$\mathscr{H}_\infty$ Control Tunning to Guarantee the Output Performance of LTI Second-Order Systems

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Abstract: Settling time and input/output specifications are general for optimal operation of control systems and for their avoiding irreversible damages. We present an $\mathscr{H}_\infty$ control design whose parameters are tuned, not only to achieve the robustness property but also to meet the step time response characteristic of a linear time-invariant second-order system. We present explicit formulas in terms of the settling time and overshoot response. Simulation and experimental evidence corroborate the results.

1. INTRODUCTION

All closed-loop systems should meet output time response characteristics such as saturation level, overshoot, or settling time for their optimal operation or for avoiding irreversible damages. Relevant procedures and formulas for tuning controllers are still attractive and open topics for the control community (see, e.g., Lotufo et al. (2019); Kumar et al. (2016); Mak and Poon (2017); Wang (2018)). Typically, controller gains are given as a set of values satisfying the stability criteria only, without taking into account their effect on the output response. Based on the time-domain approach, the present paper resides on how to tune an $\mathscr{H}_\infty$ controller for attaining desired specifications and robustness in the closed-loop.

In the state-space representation in the time domain, particularly in the $\mathscr{H}_\infty$ control, which is based on the dissipativity concept, there exist two free parameters named the *attenuation level* and the *penalty constant* Francis (1998). The latter constant defines the tradeoff between good disturbance rejection at the output and control effort. Usually, these two parameters are arbitrarily tuned to satisfy the asymptotical and robust stability conditions. However, those parameters can affect the transient motion of the system.

The main *contribution* of the paper resides into present explicit formulas in terms of the required output settling time and overshoot responses specifications to find the $\mathscr{H}_\infty$ control gain parameters for time-invariant second-order linear systems for any initial conditions. The objective is in that the output meets the time response specifications (see, e.g., Nise (2007)), and the closed-loop system satisfies the robustness requirements despite the external disturbances and unknown exact physical parameter values of the plant. The present development considers the $\mathscr{H}_\infty$ control synthesis based on the dissipativity concept of Isidori and Astolfi (1992) and van der Schaft (1992). Standard necessary and sufficient conditions of the $\mathscr{H}_\infty$-suboptimal control problem to have a solution with a disturbance attenuation level are given in terms of the existence of positive semidefinite solution of certain algebraic Riccati equation. While requiring this expression to be negative definite as proposed in Orlov and Aguilar (2014) (i.e., to be in the form of inequalities rather than equations), an appropriate $\mathscr{H}_\infty$ design procedure becomes applicable with no a priori-imposed stabilizability condition on the control system, and extra work on verification of this condition is thus obviated. It is worth noticing that the results appear to be straightforwardly applicable to higher-order linear systems by using the dominant-pole approach.

2. PRELIMINARIES

This section reviews some preliminaries in $\mathscr{H}_\infty$ control problem design and the step time response characteristics.

2.1 $\mathscr{H}_\infty$ Control Problem Design

Let us consider the following state-space representation of a linear time invariant system

$$G : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u, \quad x(0) = x_0, \\ z = C_1 x + D_{12} u, \\ y = C_2 x + D_{21} w, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ are the states, $t \in \mathbb{R}_{\geq 0}$ is the time, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is a set of measured variables, $z(t) \in \mathbb{R}^s$ defines a penalty variable, which may include a tracking error as well as a cost of the input $u(t)$ needed to achieve the prescribed control goal, and $w(t) \in \mathbb{R}^r$ is an unknown disturbance of class $\mathscr{L}_2$ to be rejected. In other words, an admissible r-vector function $w(t)$ is such that

$$\int_0^\infty \|w(s)\|^2 ds < \infty. \tag{2}$$

The standard assumptions from Doyle et al. (1989) are imposed on system (1) throughout:

(A1): (A, B_2) is stabilizable and (C_1, A) is detectable,
(A2): $D_{12}^T C_1 = 0$ and $D_{12}^T D_{12} = I$,
(A3): $B_1 D_{21}^T = 0$ and $D_{21} D_{21}^T = I$.

Assumption (A1) guarantees that internal stability of G is equivalent to the input-output stability from w to z. As shown in Orlov and Aguilar (2014), this assumption can be relaxed by the existence of a symmetric positive-semidefinite solution to the algebraic Riccati equation

$$A^T P + PA + C_1^T C_1 + P\left(\frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T\right) P = 0. \tag{3}$$

Assumption (A2) means that $C_1 x$ and $D_{12} u$ are orthogonal thereby yielding a nonsingular, normalized penalty on the control $u(t)$ in the $\mathscr{L}_2$ norm of the output $z = C_1 x + D_{12} u$ representing the system performance. Assumption (A3) is dual to (A2) and concerns how the exogenous signal $w(t)$ enters G. Being nonsingular, the sensor noise weighting is normalized and it is orthogonal to the plant disturbance. Recall the following Doyle et al. (1989).

Theorem 1. Let Assumptions (A1)-(A3) be satisfied for system (1) and let P be a positive semidefinite symmetric solution of (3). Let (1) be driven by the state feedback controller

$$u = -B_2^T P x. \tag{4}$$

Then, the origin of the closed-loop system is asymptotically stable and its perturbed version possesses the $\mathscr{L}_2$-gain less than γ, i.e., the inequality

$$\int_0^\infty \|z(t)\|^2 dt \le \gamma^2 \int_0^\infty \|w(t)\|^2 dt \tag{5}$$

holds for all piecewise-continuous function $w(t)$ for which trajectory $x(t)$ of the closed-loop system, starting from the origin, remains in some neighborhood of the origin for all $t \ge 0$.

Besides, using the strict bounded real lemma (see, e.g., Anderson and Vreugdenhil (1973)), and letting assumptions (A1)-(A3) to be in force, it follows that there exists a positive constant ε such that the perturbed algebraic Riccati equation (ARE)

$$\begin{aligned} &A^T P_\varepsilon + P_\varepsilon A + C_1^T C_1 \\ &\quad + P_\varepsilon \left(\frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T\right) P_\varepsilon + \varepsilon I = 0, \end{aligned} \tag{6}$$

has a unique symmetric positive definite solution P_ε for each $\varepsilon \in (0, \varepsilon_0)$. The result, given below, is extracted from Orlov and Aguilar (2014) to reproduce the above conclusion.

Theorem 2. Suppose Assumptions (A1)-(A3) are satisfied for system (1) and P_ε is a positive definite symmetric solution of (6). Let (1) be driven by the state-feedback controller

$$u = -B_2^T P_\varepsilon x. \tag{7}$$

Then, the origin of the disturbance-free closed-loop system (1), (7) is asymptotically stable and its perturbed version possesses the $\mathscr{L}_2$-gain less than γ thereby yielding the inequality (5) to hold true for all piecewise-continuous function $w(t)$ for which trajectory $x(t)$ of the closed-loop system, starting from the origin, remains in some neighborhood of the origin for all $t \ge 0$.

In the sequel, we use Theorem 2 to find a solution to the state-feedback $\mathscr{H}_\infty$ control problem for the system (1).

2.2 *Time Responses of Second-Order Systems*

Let us consider the stable second order transfer function

$$g(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{8}$$

where $\omega_n > 0$ is the natural frequency, $\zeta > 0$ is the damping ratio, and $s = j\omega$ is the complex variable ($j = \sqrt{-1}$).

From the classical control theory, the quality of the step time response can be quantified by the rise time t_r, the settling time t_s, and the percent overshoot $\%OS$ (see Fig. 1). These indexes

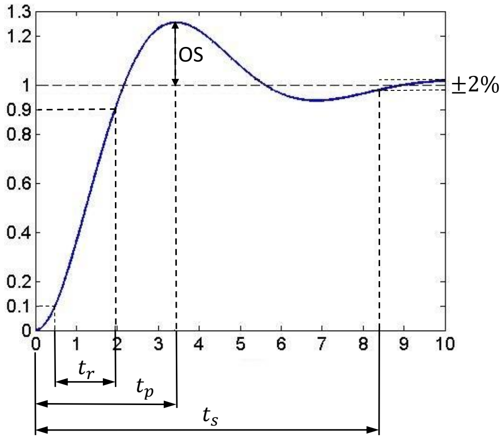

Fig. 1. Step response of second order systems where t_s is the stabilization time, t_p is the time to reach the maximum value, and t_r is time evolution from the 10 to the 90 percent of the signal.

can be approximately computed as Nise (2007):

$$t_r = \frac{1}{\omega_n\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{-\zeta\omega_n}\right), \tag{9}$$

$$t_s = \frac{4}{\zeta\omega_n}, \quad \text{(2\% criteria)} \tag{10}$$

$$\%OS = \exp\left\{-\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\pi\right\} \times 100\%. \tag{11}$$

The key aspects of the latter equations are 1) the speed of the system response is proportional to the natural frequency ω_n, and 2) the overshoot of the system response is determined only by the damping ratio ζ.

3. PROBLEM STATEMENT

Let us consider the following state-space representation of a second-order system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a_1 x_1 - a_2 x_2 + b(w_x + u), \\ y_1 &= x_1 + w_1, \\ y_2 &= x_2 + w_2, \end{aligned} \tag{12}$$

where $x(t) = [x_1\ x_2]^T \in \mathbb{R}^2$ is the state vector, $y = [y_1\ y_2]^T \in \mathbb{R}^2$ is the output vector, $w(t) = [w_x\ w_1\ w_2]^T \in \mathbb{R}^3$ is composed of the external disturbances and noise, corrupting the measurements, $u(t) \in \mathbb{R}$ is the control input, $t \in \mathbb{R}_{\ge 0}$ is the time, a_1 and a_2 are positive scalars, and $b > 0$ reflects the control gain.

The system (12) can be represented in the form (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ b \end{bmatrix}. \tag{13}$$

The matrices of the penalty variable $z(t)$ are

$$C_1 = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{14}$$

where ρ_1 and ρ_2 are positive scalars. Although setting $\rho_1 = \rho_2$ is possible, however, numerical and experimental evidences have demonstrated that the solution to the perturbed Riccati equation (6) can be obtained with lower values of γ (thereby enhancing the disturbance attenuation), if ρ_1 and ρ_2 have distinct values (see, e.g., Ponce et al. (2017)). The output $y(t) = C_2x + D_{21}w$ contains the full-state measurements corrupted by additive noise where

$$C_2 = I_2, \quad D_{21} = [0_{2\times 1} \;\; I_2]. \tag{15}$$

Here, I_2 defines the 2×2 identity matrix and $0_{2\times 1}$ is the 2×1 matrix of zeros.

The *control objective* is to asymptotically stabilize the origin of the closed-loop system (7), (12) while the perturbed version satisfy the inequality (5). Moreover, the closed-loop system (7), (12) should meet a certain performance, given in terms of (9)–(11), that is, ρ_1, ρ_2, ε, and γ should be tuned to arrive at the desired settling time $t_s^d = t_s(\omega_n^d, \zeta^d)$ and the desired percent overshoot $\%OS^d = \%OS(\zeta^d)$. Hereinafter, ζ^d is the desired damping ratio according to the desired time response overshoot $\%OS^d$ to a step input signal.

Under the Assumption that the matrix A is Hurwitz, the first part of the control objective can be straightforwardly solved by applying Theorem 2 with arbitrary values of ρ_1, ρ_2, and ε and using an iterative algorithm to find the optimal γ (γ^*). However, the following question arises as to how the given values of ρ_1, ρ_2, ε, and $\gamma \geq \gamma^*$ affect the output response of the closed-loop system? This question is motivated by real physical applications because many systems require that their outputs meet strict settling time and percent overshoot specifications for their optimal operation and/or for avoiding irreversible damages.

In the sequel, we consider (9)–(11) with the time response specifications to be achieved by means of tuning the $\mathscr{H}_\infty$ controller (7).

4. MAIN RESULT

In this section we shall detail how to tune the parameters of the $\mathscr{H}_\infty$ controller (7), through the matrix P_ε, not only to satisfy the inequality (5) but also to meet the performance specifications (9)–(11) according to the control objective.

To this end, let us consider the unperturbed ($w \equiv 0$) closed-loop system (7), (12), specified with (13), possessing the following characteristic polynomial function

$$s^2 + (a_2 + b^2 p_{22})s + (a_1 + b^2 p) = 0, \tag{16}$$

where $p \in \mathbb{R}$ and $p_{22} \in \mathbb{R}^+$ are entries of the matrix P_ε, that is

$$P_\varepsilon = \begin{bmatrix} p_{11} & p \\ p & p_{22} \end{bmatrix} > 0, \tag{17}$$

with a positive constant entry p_{11}.

According to (16), the response characteristics of the closed-loop system (7), (12) are determined by p_{22} and p. Then, to satisfy the desired closed-loop time response characteristics, denoted as t_s^d and $\%OS^d$, we may find ζ from (11) yielding

$$\zeta^d = \frac{-\ln(\%OS^d/100)}{\sqrt{\pi^2 + \ln^2(\%OS^d/100)}}. \tag{18}$$

Notice that the rise time t_r (9) and the settling time t_s (10) are both dependent on the damping ratio ζ and the natural frequency ω_n. However, only one of them can be pre-specified to uniquely determine the other. For simplicity, let us calculate the natural frequency ω_n from (10), yielding

$$\omega_n^d = \frac{4}{\zeta^d t_s^d}. \tag{19}$$

Let us consider the desired characteristic polynomial of a generalized second-order system

$$s^2 + 2\zeta^d\omega_n^d s + (\omega_n^d)^2 = 0. \tag{20}$$

If we match each coefficient of the latter equation with each coefficient of (16) and, solving for p_{22} and p, yields

$$p_{22} = \frac{2\zeta^d\omega_n^d - a_2}{b^2}, \tag{21}$$

$$p = \frac{(\omega_n^d)^2 - a_1}{b^2}, \tag{22}$$

provided that $\zeta^d\omega_n^d > a_2/2$ and $\omega_n^d > \sqrt{a_1}$. The relations on the desired step time response parameters t_s^d and $\%OS^d$

$$0 < t_s^d < \frac{8}{a_2}, \tag{23}$$

$$0 < \%OS^d < e^{-(\zeta_{\max}^d/\sqrt{1-(\zeta_{\max}^d)^2})\pi} \times 100\%, \tag{24}$$

with $\zeta_{\max}^d = a_2/(2\sqrt{a_1}) > \zeta^d$ guarantees each main diagonal entry of P_ε to be strictly positive.

Now, we need to find the conditions that makes P_ε to be positive definite and to satisfy the perturbed Riccati equation (6) with the chosen values (21)–(22).

First, we develop the perturbed Riccati equation (6). To this end, we substitute the corresponding matrices (13)–(15) into (6), thus obtaining the following set of algebraic equations:

$$(b^2\gamma^{-2} - b^2)p^2 - 2a_1 p + (\rho_1^2 + \varepsilon) = 0, \tag{25}$$

$$p_{11} - a_2 p - a_1 p_{22} + (b^2\gamma^{-2} - b^2)pp_{22} = 0, \tag{26}$$

$$(b^2\gamma^{-2} - b^2)p_{22}^2 - 2a_2 p_{22} + 2p + (\rho_2^2 + \varepsilon) = 0, \tag{27}$$

where $p = p(\zeta^d, \omega_n^d)$ and $p_{22} = p_{22}(\zeta^d, \omega_n^d)$. In terms of ρ_1, ρ_2, and p_{11}, the solutions of (25)–(27) are

$$\rho_1 = \sqrt{2a_1 p - \left(\frac{b^2}{\gamma^2} - b^2\right)p^2 - \varepsilon}, \tag{28}$$

$$\rho_2 = \sqrt{2a_2 p_{22} - 2p - \left(\frac{b^2}{\gamma^2} - b^2\right)p_{22}^2 - \varepsilon}, \tag{29}$$

$$p_{11} = a_2 p + a_1 p_{22} - \left(\frac{b^2}{\gamma^2} - b^2\right)pp_{22}. \tag{30}$$

The parameter p_{11} also depends of ζ^d and ω_n^d. The relations (28)–(29) are reals provided

$$\gamma_1 \geq \sqrt{\frac{b^2}{\frac{a_2}{p_{22}} + \frac{a_1}{p} + b^2}}, \tag{31}$$

$$\gamma_2 \geq \sqrt{\frac{b^2p^2}{2a_1p + b^2p^2 - \varepsilon}}, \tag{32}$$

and p_{11} is strictly positive if

$$\gamma_3 > \sqrt{\frac{b^2p_{22}^2}{2a_2p_{22} - 2p + b^2p_{22}^2 - \varepsilon}} \tag{33}$$

where γ_1, γ_2, and γ_3 are the corresponding set of values of γ for (28), (29), and (30), respectively. Moreover, P_ε is positive definite if γ is chosen as

$$\gamma_4 > \sqrt{\frac{b^2p_{22}^2p}{a_2p_{22}p + a_1p_{22}^2 + b^2p_{22}^2p - p^2}}. \tag{34}$$

As a conclusion, the matrix P_ε is positive definite and satisfy (6) if and only if

$$\gamma \geq \gamma^* = \max_i\{\gamma_i\}, \qquad i = 1, \dots, 4. \tag{35}$$

Summarizing, the following result is obtained.

Theorem 3. Let the conditions of Theorem 2 be satisfied for system (1) specified with matrices (13)–(15). For any given settling time t_s^d and percent overshoot $\%OS^d$ inside the domains (23)–(24), for any $\varepsilon \in (0, \varepsilon_0)$, and for any γ satisfying (35), there exists a positive definite symmetric solution of (6) given by

$$P_\varepsilon = \begin{bmatrix} a_2 p + a_1 p_{22} - \left(\dfrac{b^2}{\gamma^2} - b^2\right) p p_{22} & p \\ p & p_{22} \end{bmatrix} \tag{36}$$

with entries (21)–(22), such that (1), (13)–(15) driven by the state-feedback controller (7) makes the origin of the closed-loop system asymptotically stable and the output response meets the performance definitions (9)–(11). Moreover, its perturbed version possesses the $\mathscr{L}_2$-gain less than γ; namely the inequality (5) holds for all piecewise-continuous function $w(t)$ for which trajectory $x(t)$ of the closed-loop system, starting from the origin, remains in some neighborhood of the origin for all $t \geq 0$.

Proof: Let us consider radially unbounded function $V = x^T P_\varepsilon x$ which is positive definite provided (35). The time derivative of $V(x)$ along the solution of the closed-loop system (7)–(12) for the disturbance-free system ($w = 0$) satisfies

$$\begin{aligned} \dot V &= 2x^T P_\varepsilon \dot x = 2x^T P_\varepsilon\,(Ax + B_2 u) \\ &= x^T\left(P_\varepsilon A + A^T P_\varepsilon\right)x + 2x^T P_\varepsilon B_2 u \\ &= x^T\left(P_\varepsilon A + A^T P_\varepsilon - 2P_\varepsilon B_2 B_2^T P_\varepsilon\right)x. \end{aligned} \tag{37}$$

Since

$$\begin{aligned} P_\varepsilon A + A^T P_\varepsilon - 2P_\varepsilon B_2 B_2^T P_\varepsilon &\leq A^T P_\varepsilon + P_\varepsilon A \\ + C_1^T C_1 + P_\varepsilon\left(\frac{1}{\gamma^2}B_1 B_1^T - B_2 B_2^T\right)P_\varepsilon &\leq -\frac{\varepsilon}{2}, \end{aligned} \tag{38}$$

the relation (37) becomes

$$\dot V \leq -\frac{\varepsilon}{2}\|x\|^2 \tag{39}$$

thus concluding that the origin of the closed-loop system is asymptotically stable.

For the perturbed case, the time derivative of $V(x)$ along the solution of the closed-loop system (7)–(12) satisfies

$$\begin{aligned} \dot V(x) &= H(x, V_x, \alpha_1, \alpha_2) - \gamma^2\|w - \alpha_1\|^2 - \|z\|^2 \\ &\quad + \gamma^2\|w\|^2 - \frac{\varepsilon}{2}\|x\|^2 \end{aligned} \tag{40}$$

where $H(x, V_x, \alpha_1, \alpha_2)$ is given by the left-hand sides of (6), $V_x = \partial V/\partial x$, $\alpha_1 = -(1/\gamma^2)B_1^T P_\varepsilon x$, and $\alpha_2 = -B_2^T P_\varepsilon x$.

The latter inequality ensures that

$$\begin{aligned} V(x(t)) - V(x(0)) &\leq \frac{\varepsilon}{2}\int_0^t \|x(\tau)\|d\tau \\ + \int_0^t \left(\gamma^2\|w(\tau)\|^2 - \gamma^2\|w(\tau) - \alpha_1(x(\tau))\|^2 - \|z(\tau)\|^2\right)d\tau&. \end{aligned} \tag{41}$$

Since $V(x)$ is positive definite, it follows

$$\begin{aligned} \int_0^t (\gamma^2\|w(\tau)\|^2 - \|z(\tau)\|^2)d\tau &\geq \frac{\varepsilon}{2}\int_0^t \|x(\tau)\|d\tau \\ + V(x(t)) - V(x(0)) + \int_0^t (\gamma^2\|w(\tau) - \alpha_1(x(\tau))\|^2)d\tau &> 0, \end{aligned} \tag{42}$$

Thus, the inequality (5) is straightforwardly concluded from (42) for any $w \in \mathscr{L}_2$ and for the solutions of the closed-loop system (1), (7), (13)–(15), initialized at $x(0) = 0$. ■

Clearly, Theorem 3 results in the tuning procedure for the gain matrix P_ε and the attenuation level γ to meet the desired time response specifications. First, one needs to represent the second-order system (12) in the $\mathscr{H}_\infty$ standard representation (1). Then, the percent overshoot $\%OS^d$ and the desired settling time t_s^d are set. Second, the natural frequency ω^d and the damping ζ^d are calculated by using (18) and (19). Then, the entries p_{22} and p of P_ε are obtained from (21) and (22), respectively. Finally, choosing γ according to (35) guarantees that P_ε is positive definite and the desired output response of the closed-loop system (7), (12) attains the expected percent overshoot $\%OS^d$ and the settling time t_s^d.

Next section presents simulation and experimental evidences to corroborate Theorem 3.

5. SIMULATION AND EXPERIMENTAL RESULTS

5.1 Simulations Results

For simulation purposes, let us consider the equation of motion of a DC motor governed by

$$J\ddot q + f_v \dot q = \tau + w_q, \tag{43}$$

where $q(t) \in \mathbb{R}$ is the position of the rotor, $\dot q(t) \in \mathbb{R}$ is the angular velocity, $w_q(t) \in \mathbb{R}$ is a coupled disturbance satisfying (2), $J = 3.88 \times 10^{-5}$ kg·m^2 is the inertia, the viscous friction coefficient is set to $f_v = 4.22 \times 10^{-5}$ Nm·s/rad, and the control input τ is seek in the form

$$\tau = -k_p q + u, \tag{44}$$

where $k_p = 0.0005$ is the proportional gain and $u(t)$ is to be specified according to (7) subject to the conditions of Theorem 3.

The output to be controlled is specified by $z(t) = [q\;\dot q\;u]^T$ whereas the rotor position q and velocity $\dot q$ are assumed to be available for measurements, corrupted with additive noise $w_y = [w_1\;w_2]^T \in \mathbb{R}^2$.

Let $x_1(t) = q(t)$ and $x_2(t) = \dot q(t)$ be the states of the system and $w = [w_q\;w_1\;w_2]^T$ is a vector composed by the external and measurement disturbances. The closed-loop system (43)–(44) can thus be rewritten in the standard state-space representation (1) where

$$\begin{aligned} &A = \begin{bmatrix} 0 & 1 \\ -\dfrac{k_p}{J} & -\dfrac{f_v}{J} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{1}{J} & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \dfrac{1}{J} \end{bmatrix}. \\ &C_1 = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ &C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{45}$$

The solution to the ARE (6), given by the matrix $P_\varepsilon > 0$, is determined to satisfy t_s^d and $\%OS^d$ together with the attenuation

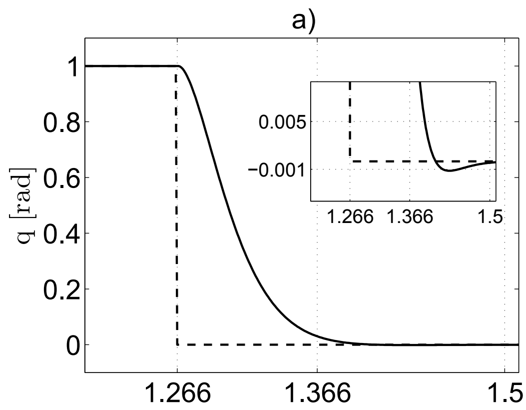

Fig. 2. Time responses for simulation 1 where $t_s = 0.1$ s and %OS=0.1 % were specified as required output.

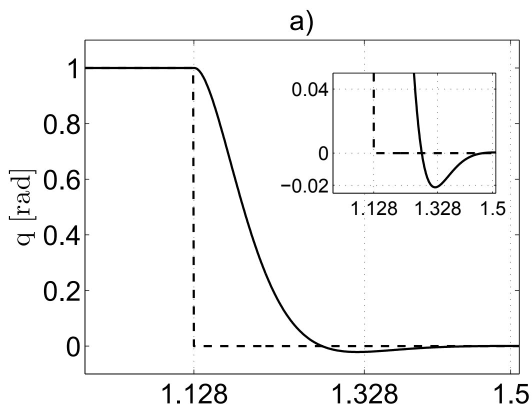

Fig. 3. Time responses for simulation 2 where $t_s = 0.2$ s and %OS=2.0 % were specified as required system output.

level γ (35). The matrix P_ε is thus conformed by (21), (22), and (30).

The simulations are performed on a personal computer with a Core–i7 processor and *Matlab/Simulink*® using the linear model (43)–(44) together with (7). Two simulations are carried out with different time response specifications. The control objective is to regulate the position at the origin, i.e., $q \equiv \dot{q} \equiv 0$ with predefined settling time t_s and percent overshoot $\%OS$.

The forward Euler integration method is applied with the sampling time 0.001 s, and the initial conditions of the test are set to $q(0) = 1$ rad and $\dot{q}(0) = 0$ rad/s. The same value $\varepsilon = 1 \times 10^{-10}$ is employed for both simulations conducted.

Simulation 1: For the first simulation, we choose the desired step input time response parameters as $t_s^d = 0.1$ s and $\%OS^d = 0.1\,\%$. Figure 2 shows the step time response of the closed-loop system (43)–(44) using the $\mathscr{H}_\infty$ controller (7). As seen in Fig. 2(a), the desired step input time response parameters t_s^d and $\%OS^d$ are achieved, where $\rho_1 = 0.0626$, $\rho_2 = 9.83 \times 10^{-4}$, and $\gamma = 1.8$ are derived from formulas (28)–(35).

Simulation 2: The desired step input time response parameters for the second simulation are $t_s^d = 0.2$ s and $\%OS^d = 2.0\,\%$. Figure 3 shows the step time response of the closed-loop (43)–(44) with $u(t)$ as in (7). As seen in Figure 3(a), the desired step

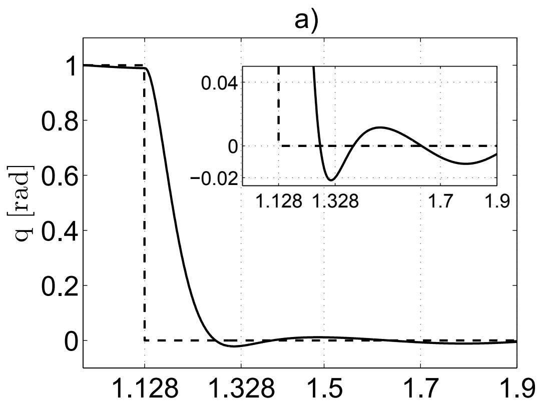

Fig. 4. Time responses for simulation 2: the perturbed case.

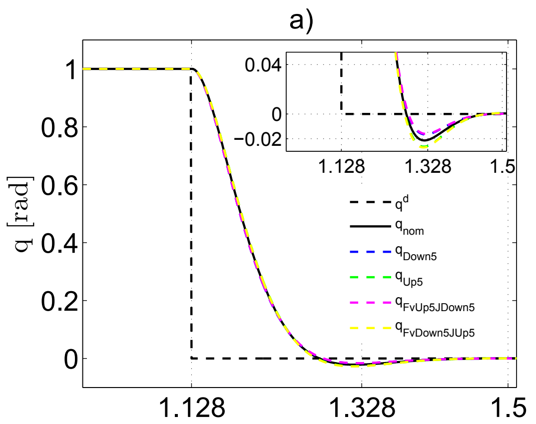

Fig. 5. Time responses for simulation 2 under uncertain parameters of the plant.

input time response parameters t_s^d and $\%OS^d$ are achieved with $\rho_1 = 0.0231$, $\rho_2 = 1.83 \times 10^{-4}$, and $\gamma = 2.3$, derived from (28)–(35).

After that, the second simulation is repeated while adding the disturbance $w_q(t) = 0.0003\sin(10t)$. Figure 4 shows the step time response of the closed-loop (43)–(44) with $u(t)$ as in (7). As seen in Figure 4, the desired step input time response parameters t_s^d and $\%OS^d$ are achieved despite the disturbance $w_q(t)$. Figure 4 shows the control input.

Finally, a deviation of the real parameters J and f_v with respect to their nominal values defined as J^{nom} and f_v^{nom} is brought into play. For this simulation, the values $J^{\mathrm{nom}} = 3.88 \times 10^{-5}$ kg·m^2 and $f_v^{\mathrm{nom}} = 4.22 \times 10^{-5}$ Nm·s/rad are chosen to subsequently compute the values of p and p_{22} in (21)–(22). The values J and f_v of the plant (43) are varied between ± 5 percent with respect to their nominal values. From Fig. 5, one can observe that the trajectories preserve the expected motion.

5.2 Experimental Results

The performance of the controller is finally tested with experiments made on a DC motor manufactured by *Leadshine*. The controller runs on a personal computer with a Core-i7 processor and *Matlab/Simulink*® coupled together with a *dSPACE® 1701* prototyping hardware. In addition, the amplifier of the motor

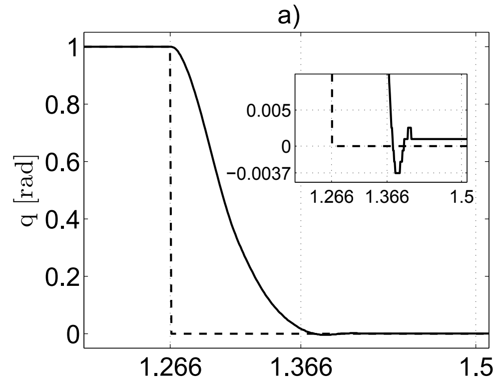

Fig. 6. Experiment: Time responses where $t_s = 0.1\,\mathrm{s}$ and %OS=0.1 % were specified as required output.

accepts a control input from the D/A converter in the range of ± 10 V. The proposed controller for experiment is

$$\tau = -F_c\,\mathrm{sign}(\dot{q}) - k_p q + u \tag{46}$$

where $F_c = 99.7 \times 10^{-4}$ N·m is the Coulomb friction coefficient and $k_p = 0.0005$ is the proportional gain.

The forward Euler integration method is applied with the sampling time 0.001 s, and the initial conditions, chosen in the experiments, are $q(0) = 1.0$ rad and $\dot{q}(0) = 0$ rad/s. The control objective is to regulate the position to the origin, i.e., $q \equiv \dot{q} \equiv 0$ with predefined settling time t_s and percent overshoot $\%OS$ as well. Two experiments are performed with different time response parameters. As in the numerical simulations, $\varepsilon = 1 \times 10^{-10}$.

Experiment 1: For the first experiment, the step input time response parameters are $t_s^d = 0.1$ s and $\%OS^d = 0.1\,\%$. Figure 6 shows the step time response of the closed-loop system (43), (46). As seen in Figure 6(a), the desired step input time response parameters t_s^d and $\%OS^d$ are achieved under $\rho_1 = 0.0626$, $\rho_2 = 9.83 \times 10^{-4}$, and $\gamma = 1.8$ obtained from (28)–(35).

Since the model of the plant is obtained from the experimental motor, the same values of ρ_1 and ρ_2 are obtained in both the simulation and experimental results. The experimental results demonstrate that the controller also preserves the predefined output in spite of the imprecisions of the physical parameters. For higher-order systems, one can use the dominant pole technique to approximate the system with a second-order system.

6. CONCLUSIONS

Tuning a linear $\mathscr{H}_\infty$ controller to meet the $\mathscr{L}_2$-gain inequality (2) and the performance specifications is developed for second-order linear time-invariant systems. The methodology consists in applying the $\mathscr{H}_\infty$ control theory of Orlov and Aguilar (2014), coupled together with the long-recognized time response approximation Francis (1998) for the linear second-order system of interest. The simulation and experimental results corroborate the theoretical development. This paper represents the first step towards the synthesis of nonlinear $\mathscr{H}_\infty$ controllers satisfying required time responses of the plant. It should be pointed out that the time response constants, defined by (9)–(11), meet the desired specifications if a_1 and a_2 are exactly known. However, from the physical point of view, the constants a_1 and a_2 typically stand for the nominal values and parametric errors can exist. Under this concern, the adaptive $\mathscr{H}_\infty$ synthesis, developed in Orlov et al. (2018) for a linear scalar system, is expected to be applicable to the second-order system as well. The work is still in progress and will be reported elsewhere. Future work also includes the formula derivations for the output feedback case for time-invariant and time-variant systems and applications to mechatronic systems.

7. ACKNOWLEDGMENTS

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