parameter, the weighting parameter with the most significant correlation can be chosen to compensate the system deviation. Second, a tuning coefficient can be determined, which relates the weighting parameter explicitly to the model parameter variation.

The proposed auto-tuning method is implemented in the form of a compensator to the real system. The objective of the auto-tuning is to minimize the performance difference between the compensator and the real system. The correlated parameters (i.e., a model parameter and a weighting parameter) are updated based on the tuning coefficient. When the computation converges, the solution will provide the estimated variation of the model parameter as well as the optimal value of the weighting parameter.

The paper is organized as follow: In Section 2, we present the problem statement and corresponding definitions on pairwise parameter correlations. In Section 3, a method for the parameter correlation analysis in LQR is introduced and its properties are discussed. In Section 4, an auto-tuning method is proposed based on the pairwise correlation analysis. In Section 5, numerical results are illustrated and discussed. The conclusions and future work are given in Section 6.

2. PROBLEM FORMULATION

Consider a linear time-invariant (LTI) system described with the following state space model

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector and $u(t) \in \mathbb{R}^{n_u}$ the control vector. $x(0) = x_0$ is the initial state. $A \in \mathbb{R}^{n_x \times n_x}, B \in \mathbb{R}^{n_x \times n_u}$ are the system and control matrices consisting of constant but uncertain model parameters, i.e., the magnitude of their variations is unknown a priori.

For an infinite-time horizon LQR, the performance index to be minimized is defined in the quadratic form:

$$J = \int_0^\infty (x^\top Q x + u^\top R u) dt \qquad (2)$$

where $Q \in \mathbb{R}^{n_x \times n_x}$ is positive semi-definite and $R \in \mathbb{R}^{n_u \times n_u}$ is positive definite. We assume that (A, B) is controllable and (A, C) with $Q = C^\top C$ is observable. In this study, we will tune the weighting parameters in these two matrices to compensate the effect of variations of the model parameters in (A, B) on the control performance.

The optimal control law derived in LQR is given as

$$u(t) = -Kx(t) \qquad (3)$$

where $K = R^{-1}B^\top P \in \mathbb{R}^{n_u \times n_x}$ is the feedback gain matrix and the matrix $P = P^\top > 0$ is the unique stabilizing solution of the algebraic Riccati equation (ARE).

$$A^\top P + PA - PBR^{-1}B^\top P = 0 \qquad (4)$$

We partition the set of matrices into two groups: the system and control matrices (A, B) contain n_v model parameters denoted as $\theta_V \in \mathbb{R}^{n_v}$ and the weighting matrices (Q, R) contain n_w weighting parameters denoted as $\theta_W \in \mathbb{R}^{n_w}$, respectively. Then, the whole parameter vector in LQR is

$$\theta = (\theta_V, \ \theta_W) \in \mathbb{R}^{n_r}, \ \text{for } n_r = n_v + v_w \qquad (5)$$

The basic idea of this study is to analyze the relationships (i.e., the correlations) between the model parameters θ_V and the weighting parameters θ_W and, based on which, to adjust the weighting parameters so as to compensate the negative effect of the varying model parameters on the control performance.

To investigate pairwise correlations, a Cartesian product is constructed as the set of all parameter pairs (θ_i, θ_j) for $\theta_i \in \theta_V$ and $\theta_j \in \theta_W$ and given as

$$\mathbb{C} = \theta_V \times \theta_W = \{(\theta_i, \theta_j) | \theta_i \in \theta_V \text{ and } \theta_j \in \theta_W\} \qquad (6)$$

First, we define some assumptions about the nominal parameter values and their variations.

A1: The nominal parameter values $\theta^0 \in \mathbb{R}^{n_r}$ and the corresponding nominal state vector $x(t; \theta^0)$ are known.
A2: The parameter variation $\Delta\theta$ is in a small neighborhood of the nominal value.

If the parameter vector changes from its nominal value θ^0 to $\theta^0 + \Delta\theta$, the deviated state vector is then denoted as $x(t; \theta^0 + \Delta\theta)$ where $\Delta\theta \in \mathbb{R}^{n_r}$ is the vector of parameter variation. Now, we define a measure to evaluate the control performance as the cost of the state trajectory deviation due to the parameter variation. The cost of the state trajectory deviation is formulated as follows:

$$D(\Delta\theta) = \sum_{l=1}^{n_x} \int_{t_0}^{t_f} \left(x_l(t; \theta^0 + \Delta\theta) - x_l(t; \theta^0) \right)^2 dt \qquad (7)$$

where t_0 and t_f are the initial and terminal time points, respectively. For the tuning purpose, the terminal time t_f should be large enough so that the state vector is at a steady state at this time point.

Consider that the negative effect on the control performance is caused by one of the model parameters $\theta_i \in \theta_V$ which varies with $\Delta\theta_i$, while the other parameters remain unchanged. The parameter variation is denoted as

$$\Delta\theta_i^p = [0, \ldots, \Delta\theta_i, \ldots, 0]$$

for $i = 1, \ldots, n_v$. The negative effect caused by $\Delta\theta_i^p$ is represented as the cost of the state trajectory deviation in (7), and denoted as $D(\Delta\theta_i^p)$.

To compensate this effect, one of the weighting parameters $\theta_j \in \theta_W$ is to be adjusted with $\Delta\theta_j$. The parameter vector involving both the model and weighting parameter variations is now denoted as

$$\Delta\theta_{i,j}^c = [0, \ldots, \Delta\theta_i, \ldots, \Delta\theta_j, \ldots, 0]$$

for $i = 1, \ldots, n_v$ and $j = n_v + 1, \ldots, n_r$. The corresponding cost of the state trajectory deviation is $D(\Delta\theta_{i,j}^c)$. Note that if $\Delta\theta_j = 0$, then $D(\Delta\theta_{i,j}^c) = D(\Delta\theta_i^p)$.

Next, we define the compensation effect in terms of the cost of the state trajectory deviation as follows

$$e_{i,j} = D(\Delta\theta_i^p) - D(\Delta\theta_{i,j}^c) \qquad (8)$$

In this study, we are interested in the pairwise correlation in terms of the compensation effect. Given the parameter vector (5), three types of correlations are defined as follows.

A pair of parameters $(\theta_i, \theta_j) \in \mathbb{C}$ is *correlated* if there exists at least one $\Delta\theta_j$ such that the compensation effect $e_{i,j}$ is larger than zero. The set of correlated parameter pairs is denoted as

$$\mathcal{R}_{corr} = \{(\theta_i, \theta_j) \in \mathbb{C} | \exists \Delta\theta_j \text{ such that } e_{i,j} > 0\} \quad (9)$$

Otherwise, θ_i and θ_j are said to be *uncorrelated* if $e_{i,j}$ is non-positive for all $\Delta\theta_j$. Therefore, the set of uncorrelated parameter pairs is a complement of $\mathcal{R}_{corr}$ in $\mathbb{C}$ and denoted as

$$\overline{\mathcal{R}}_{corr} = \{(\theta_i, \theta_j) \in \mathbb{C} | \forall \Delta\theta_j \text{ such that } e_{i,j} \leq 0\} \quad (10)$$

In addition, θ_i and θ_j are said to be *fully correlated* if there exists at least one $\Delta\theta_j$ such that $e_{i,j} = D(\Delta\theta_i^p)$ (i.e., $D(\Delta\theta_{i,j}^c) = 0$). The set of fully correlated parameter pairs is a subset of $\mathcal{R}_{corr}$ and denoted as

$$\mathcal{R}_{corr}^f = \{(\theta_i, \theta_j) \in \mathbb{C} | \exists \Delta\theta_j \text{ such that } e_{i,j} = D(\Delta\theta_i^p)\} \quad (11)$$

A correlation coefficient is introduced to describe the correlation level for a pair of parameters $(\theta_i, \theta_j) \in \mathbb{C}$, and denoted as $\rho_{i,j}$. In this way, if one of the model parameters is correlated to multiple weighting parameters, it is reasonable to choose the one which contributes the highest compensation effect. There is no difference to choose any of the weighting parameters with the identical compensation effects until we include further criteria.

For a pair of correlated parameters $(\theta_i, \theta_j) \in \mathcal{R}_{corr}$, we define a tuning coefficient $\alpha_{i,j}$ between $\Delta\theta_i$ and $\Delta\theta_j$, i.e., $\Delta\theta_j = \alpha_{i,j}\Delta\theta_i$. The purpose of the tuning coefficient is to determine the search direction for the optimal value of the weighting parameter that maximizes the compensation effect $e_{i,j}$.

3. CORRELATION ANALYSIS

3.1 First-order approximation

Based on the assumption A2, the state vector $x(t; \theta + \Delta\theta)$ can be approximated using a first-order Taylor expansion:

$$x(t; \theta^0 + \Delta\theta) \approx x(t; \theta^0) + \left.\frac{\partial x}{\partial \theta}\right|_{\theta^0} \Delta\theta \quad (12)$$

where $\partial x/\partial\theta|_{\theta^0} \in \mathbb{R}^{n_x \times n_r}$ is the parametric sensitivity matrix evaluated at the nominal parameter vector θ^0.

By substituting (12) into (7), the cost of the state trajectory deviation can be approximated as (Hearne (1985))

$$\tilde{D}(\Delta\theta) \approx \sum_{l=1}^{n_x} \int_{t_0}^{t_f} \left(x_l(t; \theta^0) + \frac{\partial x_l}{\partial \theta}\Delta\theta - x_l(t; \theta^0) \right)^2 dt$$

$$= \sum_{l=1}^{n_x} \int_{t_0}^{t_f} \left(\frac{\partial x_l}{\partial \theta}\Delta\theta \right)^2 dt$$

$$= \Delta\theta^\top \left\{ \int_{t_0}^{t_f} \left(\frac{\partial x}{\partial \theta}^\top \frac{\partial x}{\partial \theta} \right) dt \right\} \Delta\theta$$

$$= \Delta\theta^\top U \Delta\theta \quad (13)$$

In this way, the cost of the state trajectory deviation is approximated in a quadratic form with a symmetric matrix $U \in \mathbb{R}^{n_r \times n_r}$. Note that U is a Gram matrix of a set of parametric sensitivity functions $\partial x/\partial\theta_m$ for $m = 1, \ldots, n_r$ defined as follows (Sreeram and Agathoklis (1994))

$$U = [u_{mn}]$$
$$= \begin{pmatrix} <u_1, u_1> & <u_1, u_2> & \ldots & <u_1, u_{n_r}> \\ <u_1, u_2> & <u_2, u_2> & \ldots & <u_2, u_{n_r}> \\ \vdots & \vdots & & \vdots \\ <u_1, u_{n_r}> & <u_2, u_{n_r}> & \ldots & <u_{n_r}, u_{n_r}> \end{pmatrix} \quad (14)$$

where u_{mn} is the inner product of $\partial x/\partial\theta_m$ and $\partial x/\partial\theta_n$ for $m, n = 1, \ldots, n_r$ i.e.,

$$u_{mn} = <u_m, u_n> = \int_{t_0}^{t_f} \left(\frac{\partial x}{\partial \theta_m}^\top \frac{\partial x}{\partial \theta_n} \right) dt \quad (15)$$

The properties of the Gram matrix U are given as follows (Horn and Johnson (2012)):

(1) U is positive-semidefinite.
(2) The diagonal elements of U is positive.
(3) For each positive semi-definite sub-matrix

$$\tilde{U}_{m,n} = \begin{pmatrix} u_{mm} & u_{mn} \\ u_{mn} & u_{nn} \end{pmatrix} \quad (16)$$

we have

$$u_{mn} \leq \sqrt{u_{mm}u_{nn}} \quad (17)$$

3.2 Pairwise correlation analysis

To calculate the cost of the state trajectory deviation with respect to $\Delta\theta_i$ and $\Delta\theta_j$, a sub-matrix (totally $n_v \times n_w$) is extracted from (14) and given as

$$\tilde{U}_{i,j} = \begin{pmatrix} u_{ii} & u_{ij} \\ u_{ji} & u_{jj} \end{pmatrix} \quad (18)$$

Then, the approximated cost for $D(\Delta\theta_i^p)$ and $D(\Delta\theta_{i,j}^c)$ can be expressed as

$$D(\Delta\theta_i^p) \approx u_{ii}(\Delta\theta_i)^2, \quad (19a)$$

$$D(\Delta\theta_{i,j}^c) \approx u_{ii}(\Delta\theta_i)^2 + 2u_{ij}\Delta\theta_i\Delta\theta_j + u_{jj}(\Delta\theta_j)^2 \quad (19b)$$

By subtracting (19a) and (19b), the compensation effect $e_{i,j}$ in (8) is given as

$$e_{i,j} = D(\Delta\theta_i^p) - D(\Delta\theta_{i,j}^c)$$
$$\approx -\left(2u_{ij}\Delta\theta_i\Delta\theta_j + u_{jj}(\Delta\theta_j)^2 \right) \quad (20)$$

It can be seen that (20) is a quadratic function of $\Delta\theta_j$ in which the parabola opens downwards due to the nonnegative diagonal element u_{jj}, as shown in Fig. 1. The vertex of the parabola is at the point

$$\left(-\frac{u_{ij}}{u_{jj}}\Delta\theta_i, \frac{u_{ij}^2}{u_{jj}}\Delta\theta_i^2 \right) \quad (21)$$

where $-(u_{ij}^2/u_{jj})\Delta\theta_i^2$ indicates the maximum compensation effect. The slope of the quadratic function (20) at the origin (see Fig. 1b) is given as

$$\left.\frac{\partial e_{i,j}}{\partial \Delta\theta_j}\right|_{\Delta\theta_j=0} = -2u_{ij}\Delta\theta_i \quad (22)$$

which reflects the change rate of the compensation effect with respect to the weighting parameter θ_j.

Next, we identify the pairwise correlation (9-11) by analyzing the properties of the vertex from (21). Different situations of a pairwise correlation between two parameters can be schematically described by using Fig. 1. It is noted that the axis of symmetry of the parabola locates on the right plane if $-(u_{ij}/u_{jj})\Delta\theta_i > 0$ (solid lines) and on the left plane if $-(u_{ij}/u_{jj})\Delta\theta_i < 0$ (dashed lines).
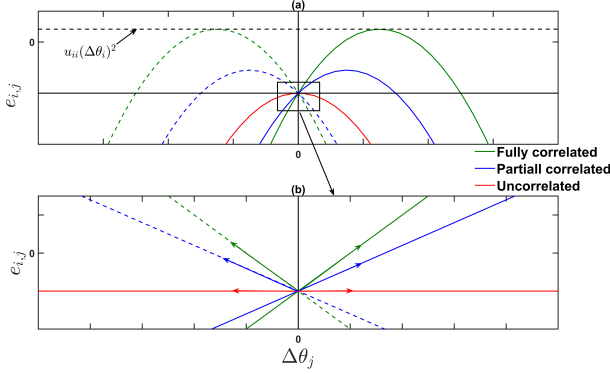
Fig. 1. Schematic graph of the three situations of a pairwise correlation. (a) The parabola describes the compensation effect $e_{i,j}$ which is a function of $\Delta\theta_j$. (b) Zoom of the part around the origin in (a) shows the slopes of the parabola with respect to different correlation situations.

Pairwise correlated/uncorrelated According to (9), if there exists at least one $\Delta\theta_j$ such that $e_{i,j} > 0$, then some part of the parabola (blue curve as shown in Fig. 1a) will be above the x-axis. Therefore, the solution $-(u_{ij}/u_{jj})\Delta\theta_i$ from (21) will be nonzero,

$$\frac{u_{ij}}{u_{jj}}\Delta\theta_i \neq 0 \qquad (23)$$

Since the diagonal element u_{jj} is positive, the following condition holds true

$$u_{ij} \neq 0 \qquad (24)$$

for a nonzero $\Delta\theta_i$. Then, the system deviation can be compensated by tuning $\Delta\theta_j$ with positive value if $-(u_{ij}/u_{jj})\Delta\theta_i > 0$ and negative value if $-(u_{ij}/u_{jj})\Delta\theta_i < 0$ (blue lines in Fig. 1b). Otherwise, the parabola (red curve in Fig. 1a) will locate below the x-axis, which means no $\Delta\theta_j$ exists such that $e_{i,j} > 0$ (red lines in Fig. 1b). In short, the condition of a pairwise correlation is as follows.

Given with the LQR described in (1) and (2) which contains a vector of parameters $\theta = (\theta_V, \theta_W)$, a pair of parameters $\theta_i \in \theta_V$ and $\theta_j \in \theta_W$ are *partially correlated*, i.e., $(\theta_i, \theta_j) \in \mathcal{R}_{corr}$ if the condition (24) is satisfied. Otherwise, $(\theta_i, \theta_j) \in \overline{\mathcal{R}}_{corr}$.

Pairwise fully correlated According to (11), if there exists at least one $\Delta\theta_j$ such that $e_{i,j} = D(\Delta\theta_i^p)$, then the parabola (green curve in Fig. 1a) intersects with the dashed line $e_{i,j} = u_{ii}(\Delta\theta_i)^2$. Then, the maximum compensation effect $(u_{ij}^2/u_{jj})\Delta\theta_i^2 = u_{ii}(\Delta\theta_i)^2$, i.e., $u_{ii}u_{jj} = u_{ij}^2$. In this situation, the effect of the model parameter variation is fully compensated (green lines in Fig. 1b). Therefore, the following condition holds true.

$$det(\tilde{U}_{i,j}) = 0 \qquad (25)$$

Thus, the condition of a fully correlated pair is as follows.

Given with the LQR described in (1) and (2) which contains a vector of parameters $\theta = (\theta_V, \theta_W)$, a pair of parameters $\theta_i \in \theta_V$ and $\theta_j \in \theta_W$ are *fully correlated*, i.e., $(\theta_i, \theta_j) \in \mathcal{R}_{corr}^f$ if the conditions (24) and (25) are satisfied.

Next, we define a correlation coefficient $\rho_{i,j}$ by normalizing $e_{i,j}$ with the cost of the state trajectory deviation
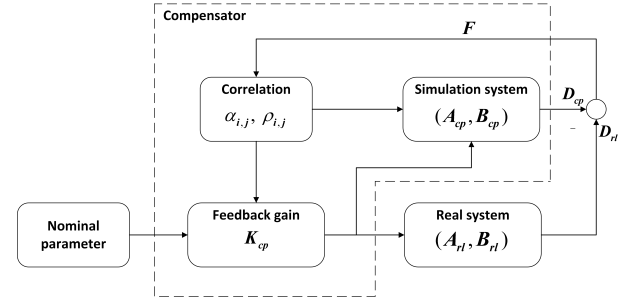

Fig. 2. Block diagram of the closed-loop system. The weighting parameters are updated based on the tuning coefficient.

$u_{ii}(\Delta\theta_i)^2$. Recall the expression of the compensation effect defined in (20), the range of $e_{i,j}$ is

$$0 \leq e_{i,j} \leq u_{ii}(\Delta\theta_i)^2 \qquad (26)$$

To bring these values into the range of $[0, 1]$, we divide (26) with $u_{ii}(\Delta\theta_i)^2$. Then $\rho_{i,j}$ is given as

$$\rho_{i,j} = \frac{e_{i,j}}{u_{ii}(\Delta\theta_i)^2} \qquad (27)$$

where $\rho_{i,j} = 1$ means that the parameter pair (θ_i, θ_j) are fully correlated and $\rho_{i,j} = 0$ uncorrelated.

As mentioned above, a tuning coefficient $\alpha_{i,j}$ is defined in terms of maximizing the compensation effect $e_{i,j}$. Thus, it is easy to see that $e_{i,j}$ reaches its maximum value at

$$\Delta\theta_j = -\frac{u_{ij}}{u_{jj}}\Delta\theta_i \qquad (28)$$

which provides the tuning coefficient stated as

$$\alpha_{i,j} = -\frac{u_{ij}}{u_{jj}} \qquad (29)$$

As a result, the following tuning method is used as a linear function whose coefficient is obtained from (29).

4. AUTO-TUNING METHOD

Our tuning approach is illustrated in Fig. 2. To carry out an iterative search scheme that converges to the optimal solution, we design a compensator next to the real system. Its goal is to reduce the performance difference between the compensator and the real system. Both share the same optimal controller but the real system is subjected to the unknown variations of the model parameter.

The dynamic model of the compensator and the real system is the same as described by (1), but the system and control matrices in the real system contain parameters with unknown variations denoted as (A_{rl}, B_{rl}). In the compensator, a LQR is implemented in which the system and weighting matrices are denoted as (A_{cp}, B_{cp}) and (Q_{cp}, R_{cp}), respectively. The feedback gain matrix in the shared optimal controller is denoted as K_{cp} and calculated based on (3). Those matrices are initialized with their nominal parameter values θ^0. The values of $\rho_{i,j}$ and $\alpha_{i,j}$ are stored in the correlation block which is used to update the correlated weighting parameter at each iteration. The costs of the state trajectory deviation in both the compensator and the real system are evaluated based on (7), denoted as D_{cp} and D_{rl}, respectively. The performance difference between the two systems is
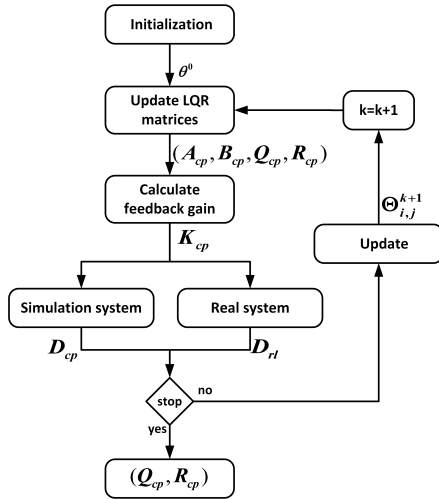
Fig. 3. Flow diagram of the auto-tuning method

measured as $F = |D_{cp} - D_{rl}|$, which is fed back to the compensator for the next iteration.

We refer $\Theta_{i,j}^* = (\Delta\theta_i^*, \Delta\theta_j^*)^\top$ as the optimal solution, where $\Delta\theta_i^*$ is the estimation on the unknown variation of θ_i in the matrices (A_{cp}, B_{cp}). $\Delta\theta_j^*$ is the amount of tuning in the weighting parameter in the matrices (Q_{cp}, R_{cp}) that minimizes the cost D_{cp}.

The objective of the iterative search scheme is to minimize the performance difference between the compensator and the real system as

$$\min_{\Theta_{i,j}} F(\Theta_{i,j}) \qquad (30)$$

The procedure of the proposed auto-tuning method is shown in Fig. 3. The iterative process begins with the nominal value (i.e., $\Theta_{i,j}^0 = (\Delta\theta_i^0, \Delta\theta_j^0)^\top = (0,0)^\top$). At each iteration k, the weighting parameter value is updated based on the tuning coefficient (i.e., $\Delta\theta_j^k = \alpha_{i,j}\Delta\theta_i^k$). As the search direction is fixed on the tuning coefficient $\alpha_{i,j}$ for all iterations, the auto-tuning process becomes a line search problem with the following update scheme:

$$\Theta_{i,j}^{k+1} = \Theta_{i,j}^k + \gamma d_{i,j} \qquad (31)$$

where $d_{i,j} = (1 \quad \alpha_{i,j})^\top$ gives the search direction and the γ is the step length. In this work, we calculate γ based on the $Armijo - Goldstein$ condition (Armijo (1966)). The process is iterated until the performance difference decreases under a user-defined tolerance, i.e., $F(\Theta_{i,j}^k) < \epsilon$.

The cost D_{cp} will be decreased by implementing the feedback gain matrix K_{cp}. When F converges, the optimal controller in the compensator will restore the control performance of the real system. Correspondingly, the model parameter variation will be estimated since (A_{rl}, B_{rl}) are approximated by (A_{cp}, B_{cp}), i.e., the optimal solution will provide the estimated model parameter variation as well as the optimal weighting parameter value.

5. CASE STUDY

In this section, we demonstrate the effectiveness of the proposed auto-tuning method with numerical experiment on an example with the system deviation caused by an unknown variation of a single parameter. The optimal
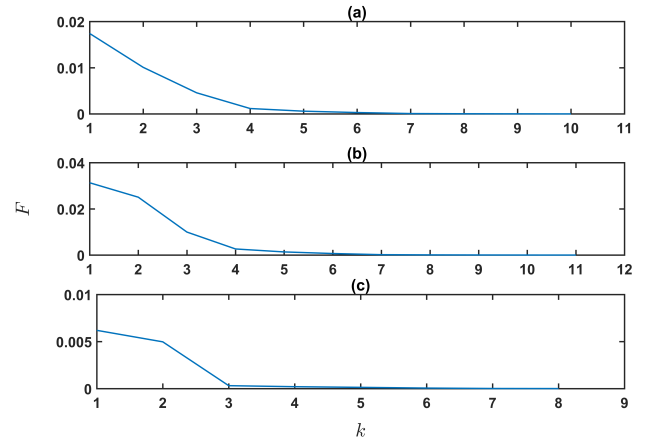

Fig. 4. The performance difference F evaluate with different parameter pairs. (a) Parameter pair $(\theta_{b_2}, \theta_{q_1})$. (b) Parameter pair $(\theta_{b_2}, \theta_{q_2})$. (c) Parameter pair (θ_{b_2}, θ_r).

control problem is formulated as the performance index described in (2) for a second order LTI system (1) with the following description:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \quad R = r$$

The elements defined in (A, B) are model parameters with physical meanings such as mass and length of a vehicle. The nominal values of the model parameters are known, but their values may vary during the system operation. Our objective is to restore the optimal control performance when the model parameters deviate from their nominal values.

In the first case, we consider an acceleration model of a rocket car with the set of matrices defined (1) and (2) given as

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ M \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad R = 1$$

where M is the mass of the object which is the model parameter involved in θ_V and denoted as θ_{b_2}. The weighting parameters θ_W are the diagonal elements of Q and R denoted as $(\theta_{q_1}, \theta_{q_2}, \theta_r)$. The initial states are $x_1(0) = 2$, $x_2(0) = 1$, respectively.

Using the correlation analysis, the correlation coefficients of the three parameter pairs are evaluated and given as

$$\rho_{b_2,q_1} = 0.815 \quad \rho_{b_2,q_2} = 0.229 \quad \rho_{b_2,r} = 1.0$$

This means that the pair (θ_{b_2}, θ_r) are fully correlated and the other two pairs are partially correlated. The corresponding tuning coefficients are given as

$$\alpha_{b_2,q_1} = -1.207 \quad \alpha_{b_2,q_2} = 2.615 \quad \alpha_{b_2,r} = 2.0$$

Next, we show the compensation effects made by $\theta_{q_1}, \theta_{q_2}$ and θ_r. The nominal value of the mass is $M = 1$kg and assumed to be underestimated. Then the vector B in the nominal and the real system, respectively, is given as

$$B^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B_{rl} = \begin{pmatrix} 0 \\ 0.833 \end{pmatrix}$$

Therefore, the (unknown) parameter variation in the real system is $\Delta\theta_{b_2} = 0.167$. The corresponding trajectory deviation is $D(\Delta\theta_{b_2}^p) = 0.046$.
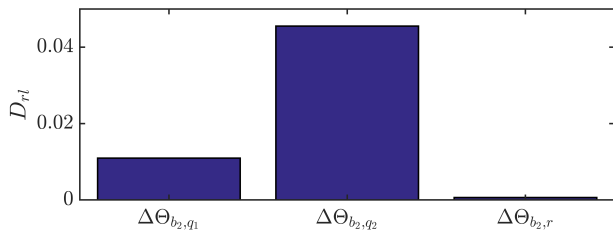
Fig. 5. The minimum cost of the state trajectory deviation D_{rl} achieved after tuning different parameter pairs.

The performance difference evaluated during the tuning process for each parameter pair is shown in Fig. 4. The x-axis indicates the number of iterations in the line search. The y-axis indicates the performance difference $F(\Theta_{i,j}^k)$ at each iteration k. It can be seen that the search scheme leads to a high convergence rate in minimizing the performance difference.

After the search scheme converges, the resulting cost of the state trajectory deviation D_{rl} is shown in Fig. 5. The largest value $D_{rl} = 0.046$ is resulted by tuning the weighting parameter using the least correlated pair $(\theta_{b_2}, \theta_{q_2})$. In addition, it is seen that there is almost no improvement by tuning θ_{q_2}. A better result comes from tuning θ_{q_1} in the pair $(\theta_{b_2}, \theta_{q_1})$ with $D_{rl} = 0.011$. The best value $D_{rl} = 6.34e^{-4}$ is obtained by tuning θ_r in the fully correlated pair (θ_{b_2}, θ_r). The corresponding optimal solution provides the values of the parameter variations $\Theta_{b_2,r}^* = (-0.167, -0.334)$. As a result, our method is able to find the optimal value of the weighting parameter to restore the optimal control performance and the a priori unknown parameter variation is accurately estimated.

6. CONCLUSIONS AND FUTURE WORK

In this paper, we propose an auto-tuning method for LQR with unknown variations of model parameters. The tuning is based on the analysis of the pairwise correlation of a model parameter with a weighting parameter in LQR which reveals the explicit relationship in terms of a tuning coefficient and a correlation coefficient. Based on the correlation analysis, the search direction for effectively tuning the weighting parameter and estimating the model parameter variation can be obtained. A line search scheme is used to iteratively search for the optimal solution so as to minimize the performance difference between the compensator and the real system.

Two aspects are expected to be studied in the future work. First, if no fully correlated parameter pairs can be found in the pairwise correlation analysis, the compensation effect will be degraded by the proposed method. Therefore, a multiple-to-multiple correlation analysis needs to be developed so that the system deviation can be compensated by tuning multiple weighting parameters. Second, a more efficient tuning strategy needs to be worked out which can converge faster and more reliable to the optimal solution.

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REFERENCES

Astrom, K.J. and Wittenmark, B. (1994). *Adaptive Control*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 2nd edition.

Bryson, A.E. and Ho, Y.C. (1969). *Applied Optimal Control*. Blaisdell, New York, NY, USA.

Clarke, D.W., Kanjilal, P.P., and Mohtadi, C. (1985). A generalized LQG approach to self-tuning control part I. Aspects of design. In *International Journal of Control*, 41(6), 1509–1523.

Clarke, D. and Gawthrop, P. (1979). Self-tuning control. In *Proceedings of the Institution of Electrical Engineers*, 126, 633–640.

Cohen, P., West, S.G., and Aiken, L.S. (2014). *Applied multiple regression/correlation analysis for the behavioral sciences*. Psychology Press.

Grimble, M. (1984). Implicit and explicit LQG self-tuning controllers. In *Automatica*, 20(5), 661–669.

Hearne, J. (1985). Sensitivity analysis of parameter combinations. In *Applied Mathematical Modelling*, 9(2), 106–108.

Hennig, P. and Schuler, C.J. (2012). Entropy search for information-efficient global optimization. In *Journal of Machine Learning Research*, 13, 1809–1837.

Horn, R.A. and Johnson, C.R. (2012). *Matrix Analysis*. Cambridge University Press, New York, NY, USA, 2nd edition.

Johnson, M.A. and Grimble, M.J. (1987). Recent trends in linear optimal quadratic multivariable control system design. In *IEE Proceedings D - Control Theory and Applications*, 134(1), 53–71.

Lazutkin, E., Geletu, A., Hopfgarten, S., and Li, P. (2015). An analytical hessian and parallel-computing approach for efficient dynamic optimization based on control-variable correlation analysis. In *Industrial & Engineering Chemistry Research*, 54(48), 12086–12095.

Li, P. and Vu, Q.D. (2013). Identification of parameter correlations for parameter estimation in dynamic biological models. In *BMC Systems Biology*, 7(1), 91.

Li, P. and Vu, Q.D. (2015). A simple method for identifying parameter correlations in partially observed linear dynamic models. In *BMC Systems Biology*, 9(1), 92.

Marco, A., Hennig, P., Bohg, J., Schaal, S., and Trimpe, S. (2016). Automatic LQR tuning based on gaussian process global optimization. In *2016 IEEE International Conference on Robotics and Automation (ICRA)*, 270–277.

Spall, J.C. (2003). *Introduction to Stochastic Search and Optimization*. John Wiley & Sons, Inc., New York, NY, USA, 1st edition.

Sreeram, V. and Agathoklis, P. (1994). On the properties of Gram matrix. In *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 41(3), 234–237.

Trimpe, S., Millane, A., Doessegger, S., and D'Andrea, R. (2014). A self-tuning LQR approach demonstrated on an inverted pendulum. In *IFAC Proceedings Volumes*, 47(3), 11281 – 11287.

Armijo, L. (1966). Minimization of functions having Lipschitz continuous first partial derivatives. In *Pacific Journal of Mathematics*, 16(1), 1–3.