Robust Finite-Time Boundedness for Linear Time-Varying Systems ⋆

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Abstract: A method for the synthesis of time-varying state-feedback gains, capable of guaranteeing robust finite-time boundedness properties for linear time-varying continuous-time systems, is proposed in this paper. The strategy relies on the computation of discrete-time gains for a discretized version of the open-loop system. If some sufficient conditions are satisfied, the desired continuous-time gain is obtained. A numerical example illustrates the validity of the technique.

Keywords: Finite-time boundedness, time-varying systems, robust control, continuous-time, discretization.

1. INTRODUCTION

The concept of finite-time stability, first introduced in Dorato (1961), of great importance mainly when dealing with physical systems, has been widely investigated. A system is finite-time stable (FTS) if the trajectories, starting from a given set of initial conditions, remain within a predefined domain during a finite time interval (Amato et al., 2014). It is worth noting that asymptotic stability does not imply on finite-time stability, and vice-versa (Amato et al., 2013), thus the development of techniques related to FTS systems is important mainly in applications where the trajectories cannot exceed some given limits.

There is a fair amount of papers in the literature related to FTS systems. The paper of Garcia et al. (2009) proposes the synthesis of a state-feedback controller based on the resolution of parameterized Lyapunov matrix differential equations to ensure the finite-time stability of linear time-varying (LTV) systems. A method to synthesize switched state-feedback gains for switched linear systems is proposed in Du et al. (2009) based on linear matrix inequalities (LMIs). Another LMI-based technique is presented in Borges et al. (2013), where the system is supposed to be linear and depending on time-varying bounded parameters. In Amato et al. (2006), a strategy for the synthesis of time-varying state and output-feedback gains to guarantee the controlled LTV system to be FTS is proposed. The methodology is based on the resolution of a set of differential linear matrix inequalities (DLMIs), which was later extended, to cite a few examples, in Amato et al. (2010) to compute time-varying dynamical controllers, in Amato et al. (2013) to deal with impulsive time-varying systems and in Amato et al. (2015) to ensure the finite-time stability of stochastic LTV systems. The main issue with the DLMI strategy is that the conditions are infinite-dimensional problems, and a discretization is needed. An alternative procedure is presented in Agulhari (2018), where a discrete-time LTV system is obtained from the continuous-time system to be controlled and, if a set of conditions relating the finite-time stability of both systems is satisfied, then a piecewise constant state-feedback gain is obtained.

In more realistic situations, systems can be affected by external noises. To deal with this case, the concept of finite-time boundedness was introduced in Amato et al. (2001). A system is finite-time bounded (FTB) if it is FTS even under the effects of bounded exogenous inputs, such as noises. The paper of Amato et al. (2001) also presents LMI conditions to compute state-feedback gains that ensure the controlled system to be FTB. Alternative FTB analysis conditions are presented in Ichihara and Katayama (2009), and an extension of the finite-time boundedness to deal with input-to-output stability is proposed in Amato et al. (2002). The work of Meng and Shen (2009) considers not only external noises, but also uncertainties, representing a wider class of dynamic systems, that is also the focus of the present paper. Meng and Shen (2009) propose sufficient conditions for the synthesis of state-feedback controllers that guarantee the robust finite-time boundedness of the system, *i.e.*, both the finite-time boundedness and an upper bound for the $\mathcal{H}_\infty$ norm of the closed-loop system. A similar robust treatment is also presented in Amato et al. (2011) and Amato et al. (2014), but without explicitly considering the finite-time boundedness property.

The main contribution of this paper is to propose a method for the synthesis of state-feedback gains that guarantee the robust finite-time boundedness of a continuous-time LTV system. The method is an extension of the technique presented in Agulhari (2018), where a discretized system is first obtained and, then, a discrete-time state-

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feedback gain is computed and used to generate the desired continuous-time gain. The main advantage of the approach is the utilization of discrete-time conditions, which do not resort to time-derivatives and, therefore, can be numerically addressed in a simpler way.

The paper is organized as follows. Section 2 presents the preliminaries, which are important for the development of the main results proposed in Section 3. A numerical example illustrates the validity of the synthesis technique in Section 4, and Section 5 concludes the paper.

2. PRELIMINARIES

Consider the LTV continuous-time system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_u(t)u(t) + B_w(t)w(t) \\ z(t) &= C(t)x(t) + D_u(t)u(t) + D_w(t)w(t), \end{aligned} \tag{1}$$

with $x(t) \in \mathbb{R}^{n_x}$ being the states of the system, $u(t) \in \mathbb{R}^{n_u}$ the control inputs, $w(t) \in \mathbb{R}^{n_w}$ the exogenous inputs and $z(t) \in \mathbb{R}^{n_z}$ the controlled outputs. The system matrices are supposed to be integrable functions of t (Zadeh and Desoer, 1963) with appropriate dimensions. Consider also the open-loop transition matrix $\Phi(t, t_0)$ solution of

$$\frac{d\Phi(t,t_0)}{dt} = A(t)\Phi(t,t_0), \quad \Phi(t_0,t_0) = I, \quad t_0 \text{ known.}$$

The main properties of the transition matrix can be found, for example, in Zadeh and Desoer (1963); D'Angelo (1970).

The definition of finite-time stability, as enunciated in Amato et al. (2014), is reproduced below.

Definition 1. Given an initial time t_0, a positive scalar T, a positive definite matrix R and a positive definite matrix-valued function $\Gamma(t)$, defined over $[t_0, t_0+T]$ such that $\Gamma(t_0) < R$, the time-varying linear system (1), with $w(t) = 0$ and a given $u(t)$, is said to be FTS with respect to $(t_0, T, R, \Gamma(t))$ if, $\forall t \in [t_0, t_0+T]$,

$$x(t_0)^T R x(t_0) \le 1 \;\Rightarrow\; x(t)^T \Gamma(t) x(t) < 1. \tag{2}$$

Consider also the LTV discrete-time system defined by

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}_u(k)\hat{u}(k) + \hat{B}_w(k)\hat{w}(k) \\ \hat{z}(k) &= \hat{C}(k)\hat{x}(k) + \hat{D}_u(k)\hat{u}(k) + \hat{D}_w(k)\hat{w}(k), \end{aligned} \tag{3}$$

where all the matrices present the same dimensions as their continuous-time counterparts in (1). The definition of finite-time stability for system (3), according to Amato et al. (2014), is given as follows.

Definition 2. Given an initial time k_0, a positive integer N, a positive definite matrix R, and a positive definite matrix-valued sequence $\hat{\Gamma}(k)$, defined over $\{k_0, \ldots, k_0+N\}$ such that $\hat{\Gamma}(k_0) < R$, the discrete-time LTV system (3), with $\hat{w}(k) = 0$ and a given $\hat{u}(k)$, is said to be FTS with respect to $(k_0, N, R, \hat{\Gamma}(k))$ if, $\forall k \in \{k_0, \ldots, k_0+N\}$,

$$\hat{x}(k_0)^T R \hat{x}(k_0) \le 1 \;\Rightarrow\; \hat{x}(k)^T \hat{\Gamma}(k) \hat{x}(k) < 1. \tag{4}$$

Lemma 3 briefly reproduces an important result from Agulhari (2018), which is a method for generating a state-feedback control law $u(t) = K(t)x(t)$ to ensure the finite-time stability of system (1) with $w(t) = 0$. The technique is based on the synthesis of a discrete-time state-feedback gain such that $\hat{u}(k) = \hat{K}(k)\hat{x}(k)$ guarantees that a related discretized system (3), with $\hat{w}(k) = 0$, is FTS.

Lemma 3. Consider the continuous-time and the discrete-time LTV systems (1) and (3), respectively with $w(t) = 0$ and $\hat{w}(k) = 0$. Let the discrete-time matrices from (3) be given by

$$\hat{A}(k) = \Phi(t_{k+1}, t_k), \quad \hat{B}_u(k) = B_u(t_{k+1}), \tag{5}$$

being $t_k = t_0 + k\delta$ and δ a predefined scalar. The piecewise constant state-feedback control law

$$u(t) = K(t)x(t),\; K(\tilde{t}) = \frac{1}{\delta}\hat{K}(k)\Phi(t_k, t_{k+1}),\; t_k \le \tilde{t} < t_{k+1}, \tag{6}$$

guarantees the finite-time stability of system (1) with $w(t) = 0$ and with respect to $(c_1, c_2, T, \Gamma(t))$ if there exists a $\delta > 0$ and a bounded discrete-time state-feedback gain $\hat{K}(k)$ that ensures the finite-time stability of system (3) with $\hat{w}(k) = 0$ and with respect to $(c_1, c_2, N, c_\delta^{cl}(k)\Gamma(t_k))$, being $\delta = T/N$ and $c_\delta^{cl}(k)$ a scalar sequence satisfying

$$\begin{aligned} \Phi_{cl}(\tilde{t}, t_k)^T \Gamma(\tilde{t}) \Phi_{cl}(\tilde{t}, t_k) &\le c_\delta^{cl}(k)\Gamma(t_k), \\ \tilde{t} \in [t_k, t_{k+1}), &\quad k = 0, \ldots, N-1, \end{aligned} \tag{7}$$

where $\Phi_{cl}(\cdot,\cdot)$ is the transition matrix of the closed-loop system (1) with the control law (6).

The results from the paper of Agulhari (2018) ensure that the controlled states remain within a predefined set during a given time interval, but considering $w(t) = 0$, *i.e.*, no noises or external disturbances are supposed to affect the system. In a more general setting, it is important to consider such issues. Since several classes of external disturbances can be modeled as given by (1) (Boyd et al., 1994; Green and Limebeer, 1995), it is interesting to develop a technique to generate controllers guaranteeing FTS-based properties for systems affected by uncertainties, also known as finite-time boundedness as defined in Amato et al. (2001, 2014). An adaptation of such definition is presented next.

Definition 4. Given an initial time t_0, a positive scalar T, a positive definite matrix R and a positive definite matrix-valued function $\Gamma(t)$, defined over $[t_0, t_0+T]$ such that $\Gamma(t_0) < R$. The continuous-time LTV system (1) is said to be robust finite-time bounded (RFTB) with respect to $(t_0, T, R, \Gamma(t), \gamma, d)$ if, for all $w(t) \in \mathcal{L}_2$ such that

$$||w(t)||_2^2 \le d, \tag{8}$$

one has

$$||z(t)||_2^2 \le \gamma^2 ||w(t)||_2^2 \tag{9}$$

and, $\forall t \in [t_0, t_0+T]$,

$$x(t_0)^T R x(t_0) \le 1 \;\Rightarrow x(t)^T \Gamma(t) x(t) < 1. \tag{10}$$

The corresponding definition of RFTB for discrete-time systems is stated in the sequence.

Definition 5. Given an initial time k_0, a positive integer scalar N, a positive definite matrix R and a positive definite matrix-valued function $\hat{\Gamma}(k)$, defined over $\{k_0, \ldots, k_0+N\}$, such that $\hat{\Gamma}(k_0) < R$. The discrete-time LTV system (3) is said to be RFTB with respect to $(k_0, N, R, \hat{\Gamma}(k), \hat{\gamma}, \hat{d})$ if, for all $\hat{w}(k) \in \ell_2$ such that

$$||\hat{w}(k)||_2^2 \le \hat{d}, \tag{11}$$

one has

$$||\hat{z}(k)||_2^2 \le \hat{\gamma}^2 ||\hat{w}(k)||_2^2 \tag{12}$$

and, $\forall k \in \{k_0, \ldots, k_0+N\}$,

$$\hat{x}(k_0)^T R \hat{x}(k_0) \le 1 \;\Rightarrow \hat{x}(t)^T \hat{\Gamma}(t) \hat{x}(t) < 1. \tag{13}$$

The main contribution of this paper is to propose a technique for the synthesis of a continuous-time state-feedback controller $u(t) = K(t)x(t)$ such that the closed-loop system (1) is RFTB with respect to $(t_0, T, R, \Gamma(t), \gamma, d)$, being $K(t)$ computed from an appropriate state-feedback gain $\hat{K}(k)$ that ensures the robust finite-time boundedness of a related discrete-time system (3), as presented in the next section.

3. MAIN RESULTS

The following theorem is an adaptation of Lemma 3.

Theorem 6. Given the positive scalars $T \in \mathbb{R}$ and $N \in \mathbb{Z}$, define $\delta = T/N$, $t_k = t_0 + k\delta$ and consider the discrete-time LTV system given by

$$\begin{aligned} \hat{A}(k) &= \Phi(t_{k+1}, t_k), \quad \hat{B}_u(k) = B_u(t_{k+1}), \\ \hat{B}_w(k) &= \sqrt{\delta} B_w(t_{k+1}), \quad \hat{C} = \sqrt{\delta} C(t_{k+1}), \\ \hat{D}_u(k) &= D_u(t_{k+1}), \quad \hat{D}_w(k) = D_w(t_{k+1}). \end{aligned} \tag{14}$$

The piecewise constant state-feedback gain control law $u(t) = K(t)x(t)$, with $K(t)$ given by

$$K(\tilde{t}) = \frac{1}{\delta} \hat{K}(k) \Phi(t_k, t_{k+1}), \quad t_k \le \tilde{t} < t_{k+1} \tag{15}$$

guarantees system (1) to be RFTB with respect to $(t_0, T, R, \Gamma(t), \hat{\gamma} + \epsilon(\delta), d)$ if the pair $\{A(t), B_u(t)\}$ from (1) is uniformly completely controllable, and if there exists a bounded discrete-time state-feedback gain $\hat{K}(k)$ such that system (14) is RFTB with respect to $(0, N, R, c_\delta^{cl}(k)\Gamma(t_k), \hat{\gamma}, d)$ for a certain function $\epsilon(\cdot)$, being $c_\delta^{cl}(k)$ a scalar sequence satisfying (7).

Proof. Suppose that $\hat{K}(k)$ is a state-feedback gain guaranteeing

$$||\hat{z}(k)||_2^2 - \hat{\gamma}^2 ||\hat{w}(k)||_2^2 \le 0, \hat{w}(k) \in \ell_2, \forall k \ge k_0. \tag{16}$$

Using some properties for discrete-time LTV systems (Rugh, 1996), (16) can be rewritten as

$$\Big|\Big| \sum_{j=0}^{k-1} \hat{C}(k) \hat{\Phi}_{cl}(k, j+1) \hat{B}_w(j) \hat{w}(j) + \hat{D}_w(k) \hat{w}(k) \Big|\Big|_2^2 - \hat{\gamma}^2 \Big|\Big| \hat{w}(k) \Big|\Big|_2^2 \le 0, \tag{17}$$

begin $\hat{\Phi}_{cl}(\cdot,\cdot)$ the transition matrix of the closed-loop discrete-time system. The last inequality is valid for all $\widetilde{\gamma} \ge \hat{\gamma}$ and is equivalent to

$$\Big|\Big| \sum_{j=0}^{k-1} C(t_{k+1}) \Phi(t_k, t_{j+1}) B_w(t_{j+1}) \hat{w}(j) \delta + D_w(t_{k+1}) \hat{w}(k) \Big|\Big|_2^2 - \widetilde{\gamma}^2 \Big|\Big| \hat{w}(k) \Big|\Big|_2^2 \le 0. \tag{18}$$

For $\delta \to 0$ and setting $\hat{w}(k) = w(t_{k+1})$, the Riemann sum (18) tends to (Stewart, 2008)

$$\underbrace{\Big|\Big| \int_{t_0}^{t} C(t) \Phi(t, \tau) B_w(\tau) w(\tau) d\tau + D_w(t) w(t) \Big|\Big|_2^2}_{p(t,t_0)} - \widetilde{\gamma}^2 \Big|\Big| w(t) \Big|\Big|_2^2 \le 0, \quad \forall t \in [t_0, t_0 + T],$$

with the equality attained for $\widetilde{\gamma} = \hat{\gamma}$. However, for $\delta > 0$, (18) is approximated by

$$||p(t, t_0) + f(t, \delta)||_2^2 - \widetilde{\gamma}^2 ||w(t) + g(t, \delta)||_2^2 \le 0, \tag{19}$$

being $f(t, \delta)$ and $g(t, \delta)$ the approximation errors. Note that, since $w(t) \in \mathcal{L}_2$ and $\hat{w}(k) \in \ell_2$, then $g(t, \delta)$ can be disregarded for the worst-case analysis. The left side of (19) is therefore upper-bounded by

$$||p(t, t_0)||_2^2 + ||f(t, \delta)||_2^2 - \widetilde{\gamma} ||w(t)||_2^2. \tag{20}$$

The function $f(t, \delta)$ is bounded for fixed values of δ, since the system matrices are considered to be integrable. Thus, there exists a function $\epsilon(\delta)$ such that (20) with $\widetilde{\gamma} = \hat{\gamma} + \epsilon(\delta)$ is still a valid upper bound for (19) and, consequently,

$$||z(t)||_2^2 \le (\hat{\gamma} + \epsilon(\delta))^2 ||w(t)||_2^2, \tag{21}$$

which is equivalent to (9) for $\gamma = \hat{\gamma} + \epsilon(\delta)$.

Additionally, comparing hypothesis (8) with (11) yields

$$||\hat{w}(k)||_2^2 \le \hat{d} \le d.$$

Therefore, it is sufficient to consider d as a bound for both the continuous and the discretized exogenous inputs. Finally, if (7) is satisfied then, according to Lemma 3, the state-feedback gain (15) ensures (10).

Remark 7. Note that, from (19) and (20),

$$\delta_1 < \delta_2 \;\Rightarrow\; \epsilon(\delta_1) \le \epsilon(\delta_2). \tag{22}$$

The following theorem presents LMI conditions to generate the discrete-time state-feedback gain $\hat{K}(k)$ used, in Theorem 6, for the computation of the desired control law.

Theorem 8. If there exist matrices $W(k) = W(k)^T > 0$, $Z(k)$, a scalar $\hat{\gamma}$ and a scalar sequence $c_\delta^{cl}(k)$ satisfying, for $k = 0, \ldots, N-1$,

$$\begin{bmatrix} W(k) & W(k)\hat{A}(k)^T + Z(k)^T \hat{B}_u(k)^T & W(k)\hat{C}(k)^T + Z(k)^T \hat{D}_u(k)^T & 0 \\ \star & W(k+1) & 0 & \hat{B}_w(k) \\ \star & \star & I & \hat{D}_w(k) \\ \star & \star & \star & \hat{\gamma}^2 I \end{bmatrix} > 0, \tag{23}$$

$$\begin{bmatrix} W(k) & W(k)\Gamma(t_k)c_\delta^{cl}(k) \\ \star & \Gamma(t_k)c_\delta^{cl}(k) \end{bmatrix} \ge 0, \tag{24}$$

$$\begin{bmatrix} (1 - \hat{\gamma}^2 N \hat{d})R & I \\ \star & W(0) \end{bmatrix} > 0, \tag{25}$$

then the state-feedback gain $\hat{K}(k) = Z(k)W(k)^{-1}$ ensures that system (3) is RFTB with respect to $(k_0, N, R, \Gamma(t_k)c_\delta^{cl}(k), \hat{\gamma}, \hat{d})$.

Proof. Setting $Z(k) = \hat{K}(k)W(k)$, $\hat{A}_{cl}(k) = \hat{A}(k) + \hat{B}_u(k)\hat{K}(k)$ and $\hat{C}_{cl}(k) = \hat{C}(k) + \hat{D}_u(k)\hat{K}(k)$, (23) is equivalent to

$$\begin{bmatrix} W(k) & W(k)\hat{A}_{cl}(k)^T & W(k)\hat{C}_{cl}(k)^T & 0 \\ \star & W(k+1) & 0 & \hat{B}_w(k) \\ \star & \star & I & \hat{D}_w(k) \\ \star & \star & \star & \hat{\gamma}^2 I \end{bmatrix} > 0. \tag{26}$$

Consider now the Lyapunov function

$$V(\hat{x}(k)) = \hat{x}(k)^T P(k) \hat{x}(k)$$

and define $W(k) = P(k)^{-1}$. If (26) is satisfied, then

$$\begin{aligned} \Psi(k) \triangleq\; & V(\hat{x}(k+1)) - V(\hat{x}(k)) \\ & + \hat{z}(k)^T \hat{z}(k) - \hat{\gamma}^2 \hat{w}(k)^T \hat{w}(k) < 0 \end{aligned} \tag{27}$$

and, from Boyd et al. (1994),

$$||\hat{z}(k)||_2^2 \leq \hat{\gamma}^2||\hat{w}(k)||_2^2. \quad (28)$$

The application of the Schur complement (Boyd et al., 1994) in (24) yields

$$W(k) - W(k)\Big(\Gamma(t_k)c_\delta^{cl}(k)\Big)W(k) \geq 0$$

that, multiplying both sides by $P(k) = W(k)^{-1}$ on the left and on the right, implies

$$\hat{x}(k)^T\Gamma(t_k)c_\delta^{cl}(k)\hat{x}(k) \leq \hat{x}(k)^T P(k)\hat{x}(k). \quad (29)$$

Note that, since

$$\begin{aligned}\sum_{i=0}^{k-1}\Psi(i) &= V(\hat{x}(k)) - V(\hat{x}(0)) \\ &\quad + \sum_{i=0}^{k-1}\Big(\hat{z}(i)^T\hat{z}(i) - \gamma^2\hat{w}(i)^T\hat{w}(i)\Big) \\ &= \hat{x}(k)^T P(k)\hat{x}(k) - \hat{x}(0)^T P(0)\hat{x}(0) \\ &\quad + \sum_{i=0}^{k-1}\Big(\hat{z}(i)^T\hat{z}(i) - \hat{\gamma}^2\hat{w}(i)^T\hat{w}(i)\Big) \quad (30)\end{aligned}$$

then, for all $k \in [0,\dots,N]$, due to hypothesis (11),

$$\begin{aligned}&\hat{x}(k)^T P(k)\hat{x}(k) \\ &< \hat{x}(0)^T P(0)\hat{x}(0) - \sum_{i=0}^{k-1}\Big(\hat{z}(i)^T\hat{z}(i) - \hat{\gamma}^2\hat{w}(i)^T\hat{w}(i)\Big) \\ &< \hat{x}(0)^T P(0)\hat{x}(0) + \sum_{i=0}^{k-1}\Big(\hat{\gamma}^2\hat{w}(i)^T\hat{w}(i)\Big) \\ &< \hat{x}(0)^T P(0)\hat{x}(0) + \hat{\gamma}^2 N\hat{d}. \quad (31)\end{aligned}$$

On the other hand, applying the Schur complement over condition (25) yields

$$P(0) < (1 - \hat{\gamma}^2 N\hat{d})R. \quad (32)$$

Consequently, considering the hypothesis $\hat{x}(0)^T R\hat{x}(0) < 1$ and using inequalities (29), (31) and (32), one has

$$\begin{aligned}\hat{x}(k)^T\Gamma(t_k)c_\delta^{cl}(k)\hat{x}(k) &< \hat{x}(k)^T P(k)\hat{x}(k) \\ &< \hat{x}(0)^T P(0)\hat{x}(0) + \hat{\gamma}^2 N\hat{d} \\ &< \big(1 - \hat{\gamma}^2\hat{d}\big)\hat{x}(0)^T R\hat{x}(0) + \hat{\gamma}^2 N\hat{d} < 1. \quad (33)\end{aligned}$$

Therefore, for $k = 0,\dots,N-1$,

$$\hat{x}(0)^T R\hat{x}(0) < 1 \Rightarrow \hat{x}(k)^T\Gamma(t_k)c_\delta^{cl}(k)\hat{x}(k) < 1,$$

thus system (3) is RFTB with respect to $(0, N, R, \Gamma(t_k)c_\delta^{cl}(k), \hat{\gamma}, \hat{d})$.

Theorem 8 proposes a strategy to compute a discrete-time state-feedback gain $\hat{K}(k)$ that ensures the RFTB of a discretized version of system (1), while Theorem 6 enunciates a way to adapt such gain to generate the desired continuous-time control law $u(t) = K(t)x(t)$. However, the theorems do not guarantee condition (7) to be satisfied, since it depends on the closed-loop transition matrix; thus, an *a posteriori* verification would be necessary. On the other hand, it is possible to resort to properties of transition matrices to generate sufficient conditions to certify (7) during the synthesis procedure, as proposed in the following theorem.

Theorem 9. For given scalars $\hat{\gamma}$ and d, if system (1) is uniformly completely controllable (Kalman, 1960), and if there exist matrices $W(k) = W(k)^T > 0$, $Z(k)$, a sufficiently small scalar δ and a scalar sequence $c_\delta^{cl}(k)$ satisfying conditions (23), (24), (25) and

$$\begin{bmatrix} W(k)c_\delta(k)\Gamma(t_k)W(k) & (W(k) + \delta Z(k)^T\hat{B}_u(k)^T)\alpha(\delta,k)\bar{\Gamma}(k) \\ \star & \bar{\Gamma}(k)\alpha(\delta,k)\end{bmatrix} \geq 0,$$
$$k = 0,\dots,N-1, \quad (34)$$

with $\hat{A}(k)$, $\hat{B}_u(k)$, $\hat{B}_w(k)$, $\hat{C}(k)$, $\hat{D}_u(k)$, and $\hat{B}_w(k)$ given by (14), $\bar{\Gamma}(k) = \bar{\sigma}(k)I$,

$$\bar{\sigma}(k) = \max_{t\in[t_k,t_{k+1})} ||\Gamma(t)||_2 \quad (35)$$

and $\alpha(\delta,k)$ a time-varying scalar satisfying

$$||\Phi(t,t_k)||_2 \leq \alpha(\delta,k), \quad \forall t \in [t_k, t_{k+1}), \quad (36)$$

then the piecewise constant state-feedback control law $u(t) = K(t)x(t)$, with $K(t)$ given by

$$K(\tilde{t}) = \frac{1}{\delta}Z(k)W(k)^{-1}\Phi(t_k,t_{k+1}), \quad t_k \leq \tilde{t} < t_{k+1} \quad (37)$$

ensures that system (1) is RFTB with respect to $(t_0, T, R, \Gamma(t), \hat{\gamma} + \epsilon(\delta), d)$ for a certain function $\epsilon(\cdot)$.

Proof. The bound $\alpha(\delta,k)$ satisfying (36) exists if system (1) is uniformly completely controllable (Kalman, 1960). Using the results of theorems 6 and 8, if the discrete-time state-feedback gain $\hat{K}(k) = Z(k)W(k)^{-1}$ guarantees system (3) to be RFTB with respect to $(0, N, R, \Gamma(t_k)c_\delta^{cl}(k), \hat{\gamma}, \hat{d})$, then system (1) is RFTB with respect to $(t_0, T, R, \Gamma(t), \hat{\gamma} + \epsilon(\delta), d)$, for a certain function $\epsilon(\cdot)$, if condition (7) is satisfied.

It remains to show that the feasibility of (34) implies on condition (7). To do so, first note (Agulhari, 2018) that the transition matrix $\Phi_{cl}(t,t_0)$ of the closed-loop system, solution of

$$\frac{d\Phi_{cl}(t,t_0)}{dt} = \big(A(t) + B_u(t)K(t)\big)\Phi_{cl}(t,t_0),$$

can be rewritten as

$$\Phi_{cl}(t,t_k) = \Phi(t,t_k)M(t,t_k)$$

with $\Phi(t,t_0)$ being the open-loop transition matrix and $M(\cdot,\cdot)$ solution of

$$\frac{dM(t+\delta,t)}{d\delta} = \Phi(t,t+\delta)B(t+\delta)K(t+\delta)\Phi(t+\delta,t). \quad (38)$$

By applying the first order Euler approximation (Lu, 2000) of matrix $M(t+\delta,t)$ and considering (38), one has

$$\Phi_{cl}(t,t_k) \approx \big(I + \delta B_u(t)K(t)\big)\Phi(t,t_k).$$

Therefore, for a small value of δ, condition (7) can be approximated by

$$\begin{aligned}&\big((I+\delta B_u(t)K(t))\Phi(t,t_k)\big)^T\Gamma(t)\big((I+\delta B_u(t)K(t))\Phi(t,t_k)\big) \\ &\leq ||(I+\delta B_u(t)K(t))\Phi(t,t_k)||_2^2\Gamma(t) \leq c_\delta^{cl}(k)\Gamma(t_k).\end{aligned}$$

Note also, following the lines from Agulhari (2018), that

$$\begin{aligned}&\big((I+\delta B_u(t)K(t))\Phi(t,t_k)\big)^T\Gamma(t)\big((I+\delta B_u(t)K(t))\Phi(t,t_k)\big) \\ &\leq \Omega(k)^T\bar{\Gamma}(k)\Omega(k)\alpha(\delta,k),\end{aligned}$$

with

$$\Omega(k) = \big((I + \delta\hat{B}_u(k)Z(k)W(k)^{-1})\Phi(t,t_k)\big).$$

If

$$\Omega(k)^T \bar{\Gamma}(k) \Omega(k) \alpha(\delta, k) \leq c_\delta^{cl}(k) \Gamma(t_k), \quad (39)$$

then condition (7) is satisfied. Rewriting inequality (39), one has

$$\left(W(k)+\delta \hat{B}_u(k) Z(k)\right)^T \bar{\Gamma}(k) \Big(W(k)+\delta \hat{B}_u(k) Z(k)\Big) \alpha(\delta, k) \leq W(k) c_\delta^{cl}(k) \Gamma(t_k) W(k),$$

and a Schur complement yields (34), concluding the proof.

The conditions stated in Theorem 9 present bilinearities, mainly due the product between $c_\delta^{cl}(k)$ and other matrix variables. In fact, high values for $c_\delta^{cl}(k)$ could be fixed to deal with inter-sample issues that could occur by using the discrete-time approach proposed in the paper, potentially causing problems to compute a feasible discrete-time state-feedback gain $\hat{K}(k)$. The following algorithm is proposed to deal with such problems.

Algorithm 1.

(1) Set $c_\delta^{cl}(k) = 1$, for all $k \in \{0, \ldots, N-1\}$;
(2) With the current $c_\delta^{cl}(k)$, compute $W(k)$ and $Z(k)$ satisfying conditions (23)–(25);
(3) If no feasible solution is found, then the proposed strategy fails to find a suitable solution; stop. Otherwise, proceed to the next step;
(4) Verify the validity of condition (7) with the current value of $c_\delta^{cl}(k)$. An alternative for such verification is to use the results from Lemma 1 of Agulhari (2018). If (7) holds, then the computed gain satisfies the conditions of Theorem 9, ending the algorithm with a proper result. Otherwise, using the obtained $W(k)$ and $Z(k)$, compute $c_\delta^{cl}(k)$ as the solution of

$$\min_{c_\delta^{cl}(k)} \sum_{i=0}^{N-1} c_\delta^{cl}(i) \qquad \text{s.t.} \quad (34);$$

(5) With the new $c_\delta^{cl}(k)$, return to Step 2.

4. ILLUSTRATIVE EXAMPLE

Consider the problem of regulating a physical system whose mass varies over time, adapted from Forbes and Damaren (2010) and given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m(t)} & \frac{\dot{m}(t)+c}{m(t)} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m(t)} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t),$$
$$z(t) = [1 \;\; 1]\, x(t) + 0.2 w(t), \quad (40)$$

with $m(t) = m_f e^{-at} + m_i$, $m_f = 1$ kg, $m_i = 1.5$ kg, $a = 0.5t^{-1}$, $c = 10^{-5}$ Ns/m and $k = 5$ N/m. The exogenous input $w(t)$ is supposed to be a Gaussian random variable with zero mean, unitary variance and such that $||w(t)||_2^2 \leq 1$.

The objective of this example is to determine, for each value of $\delta \in \Lambda$, with

$$\Lambda = \{0.02, \;\; 0.04, \;\; 0.06, \;\; 0.08, \;\; 0.1, \;\; 0.12, \;\; 0.14, \;\; 0.16, \;\; 0.18, \;\; 0.20, \;\; 0.3, \;\; 0.4, \;\; 0.5\},$$

the minimum value of σ (computed through a bisection method) for which Algorithm 1 yields controllers guaranteeing system (40) to be RFTB with respect to $(0, 10, \sigma I, I, \gamma, 1)$ for some value of γ. Table 1 presents such results, along with the minimum value γ^* for each optimal σ^* obtained. The numbers of LMI rows and decision variables of conditions (23)–(25), important for assessing the computational complexity of the technique, are also presented in Table 1. The maximum values of $x(t)^T x(t)$ for the trajectories from 100 random initial values satisfying $\sigma^* x(0)^T x(0) \leq 1$ are depicted in Figure 1, using the system controlled by the state-feedback gains computed considering $\delta = 0.02$ and $\delta = 0.5$.

Table 1. Minimum values of σ and γ such that Algorithm 1 yields feasible results for each value of δ, along with the numbers of LMI rows and decision variables of conditions (23)–(25).

δ	σ^*	γ^*	Rows	Variables
0.02	1.8218	0.2408	5000	2504
0.04	1.1918	0.3402	2500	1254
0.06	1.3384	0.3406	1660	834
0.08	1.4351	0.3769	1250	629
0.10	1.5317	0.4126	1000	504
0.12	1.6284	0.4562	830	419
0.14	1.7493	0.4741	710	359
0.16	1.8686	0.5049	620	314
0.18	2.0151	0.5101	550	279
0.20	2.1481	0.5363	500	254
0.30	3.0182	0.6302	330	169
0.40	4.3354	0.7141	250	129
0.50	6.6558	0.7882	200	104

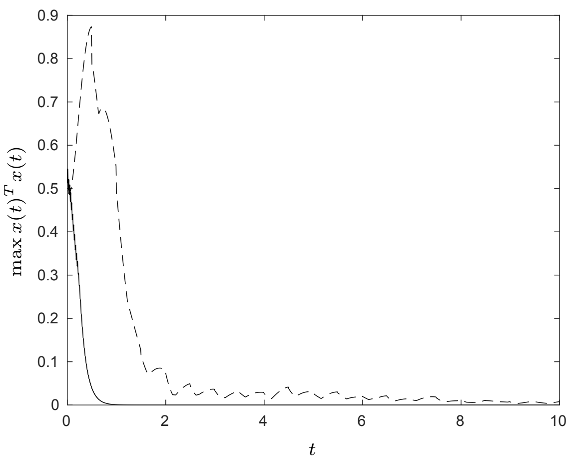

Fig. 1. Bounds for the maximum values of $x(t)^T x(t)$ from 100 random initial conditions applied to system (40), using the state-feedback controller computed from Algorithm 1 with $\delta = 0.5$ (solid line) and $\delta = 0.02$ (dashed line).

Note that higher values of δ resulted in larger values for σ^*, meaning that the set of valid initial conditions for which $x(t)^T x(t) < 1$ is more constrained as δ increases. Such effect is expected since a larger δ implies on higher approximation errors in converting the discrete-time to the continuous-time state-feedback gain $K(t)$. Also, since $K(t)$ is piecewise constant for a δ interval, the controlled trajectories may present a unstable behavior for some $t_k < t \leq t_{k+1}$, as can be seen in Figure 1 for $0 < t < 0.5$.

The trajectories for $\delta = 0.02$ are smoother due to the smaller δ.

The minimum bound γ^* also increases proportionally with δ, mainly due to the effect of $\epsilon(\delta)$, which is a consequence of approximation errors in the discretization procedure and, according to Remark 7, grows with δ. Figure 1 also depicts such effect, since lower values for γ imply on a greater robustness against $w(t)$, which hinder the trajectories for $\delta = 0.5$ to reach the origin, although still guaranteeing the desired boundedness. On the other hand, smaller values of δ imply on a higher number of LMI rows and decision variables, as shown in Table 1, thus increasing the numerical complexity of the technique. The analysis of the obtained results evidences that δ represents a tradeoff factor between computational complexity and the quality of the solution, inspiring further researches for future works.

5. CONCLUSION

A method for the synthesis of time-varying state-feedback gains, capable of guaranteeing the RFTB of continuous-time LTV systems, is presented in this paper. The strategy resorts to the discretization of the system, followed by a procedure to compute a discrete-time state-feedback gain satisfying a set of conditions. The discretized gain is then used to generate the desired continuous-time gain. Since the technique is based on a discrete-time system, the conditions can be implemented in a straightforward way through an algorithm. Due to the discretization procedure, there is a tradeoff between numerical complexity and the quality of the resulting controller, which can be regulated by the user through a scalar parameter. A numerical experiment is also presented to illustrate the main features of the proposed method.

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