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# Algebraic certificates for the structural properties of parametric linear systems ⋆

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**Abstract:** In this paper, by exploiting the concept of polynomial greatest common divisor, some algebraic tests are proposed to certify the structural properties of both discrete-time and continuous-time linear systems. Furthermore, by exploiting the concept of parametric greatest common divisor, such results are extended to certify the structural properties of systems whose dynamical matrices depend polynomially on some parameters.

*Keywords:* Linear systems, Structural Properties, Reachability, Controllability, Non-stabilizable systems, Algebraic tests.

## 1. INTRODUCTION

The Popov-Belevitch-Hautus (PBH) tests are powerful tools that allow one to characterize the properties of discrete-time (Kalman et al., 1969; Zabczyk, 2009; Sontag, 2013; Hespanha, 2018), continuous-time (Belevitch, 1968; Popov and Posehn, 1974), and hybrid (Possieri and Teel, 2016) linear systems. In particular, such a family of tests allows one to certify reachability, controllability, stabilizability, observability, constructability, and detectability of linear systems by evaluating the rank of the matrices

$$[\, A - \lambda I \;\; B \,], \qquad \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}, \tag{1}$$

for all $\lambda \in \mathbb{C}$, where $A$, $B$, and $C$ are the matrices governing the dynamics of the plant. Nonetheless, using numerical methods to implement such tests may lead to errors, as shown in the following example.

**Example 1.** Consider the discrete-time system

$$x(k+1) = A\,x(k) + B\,u(k), \tag{2}$$

with

$$A = \begin{bmatrix} 1 & -4 & -1 & 0 & -5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & -5 \\ 0 & 4 & 1 & 1 & 5 \\ 0 & -3 & -2 & -1 & -4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

By computing numerically the *spectrum* $\sigma(A)$ of $A$, one obtains the following numerical values

$$\sigma(A) = \{-4.19,\, 1.00,\, -0.802,\, -0.004 + 0.771\imath,\, -0.004 - 0.771\imath\}. \tag{3}$$

Thus, by evaluating numerically rank$[\, A - \lambda I \;\; B \,]$ for all $\lambda \in \sigma(A)$ using the `Matlab` command `rank` with tolerance $10^{-20}$ (that is lower than the precision to which the spectrum $\sigma(A)$ has been computed), one obtains that rank$[\, A - \lambda I \;\; B \,] = 5$ for all $\lambda \in \sigma(A)$, thus concluding that system (2) is reachable. Nonetheless, by computing the reachability matrix $[\, B \;\cdots\; A^4 B \,]$ of system (2), it can be easily derived that it is not reachable since

$$\mathrm{rank}([\, B \;\cdots\; A^4 B \,]) = 1.$$

The issue highlighted in Example 1 is intrinsic in the numerical application of the PBH tests. In fact, the matrix $A - \lambda I$ loses rank if and only if $\lambda \in \sigma(A)$, Therefore, if such a spectrum is computed numerically via some algorithm using finite precision, the roundoff error may lead to critical evaluation errors, as in Example 1.

In view of such an issue, the main objective of this paper is to derive tests that allows one to certify algebraically the structural properties of linear time-invariant system by using algebraic geometry tools. It is worth pointing out that these techniques have been already used in the literature to verify the structural properties of linear systems. For instance, in Habets (1993), Gröbner bases have been used to determine whether the matrix $[\, A - \lambda I \;\; B \,]$ is right-invertible (thus ensuring reachability of the corresponding system) and to compute a right-inverse of $[\, A - \lambda I \;\; B \,]$, to be used to design a compensator.

In this paper, it is shown that the structural properties of both discrete-time (see Section 3.1) and continuous-time (see Section 3.2) linear systems can be certified algebraically by using the concept of polynomial greatest common divisor (that is briefly reviewed in Section 2), which can be computed more efficiently than Gröbner bases. Furthermore, by exploiting the notion of parametric

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greatest common divisor, such results are extended to certify the structural properties of systems depending polynomially on some parameters (see Section 4). Examples of application of the proposed techniques are given all throughout the paper to illustrate the theoretical results.

## 2. ALGEBRAIC GEOMETRY CONCEPTS

Let $\mathbb{K}$ be a *field* (*e.g.*, the sets $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ of rational, real, and complex numbers). Let $\mathbb{K}[x]$ be the *ring* of all the polynomials in the single variables $x = [\, x_1 \cdots x_n \,]$ with coefficients in $\mathbb{K}$.

Given $p_1, \ldots, p_\ell \in \mathbb{K}[x]$, the *ideal of* $p_1, \ldots, p_\ell$ is the set of all the polynomials in $\mathbb{K}[x]$ that can be written as a linear combination with polynomial coefficients of $p_1, \ldots, p_\ell$,

$$\langle p_1, \ldots, p_\ell \rangle := \{\textstyle\sum_{i=1}^{\ell} q_i p_i,\ q_i \in \mathbb{K}[x],\ i = 1, \ldots, \ell\},$$

whereas the *variety of* $p_1, \ldots, p_\ell$ is the subset of $\mathbb{K}$ where such polynomials vanish jointly,

$$\mathbf{V}(p_1, \ldots, p_\ell) := \{x \in \mathbb{K} : p_i(x) = 0, i = 1, \ldots, \ell\}.$$

A set $\mathcal{V}$ is *constructible* if there exist varieties $\mathcal{Z}_i \subset \mathcal{W}_i$, $i = 1, \ldots, \varsigma$, such that $\mathcal{V} = \bigcup_{i=i}^{\varsigma} \mathcal{W}_i \setminus \mathcal{Z}_i$.

Given ideals $\mathcal{I}$ and $\mathcal{J}$ of $\mathbb{K}[x]$, the *saturation of* $\mathcal{I}$ *with respect to* $\mathcal{J}$ is defined as

$$\mathcal{I} : \mathcal{J}^\infty := \{f \in \mathbb{K}[x] : \forall p \in \mathcal{J}, \exists N \geqslant 0 \text{ s.t. } f\, p^N \in \mathcal{I}\}.$$

By Proposition 9 at page 202 of Cox et al. (2015), the saturation $\mathcal{I} : \mathcal{J}^\infty$ is an ideal of $\mathbb{K}[x]$.

Letting $\lambda$ be a single variable, a *greatest common divisor of* $p_1, \ldots, p_\ell \in \mathbb{K}[\lambda]$ is a polynomial $h \in \mathbb{K}[\lambda]$ that divides $p_1, \ldots, p_\ell$ and such that if there is another $f \in \mathbb{K}[\lambda]$ that divides $p_1, \ldots, p_\ell$, then $f$ divides $h$. Since, by Proposition 8 at page 44 of Cox et al. (2015), such a polynomial exists and is unique up to multiplication by a nonzero constant in $\mathbb{K}$, there is a unique monic $h \in \mathbb{K}[\lambda]$ that is a greatest common divisor of $p_1, \ldots, p_\ell$, denoted $\gcd(p_1, \ldots, p_\ell)$.

By Corollary 4 at page 41 of Cox et al. (2015), every ideal $\mathcal{I}$ of $\mathbb{K}[\lambda]$ is *principal*, *i.e.*, there is a single $p \in \mathbb{K}[\lambda]$ such that $\mathcal{I} = \langle p \rangle$. Such a polynomial is unique, up to a multiplication by a constant in $\mathbb{K}$. Thus, given an ideal $\mathcal{I}$ of $\mathbb{K}[\lambda]$, there is a unique monic $p$ such that $\mathcal{I} = \langle p \rangle$, which is referred to as *basis of* $\mathcal{I}$. In particular, by Proposition 8 at page 44 of Cox et al. (2015), one has that $\gcd(p_1, \ldots, p_\ell)$ is a basis of $\langle p_1, \ldots, p_\ell \rangle$. A polynomial $p \in \mathbb{C}[\lambda]$ is *Hurwitz* if all its roots have negative real part, whereas it is *Schur* if all its roots have absolute value lower than 1.

Letting $\beta = [\, \beta_1 \cdots \beta_w \,]^\top$ be a vector of parameters, let $\mathbb{K}[\beta][\lambda]$ be the ring of the polynomials in $\lambda$ over the parameter ring $\mathbb{K}[\beta]$. Given $p \in \mathbb{K}[\beta][\lambda]$, the polynomial $p(\lambda, \hat{\beta}) \in \mathbb{K}[\lambda]$ obtained by fixing the parameters as $\beta = \hat{\beta} \in \mathbb{K}^w$ is called a *specialization* of $p$. Given $p_1, \ldots, p_\ell \in \mathbb{K}[\beta][\lambda]$, the *parametric greatest common divisor* (briefly, *GCD*) *of* $p_1, \ldots, p_\ell$ is a finite sequence $\{(\mathcal{A}_i, g_i)\}_{i=1}^{\zeta}$ such that (Abramov and Kvashenko, 1993):

- each $\mathcal{A}_i$, $i = 1, \ldots, \zeta$, is a constructible set;
- $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for each $i, j \in \{1, \ldots, \zeta\}$, $i \neq j$;
- $\bigcup_{i=i}^{\zeta} \mathcal{A}_i = \mathbb{K}^w$;
- for each $\hat{\beta} \in \mathcal{A}_i$, the polynomial $g_i(\lambda, \hat{\beta})$ is the greatest common divisor of $p_1(\lambda, \hat{\beta}), \ldots, p_\ell(\lambda, \hat{\beta})$.

See Nagasaka (2017) for algorithms capable of computing such a sequence and Kapur et al. (2018) for their implementation in `Singular` [1], an open source computer algebra system for polynomial computations, which is used hereafter.

## 3. THE NOMINAL CASE

In this section, it is shown how the tools reviewed in Section 2 can be used to certify algebraically the structural properties of linear systems in the nominal case, *i.e.*, when the parameters of the system are fixed.

### 3.1 Discrete-time systems

Consider the discrete-time linear system

$$x(k+1) = A\, x(k) + B\, u(k), \tag{4a}$$

$$y(k) = C\, x(k) + D\, u(k), \tag{4b}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times\nu}$, $C \in \mathbb{R}^{\mu\times n}$, and $D \in \mathbb{R}^{\mu\times\nu}$. Let $p_1, \ldots, p_\ell \in \mathbb{C}[\lambda]$ be the $n \times n$ minors of $[\, A - \lambda I \;\; B \,]$, $\ell = \binom{n+\nu}{n}$. The objective of this section is to provide algebraic certificates built upon the $p_1, \ldots, p_\ell$ for the analysis of the structural properties of system (4).

*Reachability and observability certificates* The following theorem provides easily verifiable conditions for certifying reachability of system (4).

**Theorem 1.** *System* (4) *is reachable if and only if*

$$\gcd(p_1, \ldots, p_\ell) = 1. \tag{5}$$

The following remark details how Theorem 1 can be adapted so to certify observability of system (4).

*Remark* 1. Theorem 1 can be easily adapted to certify observability of system (4). In fact, by classical results, system (4) is observable if and only if the system

$$x(k+1) = A^\top x(k) + C^\top u(k) \tag{6}$$

is reachable. Therefore, by Theorem 1, letting $\bar{p}_1, \ldots, \bar{p}_\rho \in \mathbb{C}[\lambda]$ be $n \times n$ minors of $[\, A^\top - \lambda I \;\; C^\top \,]$, $\rho = \binom{n+\mu}{n}$, system (4) is observable if and only if $\gcd(\bar{p}_1, \ldots, \bar{p}_\rho) = 1$.

*Controllability and reconstructibility certificates* The following theorem provides easily verifiable conditions for certifying controllability of system (4).

**Theorem 2.** *System* (4) *is controllable if and only if*

$$\gcd(p_1, \ldots, p_\ell) = \lambda^\alpha, \tag{7}$$

*for some* $\alpha \in \mathbb{Z}$, $\alpha \geqslant 0$

The following corollary provides an alternative approach to certify that system (4) is controllable.

**Corollary 1.** *System* (4) *is controllable if and only if*

$$\langle \gcd(p_1, \ldots, p_\ell) \rangle : \langle \lambda \rangle^\infty = \langle 1 \rangle = \mathbb{C}[\lambda]. \tag{8}$$

The next remark details how the tools given in this section can be used to certify reconstructibility of system (4).

[1] Available at the following link
`https://www.singular.uni-kl.de`.

*Remark* 2. Since system (4) is reconstructible if and only if system (6) is controllable, letting $\bar{p}_1, \ldots, \bar{p}_\rho$ be defined as in Remark 1, system (4) is reconstructible if and only if

$$\langle \gcd(\bar{p}_1, \ldots, \bar{p}_\rho)\rangle : \langle \lambda \rangle^\infty = \langle 1 \rangle.$$

*Stabilizability and detectability certificates* Let

$$\mathbb{C}_d := \{\lambda \in \mathbb{C} : |\lambda| < 1\}.$$

The following theorem provides easily verifiable conditions for certifying stabilizability of system (4).

**Theorem 3.** *System* (4) *is stabilizable if and only if* $\gcd(p_1, \ldots, p_\ell)$ *is Schur.*

Letting $h := \gcd(p_1, \ldots, p_\ell)$ there are several ways to check whether $h$ is Schur, as, *e.g.*, the Jury criterion (Jury, 1962), which can be carried out exactly, thus obtaining an algebraic certificate for the stabilizability of system (4).

**Example 2.** Consider the system analyzed in Example 1. By computing $h = \gcd(p_1, \ldots, p_6)$, one obtains

$$h = \lambda^4 + 5\lambda^3 + 4\lambda^2 + 3\lambda + 2,$$

which implies that system (2) is neither reachable nor controllable. Furthermore, since $h$ is not Schur, system (2) is not stabilizable.

The following remark details how Theorem 3 can be adapted so to deal with detectability of system (4).

*Remark* 3. Since system (4) is detectable if and only if system (6) is stabilizable, letting $\bar{p}_1, \ldots, \bar{p}_\rho$ be defined as in Remark 1, by Theorem 3, system (4) is detectable if and only if $\gcd(\bar{p}_1, \ldots, \bar{p}_\rho)$ is Schur.

### *3.2 Continuous-time systems*

The techniques given in Section 3.1 can be easily adapted to deal with continuous-time systems of the form

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{9a}$$
$$y(t) = C\,x(t) + D\,u(t). \tag{9b}$$

**Theorem 4.** *Let* $p_1, \ldots, p_\ell \in \mathbb{C}[\lambda]$ *be the* $n \times n$ *minors of* $[\,A - \lambda I \;\; B\,]$, $\ell = \binom{n+\nu}{n}$, *and let* $\bar{p}_1, \ldots, \bar{p}_\rho \in \mathbb{C}[\lambda]$ *be* $n \times n$ *minors of* $[\,A^\top - \lambda I \;\; C^\top\,]$, $\rho = \binom{n+\mu}{n}$. *System* (9) *is:*

- *reachable if and only if* $\gcd(p_1, \ldots, p_\ell) = 1$*;*
- *observable if and only if* $\gcd(\bar{p}_1, \ldots, \bar{p}_\rho) = 1$*;*
- *stabilizable if and only if* $\gcd(p_1, \ldots, p_\ell)$ *is Hurwitz;*
- *detectable if and only if* $\gcd(\bar{p}_1, \ldots, \bar{p}_\rho)$ *is Hurwitz.*

There are many ways to check if $\gcd(p_1, \ldots, p_\ell)$ is Hurwitz, as, *e.g.*, the Routh-Hurwitz criterion (Hurwitz, 1895).

*Remark* 4. Several software allow to compute the greatest common divisor of the univariate polynomials $p_1, \ldots, p_\ell$. For instance, it can be computed using the command `PolynomialGCD` in `Mathematica`, the script `gcd` in `Maple`, the function `gcd` in `Macaulay2`, the command `gcd` in `Sage`, or the script `gcd` in `Matlab`. On the other hand, computing the parametric greatest common divisor, which is used in the following Section 4 to characterize the properties of parametric linear systems, requires specialized software, such as the `Singular` package `parametric GCD` [2]; see Kapur et al. (2018) for further details.

[2] Available at the following link
`http://mmrc.iss.ac.cn/~dwang/software.html`.

## 4. THE PARAMETRIC CASE

In this section, the results given in Section 3 for nominal plants are extended to the case of systems depending on some parameters by using the concept of parametric GCD.

### *4.1 Discrete-time parametric systems*

Consider the discrete-time parametric linear system

$$x(k+1) = A(\beta)\,x(k) + B(\beta)\,u(k), \tag{10a}$$
$$y(k) = C(\beta)\,x(k) + D(\beta)\,u(k), \tag{10b}$$

where $A \in \mathbb{R}^{n\times n}[\beta]$, $B \in \mathbb{R}^{n\times \nu}[\beta]$, $C \in \mathbb{R}^{\mu\times n}[\beta]$, and $D \in \mathbb{R}^{\mu\times \nu}[\beta]$ are parametric matrices depending polynomially on the vector of parameters $\beta = [\,\beta_1 \cdots \beta_w\,]^\top$. As in Section 3, let $p_1, \ldots, p_\ell \in \mathbb{C}[\beta][\lambda]$ be the $n \times n$ minors of the matrix $[\,A(\beta) - \lambda I \;\; B(\beta)\,]$, $\ell = \binom{n+\nu}{n}$. The objective of this section is to provide algebraic certificates built upon $p_1, \ldots, p_\ell$ for the structural properties of the specialization

$$x(k+1) = A(\hat\beta)\,x(k) + B(\hat\beta)\,u(k), \tag{11a}$$
$$y(k) = C(\hat\beta)\,x(k) + D(\hat\beta)\,u(k), \tag{11b}$$

of system (10) for some $\hat\beta \in \mathbb{R}^w$.

*Reachability and observability certificates* The following theorem provides conditions to ensure reachability of system (10) for some specialization $\hat\beta \in \mathbb{R}^w$.

**Theorem 5.** *Let* $\{(\mathcal{A}_i, g_i)\}_{i=1}^{\zeta}$, *with* $\mathcal{A}_i \subset \mathbb{C}^w$ *and* $g_i \in \mathbb{C}[\beta, \lambda]$, $i = 1, \ldots, \zeta$, *be a parametric GCD of* $p_1, \ldots, p_\ell \in \mathbb{C}[\beta][\lambda]$. *Hence, the specialization* (11) *of system* (10) *is reachable for any* $\hat\beta \in \mathcal{A}_i$ *if and only if the specialization* $g_i(\lambda, \hat\beta)$ *is a nonzero constant in* $\mathbb{C}[\lambda]$, $i = 1, \ldots, \zeta$.

The following example illustrates the application of Theorem 5 to a parametric discrete-time system.

**Example 3.** Consider system (10) with $\beta = [\,\beta_1 \;\; \beta_2\,]^\top$,

$$A(\beta) = \begin{bmatrix} 0 & \beta_1 \\ 1 & 0 \end{bmatrix}, \qquad B(\beta) = \begin{bmatrix} \beta_2 & 0 \\ 0 & 1 \end{bmatrix}.$$

By computing the parametric GCD of $p_1, \ldots, p_4$, one obtains the following three branches:

- $g_1 = 1$ and $\mathcal{A}_1 := \mathbf{V}(\phi_1) \setminus \mathbf{V}(\omega_1)$, with
$$\phi_1 = 0, \qquad \omega_1 = \beta_2;$$
- $g_2 = 1$ and $\mathcal{A}_2 := \mathbf{V}(\phi_2) \setminus \mathbf{V}(\omega_2)$, with
$$\phi_2 = \beta_2, \qquad \omega_2 = \beta_1;$$
- $g_3 = \lambda$ and $\mathcal{A}_3 := \mathbf{V}(\phi_{3,1}, \phi_{3,2}) \setminus \mathbf{V}(\omega_3)$, with
$$\phi_{3,1} = \beta_1, \qquad \phi_{3,2} = \beta_2, \qquad \omega_3 = 1.$$

Since both $g_1$ and $g_2$ equal 1, system (11) is reachable for each specialization $\hat\beta \in \mathcal{A}_1 \cup \mathcal{A}_2$. On the other hand, since $g_3 \neq 1$, system (11) is not reachable for $\beta_1 = 0$ and $\beta_2 = 0$.

The technique given in Theorem 5 can be easily adapted to certify observability of a parametric discrete-time linear system, as detailed in the following remark.

*Remark* 5. Let $\bar{p}_1, \ldots, \bar{p}_\rho \in \mathbb{C}[\beta][\lambda]$ be $n \times n$ minors of $[\,A^\top(\beta) - \lambda I \;\; C^\top(\beta)\,]$, $\rho = \binom{n+\mu}{n}$. Let $\{(\bar{\mathcal{A}}_i, \bar{g}_i)\}_{i=1}^{\bar\zeta}$, with $\bar{\mathcal{A}}_i \subset \mathbb{C}^w$ and $\bar{g}_i \in \mathbb{C}[\beta, \lambda]$, $i = 1, \ldots, \bar\zeta$, be a parametric GCD of $\bar{p}_1, \ldots, \bar{p}_\rho$. By the same reasoning given in Remark 1 and Theorem 5, the specialization (11) of system (10) is observable for some $\hat{\beta} \in \bar{\mathcal{A}}_i$ if and only if the corresponding specialization $\bar{g}_i(\lambda, \hat{\beta})$ is a nonzero constant in $\mathbb{C}[\lambda]$, $i = 1, \ldots, \zeta$.

The following example illustrates the application of Remark 5 to a parametric discrete-time system.

**Example 4.** Consider system (10) with $\beta = [\,\beta_1 \;\; \beta_2\,]^\top$,

$$A(\beta) = \begin{bmatrix} \beta_1 & \beta_1 - \beta_2 \\ 1 & \beta_1 + \beta_2 \end{bmatrix}, \quad C(\beta) = \begin{bmatrix} \beta_1 - \beta_2 & 1 \\ 0 & \beta_2 - \beta_1 \end{bmatrix}.$$

By computing the parametric GCD of $p_1, \ldots, p_6$, one obtains the following three branches:

- $\bar{g}_1 = 1$ and $\bar{\mathcal{A}}_1 := \mathbf{V}(\bar{\phi}_1) \setminus \mathbf{V}(\bar{\omega}_1)$, with $\bar{\phi}_1 = 0$, and
$$\bar{\omega}_1 = \beta_1^4 - 4\beta_2\beta_1^3 + 6\beta_2^2\beta_1^2 + \beta_2\beta_1^2 - 4\beta_2^3\beta_1 - 2\beta_2^2\beta_1 - \beta_1 + \beta_2^4 + \beta_2^3 + \beta_2;$$
- $\bar{g}_2 = 1$ and $\bar{\mathcal{A}}_2 := \mathbf{V}(\bar{\phi}_2) \setminus \mathbf{V}(\bar{\omega}_2)$, with
$$\bar{\phi}_2 = \beta_1 - \beta_2, \qquad \bar{\omega}_2 = 1;$$
- $\bar{g}_3 = 1$ and $\bar{\mathcal{A}}_3 := \mathbf{V}(\bar{\phi}_3) \setminus \mathbf{V}(\bar{\omega}_3)$, with
$$\bar{\phi}_3 = \beta_1^3 - 3\beta_2\beta_1^2 + 3\beta_2^2\beta_1 + \beta_2\beta_1 - \beta_2^3 - \beta_2^2 - 1,$$
$$\bar{\omega}_3 = \beta_1 - \beta_2.$$

Figure 1 depicts the corresponding partition of $\mathbb{R}^2$.

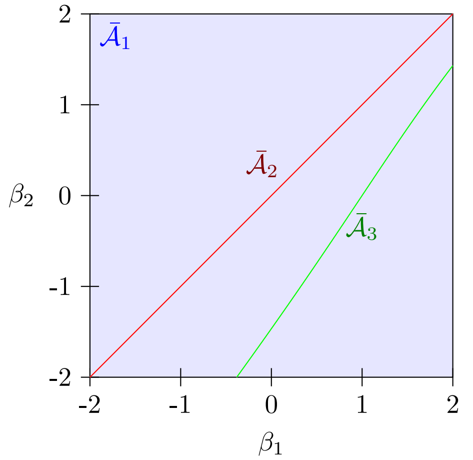


Fig. 1. Partition $\bar{\mathcal{A}}_1, \bar{\mathcal{A}}_2, \bar{\mathcal{A}}_3$ of the parameter space.

Since $g_1 = g_2 = g_3 = 1$, system (11) is observable for all specializations $\hat{\beta} \in \mathbb{R}^2$.

*Controllability and reconstructibility certificates* The following theorem provides conditions to ensure controllability of the parametric system (10).

**Theorem 6.** *Let $\{(\mathcal{A}_i, g_i)\}_{i=1}^{\zeta}$, with $\mathcal{A}_i \subset \mathbb{C}^w$ and $g_i \in \mathbb{C}[\beta, \lambda]$, $i = 1, \ldots, \zeta$, be a parametric GCD of $p_1, \ldots, p_\ell \in \mathbb{C}[\beta][\lambda]$. Hence, the specialization (11) of system (10) is controllable for some $\hat{\beta} \in \mathcal{A}_i$ if and only if $g_i(\lambda, \hat{\beta})$ equals $c\,\lambda^\alpha$ in $\mathbb{C}[\lambda]$, for some $c \in \mathbb{C}$ and $\alpha \geqslant 0$, $i = 1, \ldots, \zeta$.*

The following example illustrates Theorem 6.

**Example 5.** Consider system (10) with $\beta = [\,\beta_1 \;\; \beta_2\,]^\top$,

$$A(\beta) = \begin{bmatrix} 0 & \beta_1 + \beta_2 - 1 & \beta_1 - \beta_2 - 1 \\ 0 & 1 & \beta_1 + 2\beta_2 - 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B(\beta) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

By computing the parametric GCD of $p_1, \ldots, p_4$, one obtains the following four branches:

- $g_1 = 1$ and $\mathcal{A}_1 := \mathbf{V}(\phi_1) \setminus \mathbf{V}(\omega_1)$, with $\phi_1 = 0$ and
$$\omega_1 = (\beta_1 + 2\beta_2 - 1)\cdot (\beta_1^2 + 3\beta_2\beta_1 - 3\beta_1 + 2\beta_2^2 - 2\beta_2 + 2);$$
- $g_2 = \lambda$ and $\mathcal{A}_2 := \mathbf{V}(\phi_2) \setminus \mathbf{V}(\omega_2)$, with
$$\phi_2 = \beta_1^2 + 3\beta_2\beta_1 - 3\beta_1 + 2\beta_2^2 - 2\beta_2 + 2,$$
$$\omega_2 = \beta_1 + 2\beta_2 - 1;$$
- $g_3 = \lambda - 1$ and $\mathcal{A}_3 := \mathbf{V}(\phi_3) \setminus \mathbf{V}(\omega_3)$, with
$$\phi_3 = \beta_1 + 2\beta_2 - 1, \qquad \omega_3 = \beta_2;$$
- $g_4 = \lambda^2 - \lambda$ and $\mathcal{A}_3 := \mathbf{V}(\phi_{4,1}, \phi_{4,2}) \setminus \mathbf{V}(\omega_4)$, with
$$\phi_{4,1} = \beta_1 - 1, \qquad \phi_{4,2} = \beta_2, \qquad \omega_4 = 1.$$

Since $g_1 = 1$, system (11) is reachable (and hence controllable) for each specialization $\hat{\beta} \in \mathcal{A}_1$; whereas, since $g_2 = \lambda$, one has that system (11) is controllable (but not reachable) for each specialization $\hat{\beta} \in \mathcal{A}_2$. On the other hand, since $g_3 = \lambda - 1$ and $g_4 = \lambda^2 - \lambda$, system (11) is not controllable for each specialization $\hat{\beta} \in \mathcal{A}_3 \cup \mathcal{A}_4$.

Figure 2 depicts the partition $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ of $\mathbb{R}^2$.

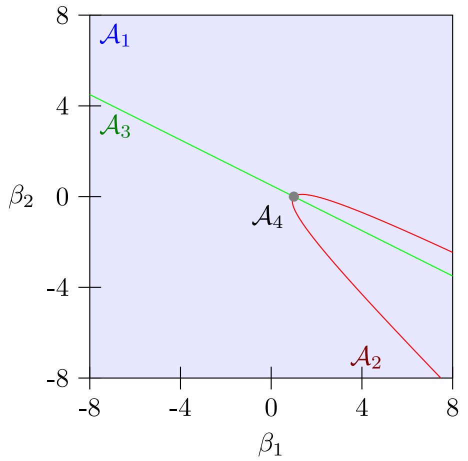


Fig. 2. Partition $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ of the parameter space.

As for observability, the technique proposed in Theorem 6 can be easily extended to certify reconstructibility of discrete-time parametric linear systems, as detailed in the following remark.

*Remark 6.* Let $\{(\bar{\mathcal{A}}_i, \bar{g}_i)\}_{i=1}^{\bar{\zeta}}$ be defined as in Remark 5. By Theorem 7, the specialization (11) of system (10) is reconstructible for some $\hat{\beta} \in \bar{\mathcal{A}}_i$ if and only if $\bar{g}_i(\lambda, \hat{\beta})$ equals $c\,\lambda^\alpha$ in $\mathbb{C}[\lambda]$, for some $c \in \mathbb{C}$ and $\alpha \geqslant 0$, $i = 1, \ldots, \bar{\zeta}$.

*Stabilizability and detectability certificates* The following theorem provides conditions to ensure stabilizability of the parametric system (10).

**Theorem 7.** *Let $\{(\mathcal{A}_i, g_i)\}_{i=1}^{\zeta}$, with $\mathcal{A}_i \subset \mathbb{C}^w$ and $g_i \in \mathbb{C}[\beta, \lambda]$, $i = 1, \ldots, \zeta$, be a parametric GCD of $p_1, \ldots, p_\ell \in \mathbb{C}[\beta][\lambda]$. Hence, the specialization (11) of system (10) is stabilizable for some $\hat{\beta} \in \mathcal{A}_i$ if and only if the specialization $g_i(\lambda, \hat{\beta})$ is Schur, $i = 1, \ldots, \zeta$.*

The following example illustrates Theorem 7.

**Example 6.** Consider system (10) with $\beta = [\,\beta_1\ \ \beta_2\,]^\top$,

$$A(\beta) = \begin{bmatrix} \beta_2 & 2\beta_2 - 1 \\ 1 & 1-\beta_1 \end{bmatrix}, \qquad B(\beta) = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

By computing the parametric GCD of $p_1, p_2, p_3$, one obtains the following three branches:

- $g_1 = 1$ and $\mathcal{A}_1 := \mathbf{V}(\phi_1) \setminus \mathbf{V}(\omega_1)$, with
$$\phi_1 = 0, \quad \omega_1 = 2\beta_2^3 + \beta_1\beta_2^2 - \beta_2^2 + \beta_1^2\beta_2 - \beta_1\beta_2 - \beta_1^2;$$
- $g_2 = \lambda\,\beta_2 + \beta_1 - \beta_2^2$ and $\mathcal{A}_2 := \mathbf{V}(\phi_2) \setminus \mathbf{V}(\omega_2)$, with
$$\phi_2 = 2\beta_2^3 + \beta_1\beta_2^2 - \beta_2^2 + \beta_1^2\beta_2 - \beta_1\beta_2 - \beta_1^2, \quad \omega_2 = \beta_2;$$
- $g_3 = \lambda^2 - \lambda + 1$ and $\mathcal{A}_3 := \mathbf{V}(\phi_{3,1}, \phi_{3,2}) \setminus \mathbf{V}(\omega_3)$, with
$$\phi_{3,1} = \beta_1, \qquad \phi_{3,2} = \beta_2, \qquad \omega_3 = 1.$$

Since $g_1 = 1$, system (11) is reachable (and hence controllable and stabilizable) for all specializations $\hat\beta \in \mathcal{A}_1$. On the other hand, since $g_2 = \lambda\,\beta_2 + \beta_1 - \beta_2^2$ and

$$\mathcal{A}_2 \cap \{\beta \in \mathbb{R}^2 : \beta_2 = 0\} = \emptyset,$$

system (11) is not reachable for all $\hat\beta \in \mathcal{A}_2$. However, since

$$\mathcal{A}_2 \cap \{\beta \in \mathbb{R}^2 : \beta_1 - \beta_2^2 = 0\} \neq \emptyset,$$

there is a specialization $\hat\beta^\circ \in \mathcal{A}_2$ such that system (11) is controllable. Nonetheless, since $g_2(\lambda, \hat\beta) \neq c\,\lambda^\alpha$ for all specializations $\hat\beta \in \mathcal{A}_2 \setminus \{\hat\beta^\circ\}$, system (11) is not controllable for all $\hat\beta \in \mathcal{A}_2 \setminus \{\hat\beta^\circ\}$. By considering that $g_2(\lambda, \hat\beta)$ is Schur for all

$$\hat\beta \in \{\beta \in \mathbb{R}^2 : |\tfrac{\beta_1 - \beta_2^2}{\beta_2}| < 1\} =: \mathcal{H},$$

system (11) is stabilizable for all $\hat\beta \in \mathcal{A}_2 \cap \mathcal{H}$, whereas, since $g_2(\lambda, \hat\beta)$ is not Schur for all

$$\hat\beta \in \{\beta \in \mathbb{R}^2 : |\tfrac{\beta_1 - \beta_2^2}{\beta_2}| \geqslant 1\} =: \mathcal{N},$$

system (11) is not stabilizable for all specializations $\hat\beta \in \mathcal{A}_2 \cap \mathcal{N}$. Finally, since $g_3 \in \mathbb{C}[\lambda]$ is not Schur and

$$\mathcal{A}_3 = \{[\,0\ \ 0\,]^\top\},$$

system (11) is not stabilizable for $\hat\beta = 0$. Figure 3 depicts the partition $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ of $\mathbb{R}^2$, the point $\hat\beta^\circ$, and the set $\mathcal{A}_2 \cap \mathcal{H}$.

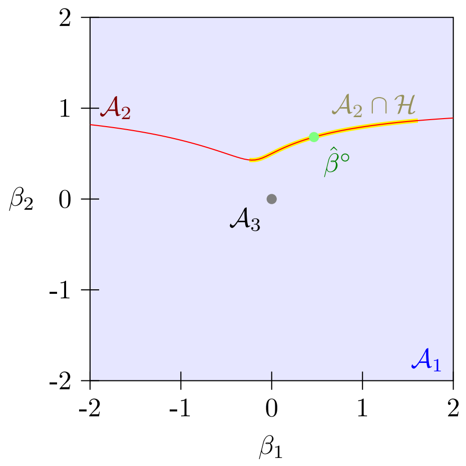


Fig. 3. Partition $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ of the parameter space, point $\hat\beta^\circ$, and set $\mathcal{A}_2 \cap \mathcal{H}$.

The next remark details how to adapt the technique given in Theorem 7 to certify algebraically detectability of a parametric discrete-time linear system.

*Remark 7.* Let $\{(\bar{\mathcal{A}}_i, \bar g_i)\}_{i=1}^{\bar\zeta}$ be defined as in Remark 5. By Theorem 7, the specialization (11) of system (10) is detectable for some $\hat\beta \in \bar{\mathcal{A}}_i$ if and only if the corresponding specialization $\bar g_i(\lambda, \hat\beta)$ is Hurwitz, $i = 1, \ldots, \bar\zeta$.

### *4.2 Continuous-time parametric systems*

The techniques given in Section 4.1 can be adapted to deal with continuous-time parametric systems of the form

$$\dot x(t) = A(\beta)\,x(t) + B(\beta)\,u(t), \tag{12a}$$
$$y(t) = C(\beta)\,x(t) + D(\beta)\,u(t), \tag{12b}$$

where $A \in \mathbb{R}^{n\times n}[\beta]$, $B \in \mathbb{R}^{n\times\nu}[\beta]$, $C \in \mathbb{R}^{\mu\times n}[\beta]$, and $D \in \mathbb{R}^{\mu\times\nu}[\beta]$ are parametric matrices depending polynomially on the vector of parameters $\beta = [\,\beta_1\ \cdots\ \beta_w\,]^\top$. Namely, letting $p_1, \ldots, p_\ell \in \mathbb{C}[\beta][\lambda]$ be the $n \times n$ minors of the matrix $[\,A(\beta) - \lambda I\ \ B(\beta)\,]$, $\ell = \binom{n+\nu}{n}$, and letting $\bar p_1, \ldots, \bar p_\rho \in \mathbb{C}[\beta][\lambda]$ be $n \times n$ minors of $[\,A^\top(\beta) - \lambda I\ \ C^\top(\beta)\,]$, $\rho = \binom{n+\mu}{n}$, consider the following theorem.

**Theorem 8.** *Let $\{(\mathcal{A}_i, g_i)\}_{i=1}^{\zeta}$, with $\mathcal{A}_i \subset \mathbb{C}^w$ and $g_i \in \mathbb{C}[\beta, \lambda]$, $i = 1, \ldots, \zeta$, be a parametric GCD of $p_1, \ldots, p_\ell \in \mathbb{C}[\beta][\lambda]$, and let $\{(\bar{\mathcal{A}}_i, \bar g_i)\}_{i=1}^{\bar\zeta}$, with $\bar{\mathcal{A}}_i \subset \mathbb{C}^w$ and $\bar g_i \in \mathbb{C}[\beta, \lambda]$, $i = 1, \ldots, \bar\zeta$, be a parametric GCD of $\bar p_1, \ldots, \bar p_\rho$. Given $\hat\beta \in \mathcal{A}_i$ for some $i \in \{1, \ldots, \zeta\}$, the specialization*

$$\dot x(t) = A(\hat\beta)\,x(t) + B(\hat\beta)\,u(t), \tag{13a}$$
$$y(t) = C(\hat\beta)\,x(t) + D(\hat\beta)\,u(t), \tag{13b}$$

*of system (10) is*

- *reachable if and only if $g_i(\lambda, \hat\beta)$ is a nonzero constant;*
- *stabilizable if and only if $g_i(\lambda, \hat\beta)$ is Hurwitz.*

*On the other hand, given $\hat\beta \in \bar{\mathcal{A}}_i$ for some $i \in \{1, \ldots, \zeta\}$, the specialization (13) of system (10) is*

- *observable if and only if $\bar g_i(\lambda, \hat\beta)$ is a nonzero constant;*
- *detectable if and only if $\bar g_i(\lambda, \hat\beta)$ is Hurwitz.*

The following example illustrates Theorem 8.

**Example 7.** Consider system (12) with $\beta = [\,\beta_1\ \ \beta_2\,]^\top$,

$$A(\beta) = \begin{bmatrix} \beta_1 + 1 & 1 \\ \beta_1 & 1 - \beta_1 \end{bmatrix},$$
$$B(\beta) = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},$$
$$C = [\,\beta_2\ \ 0\,].$$

By computing the parametric GCD of $p_1, p_2, p_3$, one obtains the following four branches:

- $g_1 = 1$ and $\mathcal{A}_1 := \mathbf{V}(\phi_1) \setminus \mathbf{V}(\omega_1)$, with
$$\phi_1 = 0, \qquad \omega_1 = \beta_1^4 - 2\beta_2\beta_1^3 - \beta_2^2\beta_1;$$
- $g_2 = 1$ and $\mathcal{A}_2 := \mathbf{V}(\phi_2) \setminus \mathbf{V}(\omega_2)$, with
$$\phi_2 = \beta_1, \qquad \omega_2 = \beta_2;$$
- $g_3 = \lambda^2 - 2\lambda + 1$, and $\mathcal{A}_3 := \mathbf{V}(\phi_{3,1}, \phi_{3,2}) \setminus \mathbf{V}(\omega_3)$,
$$\phi_{3,1} = \beta_1, \qquad \phi_{3,2} = \beta_2, \qquad \omega_3 = 1;$$

- $g_4 = \lambda\,\beta_2 + \beta_1^2 - \beta_1\beta_2 - \beta_2$ and $\mathcal{A}_4 := \mathbf{V}(\phi_4) \setminus \mathbf{V}(\omega_4)$,
$$\phi_4 = \beta_1^3 - 2\beta_2\beta_1^2 - \beta_2^2, \qquad \omega_4 = \beta_1\,\beta_2.$$

By Theorem 8, system (13) is reachable for each specialization $\hat{\beta} \in \mathcal{A}_1 \cup \mathcal{A}_2$; whereas it is neither reachable nor stabilizable for $\hat{\beta} \in \mathcal{A}_3$. Furthermore, letting
$$\mathcal{H} := \{\beta \in \mathbb{R}^2 : \tfrac{-\beta_1^2+\beta_2\beta_1+\beta_2}{\beta_2} < 0\},$$
since $g_4(\lambda, \hat{\beta})$ is Hurwitz for all $\hat{\beta} \in \mathcal{A}_4 \cap \mathcal{H}$, system (13) is stabilizable for each specialization $\hat{\beta} \in \mathcal{A}_4 \cap \mathcal{H}$. On the other hand, letting
$$\mathcal{N} := \{\beta \in \mathbb{R}^2 : \tfrac{-\beta_1^2+\beta_2\beta_1+\beta_2}{\beta_2} \geqslant 0\},$$
since $g_4(\lambda, \hat{\beta})$ is not Hurwitz for all $\hat{\beta} \in \mathcal{A}_4 \cap \mathcal{N}$, system (13) is not stabilizable for each $\hat{\beta} \in \mathcal{A}_4 \cap \mathcal{N}$. Figure 4 depicts the partition $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ of $\mathbb{R}^2$ and the set $\mathcal{A}_4 \cap \mathcal{H}$.

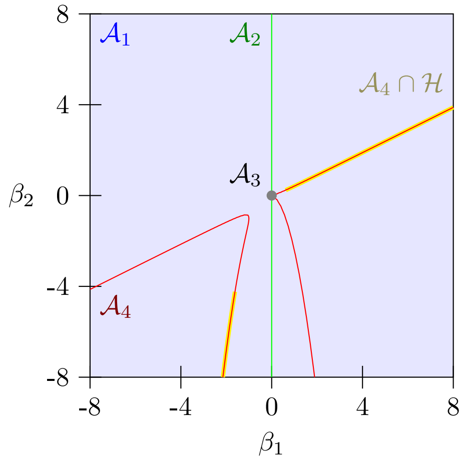


Fig. 4. Partition $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ of the parameter space and set $\mathcal{A}_4 \cap \mathcal{H}$.

On the other hand, by computing the parametric GCD of $\bar{p}_1, \bar{p}_2, \bar{p}_3$, one obtains the following two branches:

- $\bar{g}_1 = 1$ and $\bar{\mathcal{A}}_1 := \mathbf{V}(\bar{\phi}_1) \setminus \mathbf{V}(\bar{\omega}_1)$, with
$$\bar{\phi}_1 = 0, \qquad \bar{\omega}_1 = \beta_2;$$
- $\bar{g}_2 = \lambda^2 - 2\lambda - \beta_1^2 - \beta_1 + 1$ and $\bar{\mathcal{A}}_2 := \mathbf{V}(\bar{\phi}_2) \setminus \mathbf{V}(\bar{\omega}_2)$,
$$\bar{\phi}_2 = \beta_2, \qquad \bar{\omega}_2 = 1.$$

Thus, system (13) is observable for all specializations $\hat{\beta} \in \bar{\mathcal{A}}_1$. On the other hand, since $\bar{g}_2(\lambda, \hat{\beta})$ is not Hurwitz for all $\hat{\beta} \in \bar{\mathcal{A}}_2$, system (13) is not detectable for all $\hat{\beta} \in \bar{\mathcal{A}}_2$.

## 5. CONCLUSIONS

In this paper, by exploiting the concept of parametric polynomial greatest common divisor, some tests have been proposed to provide algebraic certificates for the structural properties of linear systems whose dynamical matrices depend polynomially on some parameters. Examples of application of the proposed techniques have been given all throughout the paper to illustrate the theoretical results.

Differently from other tests available in the literature, such as the ones involving the computation of the singular value decomposition of the pencil matrices in (1), the proposed tests do not asses a "degree" for a structural property, but provide an algebraic certificates about its satisfaction. However, these tests can be employed also to deal with parametric linear systems whose dynamical matrices depend polynomiallu on the parameters, as it has been shown in Section 4.

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