Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Adaptive Continuous Controllers Ensuring Prescribed Ultimate Bound for Uncertain Dynamical Systems

Christopher D. Cruz-Ancona * **Manuel A. Estrada** *
Leonid Fridman * **Hussein Obeid** ** **Salah Laghrouche** **

* *Department of Robotics and Control, Engineering Faculty, Universidad Nacional Autónoma de México (UNAM), Mexico city 04510, Mexico (e-mail: cdiegoca89@gmail.com; mannyestrada94@gmail.com; lfridman@unam.mx).*
** *Femto-ST UMR CNRS. Univ. Bourgogne Franche-Comté/UTBM, 90010, Belfort, France (e.mail: hussein.obeid@univ-fcomte.fr;salah.laghrouche@utbm.fr)*

Abstract: Within the framework of the classical problem of Lyapunov redesign, the barrier function-based adaptation control is presented. The proposed approach does not require the knowledge of an upper bound of the uncertainty. However, producing a continuous control signal adjusts the chattering problem and ensures that the solution will converge in finite time to a region with *a priori* prescribed ultimate bound.

Keywords: Sliding modes, robust control, adaptive control, barrier function

1. INTRODUCTION

In the middle of the 20th century, with the advent of the space race, optimal control theory was an intensive area of research among the scientific community. Yet, unpractical assumptions such as the complete knowledge of the parameters of the system, do not consider that a small variation on the systems' parameters, or the presence of external perturbations, disrupt an optimal design. The latter led to the theoretical study of the control of uncertain systems.

1.1 Lyapunov redesign

The term Lyapunov redesign, coined in (Khalil, 2002), refers to the classical approach developed by Gutman (1979) and Leitmann (1979) for the stabilization of uncertain systems based on the Lyapunov's second method. Under the premise that a Lyapunov function is given for the closed loop system with a stabilizable nominal controller and without the prescence of uncertainty terms, Gutman (1979) presented his approach to robustification of the system uncertainties in the derivative of the LF. This approach requires to add a discontinuous unit controller to the nominal one to robustly compensate the uncertainties. Simultaneosly, Leitmann (1979) presented the same idea for stabilization of uncertain linear systems, proving that the system with the robustifying control designed is asymptotically stable despite bounded uncertainties. The robustifying term is discontinuous on a manifold with relative degree one, defined by the derivative of the nominal Lyapunov function. Two main drawbacks of this approach are the following:

- Discontinuous control causes the undesirable *chattering effect.* The size of *chattering* grows with the level of the controller's gain (Boiko, 2008; Pérez-Ventura and Fridman, 2019).
- The control gain should be chosen bigger than the uncertainty, that is why the upper-bound of uncertainty should be known. In reality such upper bound is unknown or overestimated.

Even in the case when the upper-bound of uncertainty is known, the gain of unit controller is fixed and do not follow the uncertainty variations. This overestimation causes an unwanted energy consumption. Recently, the combination of discontinuous controllers with adaptive gains has paved the way to weaken the assumptions on the knowledge of the upper bound of perturbation to its mere existence. Moreover, if possible, to generate a smooth control signal improving the energy consumption and reducing the chattering.

1.2 Analysis of principal adaptive strategies in sliding mode controllers

The following four types of adaptation resume the efforts in that direction (Shtessel et al., 2016).

Usage of filtered value of discontinuous control. In this strategy, the controller's gain is increased such that the sliding mode (SM) is enforced. Once the SM is established, this strategy uses the modulus value of low-pass filtered discontinuous control in order to follow

⋆ The authors thank the financial support of DGAPA-UNAM (Programa de Becas Posdoctorales DGAPA en la UNAM), CONACyT (Consejo Nacional de Ciencia y Tecnología), Project 282013, and CVU 833748; PAPIIT–UNAM (Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica) IN115419

the perturbation and reduce the amplitude of chattering (Oliveira et al., 2016; Edwards and Shtessel, 2016).

It is worth mentioning that for the design of the low-pass filter, its time constant should be known, *i.e*, the upper-bound of the velocity of the perturbation. However, if this information is available, continuous sliding mode controllers (Moreno, 2018) provide better precision, and guarantee theoretically exact compensation without adaptation. Moreover, at least in the presence of fast actuators continuous sliding mode controllers provide smaller level of chattering and energy consumption (Pérez-Ventura and Fridman, 2019).

Monotonically increasing gain. This strategy consists in monotonically increasing the gain until the SM is reached. Two disadvantages of this approach are clear:

- The gain does not decrease as perturbation decreases, thus chattering level and the power consumption needed for the actuator to maintain the system in SM will be unreasonable big (Pérez-Ventura and Fridman, 2019).
- However one cannot be sure that the SM will be lost if the perturbation continues to increase (Negrete-Chávez and Moreno, 2016; Shtessel et al., 2010). SM can be lost in any moment when perturbation grows.

Increasing and decreasing gain. The main idea behind it consists in increasing the gain until the SM is reached, then the gain is decreased and follow the perturbation changes. Thus, the sliding variable is driven towards some vicinity of zero that depends on the upper bound of the perturbation (Plestan et al., 2010; Shtessel et al., 2012). However, only ultimately boundedness of the sliding variable can be ensured and this vicinity cannot be prescribed in advance.

The following challenges in adaptation are concluded from the advantages and disadvantages of above strategies:

- The gain should be adapted to follow perturbation variations without being overestimated.
- A vicinity towards the sliding variable is driven should be *a priori* prescribed.

Barrier function-based approach. This type of adaptation solves the aforementioned challenges. First the gain increases until the sliding variable reaches a value lower than a predefined $\varepsilon/2$-neighborhood of zero. Then the gain defined by barrier function ensures that the solution will never leave a prescribed domain of origin. Moreover, the gain decreases practically to the value of the norm of perturbation whenever the solution of the system decreases to the origin (Obeid et al., 2018) .

The main differences of the barrier function approach and previous efforts can be summarized as follows:

- Once the solution reaches the prescribed vicinity of the sliding manifold, the solution can never leave it.
- The adaptive gain follows the solution variations instead of the perturbations ones, *i.e.*, if the solution converges to the sliding surface, then the controller gain is close to the perturbation norm.

Notice that all the aforementioned results consider the cases of single variable control input through perturbed integrator.

This paper proposes a new approach to a *classical control problem*: barrier function-based Lyapunov redesign for a class of MIMO uncertain nonlinear systems. The main advantages the approach are the following:

- A continuous control signal is produced to overcome the chattering problem by means of the adaptation in the unitary controller's gain.
- It is ensured that the solution will converge in finite time to a region with *a priori* prescribed ultimate bound and without the knowledge of the upper bound of perturbation.
- If the solution converge to the desired manifold, the adaptive gain will converge to perturbation norm, *i.e.*, energy consumption will be diminished.

2. PRELIMINARIES

Notation. The absolute value and Euclidean vector norm are denoted by $|\cdot|$ and $\|\cdot\|$, respectively. Arguments of functions are omitted when they are understood from the context.

2.1 Barrier Functions

Definition 1. Let s denote the value of the Euclidean norm of some vector of appropriate dimension. Given $\varepsilon > 0$, multi-variable barrier functions $\beta : [0,\varepsilon) \to [\bar{\beta},\infty)$ are defined as the class of strictly increasing functions in the interval $[0,\varepsilon)$, with vertical asymptote $\lim_{s\to\varepsilon}\beta(s) = +\infty$, $s \in \mathbb{R}^+$ and a unique global minimum at zero, i.e., $\beta(0) = \bar{\beta} \geq 0$.

Let $\mathbf{w} \in \mathbb{R}^m$, a simple example of a function that satisfies Definition 1 is the following positive semi-definite barrier function (PSBF) which is used throughout this paper:

$$\beta(\|\mathbf{w}\|) = \frac{\|\mathbf{w}\|}{\varepsilon-\|\mathbf{w}\|}, \quad \bar{\beta} = 0. \tag{1}$$

Note that its time derivative is given as follows,

$$\dot{\beta}(\|\mathbf{w}\|) = \frac{\varepsilon}{(\varepsilon-\|\mathbf{w}\|)^2}\frac{\mathbf{w}^T\dot{\mathbf{w}}}{\|\mathbf{w}\|}. \tag{2}$$

3. PROBLEM FORMULATION AND MAIN RESULT

3.1 Classical Lyapunov redesign

Consider the nonlinear uncertain system (*cf.* Khalil (2002)):

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\left[\mathbf{u} + \xi(t,\mathbf{x})\right], \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$. Function $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth; the matrix function $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the control input matrix whose columns are m-linearly independent smooth vector fields $g_i(x)$, $i = 1,2,\ldots,m$. The function $\xi(t,\mathbf{x}) \in \mathbb{R}^m$ is an absolutely continuous matched and bounded uncertain term.

Assumption 1. Suppose that, for the *nominal system* corresponding to system (3) with $\xi(t,\mathbf{x}) = 0$, there exist a *nominal control law* $\mathbf{u} = \psi(\mathbf{x})$ and a Lyapunov function $V(\mathbf{x})$, satisfying

$$c_1(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq c_2(\|\mathbf{x}\|), \tag{4a}$$

$$\dot{V} = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\left[f(\mathbf{x}) + g(\mathbf{x})\psi(\mathbf{x})\right] \leq -c_3(\|\mathbf{x}\|), \tag{4b}$$

with class $\mathcal{K}_\infty$ functions c_i, for $i = 1, 2, 3$.

In the prescence of the uncertain term $\xi(t, \mathbf{x}) \neq 0$ and to recover the asymptotic properties in Assumtion 1, a compensation term $\mathbf{v}$ should be added to the control law, and the new control law will have the form $\mathbf{u} = \psi(\mathbf{x}) + \mathbf{v}$. Differentiating Lyapunov function V along the trajectories of uncertain system (3), yields to

$$\begin{aligned}\dot{V} =& \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \left[f(\mathbf{x}) + g(\mathbf{x}) \left(\psi(\mathbf{x}) + \mathbf{v} + \xi(t, \mathbf{x}) \right) \right] \\ =& \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \left[f(\mathbf{x}) + g(\mathbf{x})\psi(\mathbf{x}) \right] + \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} g(\mathbf{x}) \left(\mathbf{v} + \xi(t, \mathbf{x}) \right) \\ \leq& - c_3 \left(\|\mathbf{x}\| \right) + \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} g(\mathbf{x}) \left(\mathbf{v} + \xi(t, \mathbf{x}) \right) \ ,\end{aligned} \tag{5}$$

Define the variable $\mathbf{w}^T = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} g(\mathbf{x})$. Then it is possible to rewrite the bound of $\dot{V}$ as

$$\dot{V} \leq -c_3 \left(\|\mathbf{x}\| \right) + \mathbf{w}^T \left(\mathbf{v} + \xi(t, \mathbf{x}) \right) \ . \tag{6}$$

It is easy to see that the effect of the uncertainty term is in the same channel that the robustifying control $\mathbf{v}$.

Assumption 2. The uncertain term $\xi(t, \mathbf{x})$ is bounded, such that

$$\|\xi(t, \mathbf{x})\| \leq M \tag{7}$$

holds almost everywhere with an unknown positive bound M.

To compensate uncertainty term ξ, (Leitmann, 1979) and (Gutman, 1979) suggested a discontinuous unit control law

$$\mathbf{v} = -k(t, \mathbf{x}) \frac{\mathbf{w}}{\|\mathbf{w}\|} \ , \tag{8}$$

understanding solution of system (3) in Filippov (1988) sense. Now,

$$\begin{aligned}\dot{V} &\leq -c_3 \left(\|\mathbf{x}\| \right) + \mathbf{w}^T \mathbf{v} + \|\mathbf{w}\| \|\xi\| \\ &\leq -c_3 \left(\|\mathbf{x}\| \right) - k(t, \mathbf{x}) \|\mathbf{w}\| + M \|\mathbf{w}\|.\end{aligned} \tag{9}$$

So, choosing $k(t, \mathbf{x}) = M$, it is easy to see that the stability of the origin of system (3) is recovered despite of the presence of uncertain term $\xi(t, \mathbf{x})$. Moreover, $\dot{V}$ satisfies (4b), which is independent of the uncertainty. However, (8) produces a discontinuous control signal and moreover the knowledge of the bounds M produces a big overestimation of the gain $k(t, \mathbf{x})$ in $\mathbf{u}(t)$ and as a consequence a great amplitude of chattering and power consumption needed for the actuator.

Derive an *adaptive control law* to compensate the effect of the uncertain term in the derivative of the Lyapunov Function in (6) without overestimation of the upper-bound of the uncertain term.

3.2 Main result

The time derivative of $\mathbf{w}$ has the form

$$\dot{\mathbf{w}} = L_{\bar{f}(\mathbf{x})} \mathbf{w}(\mathbf{x}) + L_{g(\mathbf{x})} \mathbf{w}(\mathbf{x}) \left(\mathbf{v} + \xi(t, \mathbf{x}) \right) \tag{10}$$

where $L_{\bar{f}} \mathbf{w}(\mathbf{x}) := [\partial \mathbf{w} / \partial \mathbf{x}] \bar{f}(\mathbf{x})$, $\bar{f}(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})\psi(\mathbf{x})$ and $L_g \mathbf{w}(\mathbf{x}) := [\partial \mathbf{w} / \partial \mathbf{x}] g(\mathbf{x})$. To recover the stability conditions in assumption 1, it is necessary to restrict $\mathbf{w}$ in (6) to the manifold $\mathbf{w} = 0$ in finite time while compensating the effect of the uncertain term ξ in (6). This is possible whenever the upper-bound of the uncertain term ξ is exactly known and the following assumption is fulfilled.

Assumption 3. The vector relative degree of $\mathbf{w}(\mathbf{x})$ is $\mathbf{r} = \mathbf{1}_m^T$, *i.e.*, $L_{g(\mathbf{x})} \mathbf{w}(\mathbf{x})$ is invertible for all $\mathbf{x} \in \mathbb{R}^n$ and there exist $\bar{\gamma} > \underline{\gamma} > 0$ such that $0 < \underline{\gamma} \leq \|L_{g(\mathbf{x})} \mathbf{w}(x)\| \leq \bar{\gamma} < 1$.

In regard to alleviate the assumptions on the known bound of the uncertain term, consider the following adaptive gain which does not require the knowledge of the bound of ξ but its mere existence:

$$k(t, \|\mathbf{w}\|) = \begin{cases} \dot{\alpha}(t) = \bar{\alpha} \|\mathbf{w}\|, \ \bar{\alpha} > 0, \text{ if } 0 \leq t \leq \bar{t}, \\ \beta(t) = \frac{\|\mathbf{w}\|}{\varepsilon - \|\mathbf{w}\|}, \text{ if } t > \bar{t} \end{cases} \tag{11}$$

where $\bar{t} = \bar{t}(\|\mathbf{w}(\mathbf{x}(0))\|, \varepsilon)$.

Setting the control $\mathbf{v} = -\left[L_{g(\mathbf{x})} \mathbf{w}(\mathbf{x}) \right]^{-1} \left(L_{f(\mathbf{x})} \mathbf{w}(\mathbf{x}) - u_{sm} \right)$ with the adaptive unitary control law:

$$u_{sm} := -k(t, \|\mathbf{w}\|) \frac{\mathbf{w}}{\|\mathbf{w}\|}, \tag{12}$$

yields to the closed-loop dynamics of the extended variable $\mathbf{w}$, *i.e.*,

$$\dot{\mathbf{w}} = -k(t, \|\mathbf{w}\|) \frac{\mathbf{w}}{\|\mathbf{w}\|} + \delta(t, \mathbf{x}) \tag{13}$$

with $\delta := L_g \mathbf{w}(\mathbf{x}) \xi$. Note from assumptions 2 and 3 that the following holds.

Assumption 4. The uncertain term $\delta(t, \mathbf{x})$ is bounded, such that

$$\|\delta(t, \mathbf{x})\| < M$$

holds almost everywhere with an unknown positive bound M.

The closed-loop dynamics of $\mathbf{w}$ possess the following features:

- At the beginning, the adaptive gain increases (*cf.* Plestan et al. (2010)) such that the sliding variable $\mathbf{w}(\mathbf{x})$ reaches the domain $\|\mathbf{w}(\mathbf{x})\| < \varepsilon/2$ in finite time $\bar{t}$.
- Then, the adaptive gain is switched to the barrier function which makes the sliding variable not to increase beyond the predefined ε-neighborhood of zero even when the disturbance δ continues to increase, *i.e.*, $\|\mathbf{w}(\mathbf{x})\| < \varepsilon$ for all $t \geq \bar{t}$.

Note that the gain (1) increases and decreases according to the value of the solution of (13) (see Fig. 1). Within the intervals (t_1, t_2) and (t_3, t_4) the gain matches the upper-bound whenever the value of $\mathbf{w}$ is closed to the origin.

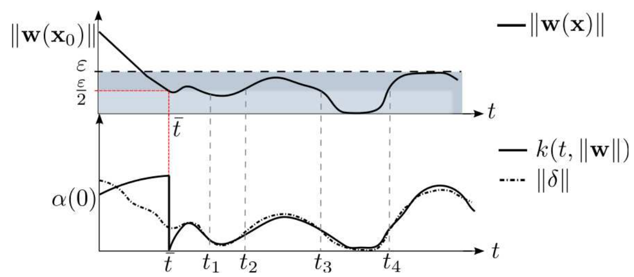

Fig. 1. Convergence of the norm of the output variable $\|\mathbf{w}(\mathbf{x})\|$ to a predefined ε-neighborhood of the origin.

The latter is resumed in the next important and main result of this paper.

Proposition 5. Consider the perturbed system (3) and suppose that the assumptions 1, 2, 3 and 4 are satisfied. Let $D \subset \mathbb{R}^n$ be a domain that contains the origin and $B_r = \{\|\mathbf{x}\| \leq r\}$. Set the control $\mathbf{u} = \psi(\mathbf{x}) + \mathbf{v}$ with

$$\mathbf{v} = -\left[L_{g(\mathbf{x})} \mathbf{w}(\mathbf{x}) \right]^{-1} \left(L_{f(\mathbf{x})} \mathbf{w}(\mathbf{x}) - u_{sm} \right) \tag{14}$$

and adaptive unitary control u_{sm} in (12), and adaptive gain $k(t, x)$ in (11). Given $\varepsilon > 0$ and $N \in \mathbb{Z}_+ \backslash \{1\}$, then for any $\|\mathbf{x}(0)\| < c_2^{-1}(c_1(r))$, there exists a finite time $\bar{t}$ such that the solutions of (3) reach the set $\|\mathbf{w}\| < \varepsilon$ for all $t \geq \bar{t}$. After that, the solution of the closed-loop system (3) satisfies,

$$\|\mathbf{x}\| \leq b(\varepsilon), \quad \forall t \geq \bar{t}$$

where

$$b(\varepsilon) := c_1^{-1}(c_2(c_3^{-1}(2\varepsilon(N-1)/N))). \tag{15}$$

4. NUMERICAL EXAMPLE

Consider the 2-Degrees of Freedom robot with the model presented in Slotine et al. (1991) pp. 396,

$$J(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + \varphi(t), \tag{16}$$

where $q = [q_1 \ q_2]$, $\tau = [\tau_1 \ \tau_2]$, $\varphi : \mathbb{R}^+ \times \mathbb{R}^2$ is an additive matched perturbation to be defined later and the matrices J, C y g,

$$J(q) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \quad g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$
$$C(q,\dot{q}) = \begin{bmatrix} -h\dot{q}_2 & -h(\dot{q}_1 + \dot{q}_2) \\ h\dot{q}_1 & 0 \end{bmatrix} \tag{17}$$

with

$$\begin{aligned}
J_{11} =& m_1 l_{c1}^2 + I_1 + m_2[l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos(q_2)] + I_2, \\
J_{22} =& m_2 l_{c2}^2 + I_2, \\
J_{12} =& J_{21} = m_2 l_1 l_{c2}\cos(q_2) + m_2 l_{c2}^2 + I_2, \\
h =& m_2 l_1 l_{c2}\sin(q_2), \\
g_1 =& m1 l_{c1} g\cos(q_1) + m_2 g[l_{c2}\cos(q_1+q_2) + l_1\cos(q_1)], \\
g_2 =& m_2 l_{c2} g\cos(q_1+q_2).
\end{aligned}$$

Taking $x_1 = q - q_d$, and $x_2 = \dot{q} - \dot{q}_d$, being $q_d \in \mathbb{C}^2$ a desired trajectory, the system can be expressed as

$$\begin{aligned}
\dot{x}_1 =& x_2, \\
\dot{x}_2 =& [J(x_1)]^{-1}\left(-C(x)x_2 - g(x_1) + \tau + \varphi(t) - \ddot{q}_d\right),
\end{aligned}$$

Which can be rewritten in compact form as

$$\dot{\mathbf{x}} = A\mathbf{x} + B\left[h(t,\mathbf{x}) + G(\mathbf{x})\tau\right], \tag{18}$$

where $A = \begin{bmatrix} 0 & \mathbb{I} \\ 0 & 0 \end{bmatrix}$, $B^T = \begin{bmatrix} 0 & \mathbb{I} \end{bmatrix}$, $G(\mathbf{x}) = [J(x_1)]^{-1}$, and $h(t,\mathbf{x}) = -[J(x_1)]^{-1}(C(x)x_2 + g(x_1) - \varphi(t) + \ddot{q}_d)$. Assume that $0 < \beta_1\mathbb{I} \leq J(q) \leq \beta_2\mathbb{I}$. Consider the feedback control law $\tau = \mathbf{u} + \mathbf{v}$, with the nominal control law $\mathbf{u}$ designed in Corless and Leitmann (1990) for the classical Lyapunov redesign,

$$\mathbf{u} = -\gamma B^T P\mathbf{x}, \tag{19}$$

with $\gamma \geq \sigma\lambda_{max}[J(\mathbf{x})]$. The control gain in (19) is designed by taking $Q = Q^T > 0$, for which P is calculated as the solution of the algebraic Ricatti equation,

$$PA + A^T P - \sigma PBB^T P + 2Q = 0\,. \tag{20}$$

Then, taking the Lyapunov function $V = \mathbf{x}^T P\mathbf{x}$, it can be proved that $\dot{V} \leq -\mathbf{x}^T Q\mathbf{x}$, when $h(t,\mathbf{x}) = \mathbf{v} = 0$. Set now the variable $\mathbf{w} = B^T P\mathbf{x}$, its time derivative is

$$\begin{aligned}
\dot{\mathbf{w}} =& B^T P\dot{\mathbf{x}} \\
=& B^T P\left[\bar{A}\mathbf{x} + B\left(h(t,\mathbf{x}) + G(\mathbf{x})\mathbf{v}\right)\right] \\
=& B^T P\bar{A}\mathbf{x} + B^T PBG(\mathbf{x})\mathbf{v} + \delta(t,\mathbf{x})\,,
\end{aligned} \tag{21}$$

with $\bar{A} = A\mathbf{x} - B\mathbf{u}$ and $\delta(t,\mathbf{x}) = B^T PBh(t,\mathbf{x})$. Selecting the control $\mathbf{v}$ such that $\mathbf{w}$ is forced to a prescribed ε-neighborhood of zero,

$$\mathbf{v} = -\left[B^T PBG(\mathbf{x})\right]^{-1}\left(B^T P\bar{A}\mathbf{x} - k(t,\|\mathbf{w}\|)\frac{\mathbf{w}}{\|\mathbf{w}\|}\right) \tag{22}$$

One can take $\sigma = 1$, and the control then is $\mathbf{u} = -B^T P\mathbf{x}$ with P solution of the Ricatti equation setting $Q = \mathbb{I}_4$. Then,

$$P = \begin{bmatrix} 3.1075 & 0 & 1.4142 & 0 \\ 0 & 3.1075 & 0 & 1.4142 \\ 1.4142 & 0 & 2.1974 & 0 \\ 0 & 1.4142 & 0 & 2.1974 \end{bmatrix}$$

Also, the trajectories requested $q(t) = [\sin(t) \ \cos(t)]^T$.

Table 1. Parameters of the simulation model

Notation	**value**	**Notation**	**value**
l_1	0.5 m	l_2	0.2 m
l_{c1}	0.32 m	l_{c2}	0.08 m
m_1	1 kg	m_2	0.5 kg
g	9.81 kg/s^2	I_1	1e-2 kg m^2
I_2	0.5e-2 kg m^2		

Moreover, consider the time dependent matched perturbation given by $\varphi(t) := a(t)\sin(2t)[1 \ \ 1]^T$ with

$$a(t) = \begin{cases} 1, & 0 \leq t \leq 22, \\ 0.6, & 22 < t \leq 40, \\ 0.3, & 40 < t \leq 60. \end{cases}$$

Table 2. Parameter values for PSBF and DLR

	PSBF	DLR1	DLR2
Parameter values	$\alpha(0) = 1$, $\varepsilon = 0.5$, $\bar{\alpha} = 0.1$	$k = 5$	$k = 15$

In simulations results, three scenarios are considered: the upper-bound of perturbation exists but it is unknown, *i.e.*, the barrier function approach (PBSF); the bound of the perturbation is not too much overestimated by using the unit control (DLR1) in (8); and the upper-bound of perturbation is overestimated with unitary control (DLR2) in (8). The parameters are given in tables 1 and 2 and simulations were made using a sampling step of $1 \times 10^{-4}\,s$. The tracking errors $\mathbf{x}$ for PBSF, DLR1 and DLR2 are shown in Figs. 2(a)-Fig. 2(c), respectively. To show the advantages of the proposed approach, an external perturbation which changes in amplitude $\varphi(t)$ was added. It decreases in amplitude at $t = 22\,s$ and $t = 40\,s$, in order to show that PSBF approach is capable to follow these changes. The effect of φ is shown in Figure 2(d). The gain in the classical LR cannot be changed, thus the *the chattering* amplitude is maintained and this gain does not follows the changes of the perturbation. In contrast, the adaptive gain in the barrier function approach will converge to the perturbation norm (see Fig. 4(d)), without overestimating it. It is worth to mention that PSBF does not use any information about the upper bound of the perturbation. Hence, the *chattering effect* is no longer present.

Fig. 3(a) illustrates that the auxiliary variable $\mathbf{w}$ in PSBF is smooth and its norm is contained in a predefined neighborhood of the origin that will never be exceeded. When the gain is $k = 5$ in DLR1 (see Fig. 3(b)), this auxiliary variable is maintained within the same neighborhood with constant level of *chattering*. However, the amplitude of *chattering* is bigger than the case in DLR1 whenever the gain is chosen as $k = 15$. In the case DLR2, note that the norm of the auxiliary variable surpass the prescribed neighborhood of the origin designed for the barrier function approach (see Fig. 3(c)).

The control signals are shown in Fig. 4. Notice that the control signal in the case of PSBF is continuous (see Fig. 4(a)) due to the combination of the barrier function in (1) and the unit control (12). Moreover, the adaptive gain depicted in Fig. 4(d) decreases in a similar fashion to the solution of the system, and it follows the perturbation variations whenever the perturbation decreases/increases, this is not the case when the gain is fixed such as in DLR1 and DLR2. On the other hand, in order to compensate the uncertainties such as the perturbation is contained in the control signal, the overestimation the unit controller produces big undesirable levels of *chattering* as shown in the scenarios DLR1 and DLR2 in Figs. 4(b)-4(c).

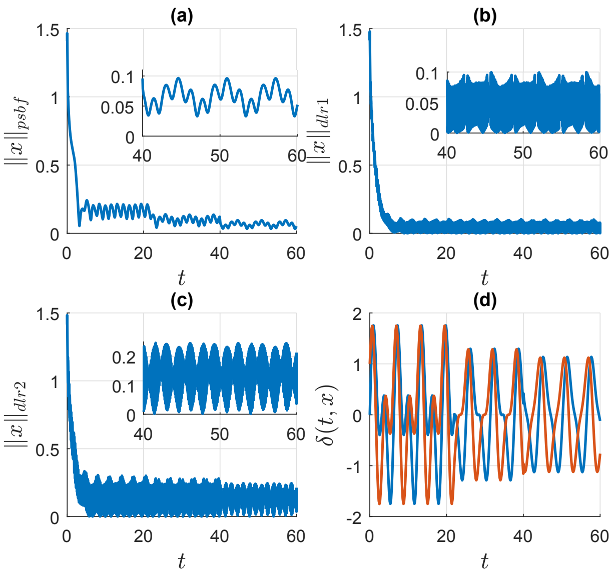

Fig. 2. Tracking error

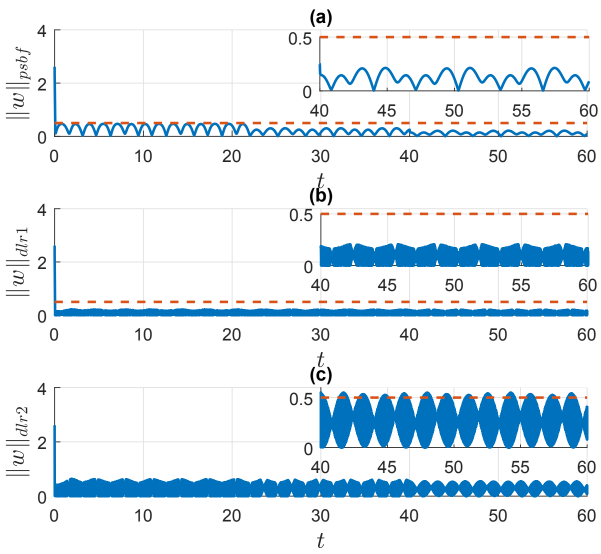

Fig. 3. Auxiliary variable

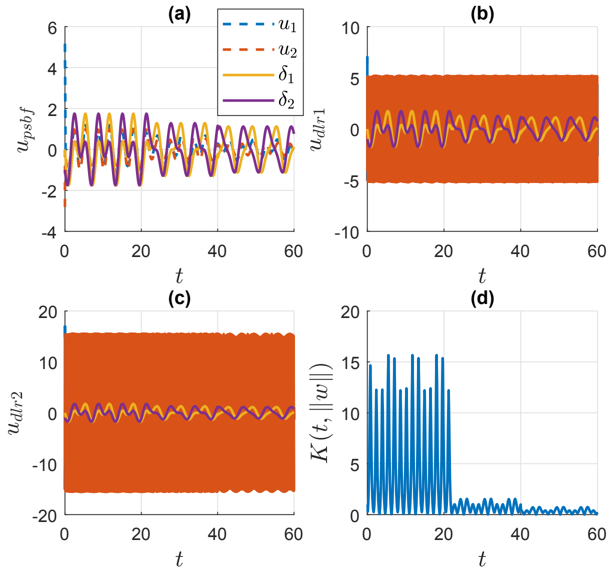

Fig. 4. Control signals and gain

5. CONCLUSIONS

In this paper a PSBF gain to the unit control law for the classical problem of Lyapunov redesign is suggested. It is supposed that the upper bound of the perturbation exists but it is unknown.

PSBF ensures the finite time convergence of the solution to a region with *a priori* prescribed ultimate bound. It is important to mention that if the positive semi-definite barrier function is selected as the gain of the unit control, it produces a continuous control signal capable to maintain the sliding variable in a predefined ε-neighborhood of the origin. The three main advantages of the PSBF in comparison with the classical unit control Lyapunov redesign are the following: no any knowledge of the upper bound of the perturbation is needed, the *chattering effect* is no longer present; the adaptive gain follows the changes in the uncertain terms considerably reducing the energy consumption in the actuator.

ACKNOWLEDGEMENTS

The authors are grateful for the financial support of DGAPA-UNAM (Programa de Becas Posdoctorales en la UNAM), CONACyT (Consejo Nacional de Ciencia y Tecnología), Project 282013, and CVU 833748; PAPIIT–UNAM (Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica) IN115419

REFERENCES

Boiko, I. (2008). *Discontinuous control systems: frequency-domain analysis and design.* Springer Science & Business Media.

Corless, M. and Leitmann, G. (1990). Deterministic control of uncertain systems: A lyapunov theory approach. In *Deterministic control of uncertain systems*, 220–251. IET.

Edwards, C. and Shtessel, Y.B. (2016). Adaptive continuous higher order sliding mode control. *Automatica*, 65, 183–190.

Filippov, A. (1988). *Differential equations with discontinuous right hand sides*. Kluwer Academic Publisher, Dordrecht.

Gutman, S. (1979). Uncertain dynamical systems–a lyapunov min-max approach. *IEEE Transactions on Automatic Control*, 24(3), 437–443.

Khalil, H.K. (2002). *Nonlinear Systems*. Prentice Hall, New Jersey.

Leitmann, G. (1979). Guaranteed asymptotic stability for some linear-systems with bounded uncertainties. *Journal of Dynamic Systems Measurement and Control-Transactions of the Asme*, 101(3), 212–216.

Moreno, J.A. (2018). Discontinuous integral control for systems with relative degree two. In *New Perspectives and Applications of Modern Control Theory*, 187–218. Springer.

Negrete-Chávez, D.Y. and Moreno, J.A. (2016). Second-order sliding mode output feedback controller with adaptation. *International Journal of Adaptive Control and Signal Processing*, 30(8-10), 1523–1543.

Obeid, H., Fridman, L.M., Laghrouche, S., and Harmouche, M. (2018). Barrier function-based adaptive sliding mode control. *Automatica*, 93, 540–544.

Oliveira, T.R., Cunha, J.P.V., and Hsu, L. (2016). Adaptive sliding mode control for disturbances with unknown bounds. In *2016 14th International Workshop on Variable Structure Systems (VSS)*, 59–64. IEEE.

Pérez-Ventura, U. and Fridman, L. (2019). When is it reasonable to implement the discontinuous sliding-mode controllers instead of the continuous ones? frequency domain criteria. *International Journal of Robust and Nonlinear Control*, 29(3), 810–828.

Plestan, F., Shtessel, Y., Bregeault, V., and Poznyak, A. (2010). New methodologies for adaptive sliding mode control. *International journal of control*, 83(9), 1907–1919.

Shtessel, Y., Fridman, L., and Plestan, F. (2016). Adaptive sliding mode control and observation. *International Journal of Control*, 89(9), 1743–1746.

Shtessel, Y., Taleb, M., and Plestan, F. (2012). A novel adaptive-gain supertwisting sliding mode controller: Methodology and application. *Automatica*, 48(5), 759–769.

Shtessel, Y.B., Moreno, J.A., Plestan, F., Fridman, L.M., and Poznyak, A.S. (2010). Super-twisting adaptive sliding mode control: A lyapunov design. In *49th IEEE conference on decision and control (CDC)*, 5109–5113. IEEE.

Slotine, J.J.E., Li, W., et al. (1991). *Applied nonlinear control*, volume 199. Prentice hall Englewood Cliffs, NJ.