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Data rate limits for the remote state estimation problem ⋆

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Abstract: In the context of control and estimation under information constraints, restoration entropy measures the minimal required data rate above which a system can be regularly observed. The observer here is assumed to receive its state information through a communication channel of a finite bit-rate capacity. In this paper, we provide a new characterization of restoration entropy which does not require to compute any temporal limit, i.e., an asymptotic quantity. Our new formula is based on the idea of finding an adapted Riemannian metric on the state space that allows to 'see' the decisive quantity that determines the restoration entropy – a certain type of Lyapunov exponent – in only one step of time.

Keywords: Entropy, Nonlinear systems, First and second Lyapunov methods

1. INTRODUCTION

The past few decades have witnessed a substantially growing attention to networked control systems (Hespanha et al., 2007; Heemels et al., 2010) and related problems of control and/or state estimation via communication channels with constrained bit-rates; for extended surveys of this area, we refer the reader to (Matveev and Savkin, 2009; Yüksel and Başar, 2013; Andrievsky et al., 2010) and references therein.

One of the fundamental concerns in this context is to find a minimal data rate between the communication peers under which remote state estimation (Wong and Brockett (1997), see also Fradkov et al. (2015) for a related problem) is feasible; in other words, the receiver is able to reconstruct the current state of the remote system in the real time regime if and only if data about this state is updated by means of a bit flow whose intensity exceeds that "threshold" rate. Loosely speaking, this communication rate has to exceed the rate at which the system "generates information", while the latter concept is classically formalized in a form of entropy-like characteristic of the dynamical system at hands. The related mathematical results are usually referred to as Data Rate Theorems (see, e.g. (Nair et al., 2007; Matveev and Savkin, 2009; Matveev and Pogromsky, 2019) and references therein) - their various versions coexist to handle various kinds of observability and models of both the plant and the constrained communication channel.

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Those results deliver a consistent message that the concept of the topological entropy (TE) of the system and its recent offshoots provide the figure-of-merit needed to evaluate the channel capacity for control applications; the mentioned modifications of TE are partly aimed to properly respond to miscellaneous phenomena crucial for control problems, like uncertainties in the observed system (Savkin, 2006; Kawan and Yüksel, 2017, 2018), implications of control actions (Colonius et al., 2013; Hagihara and Nair, 2013; Colonius and Kawan, 2009), the decay rate of the estimation error (Liberzon and Mitra, 2018), or Lipschitz-like relations between the exactness of estimation and the initial state uncertainty (Matveev and Pogromsky, 2019). Keeping in mind relevance of communication constraints in modern control engineering, constructive methods to compute or finely estimate those entropy-like characteristics take on crucial not only theoretical but also practical importance. Several steps have been done in this direction in (Pogromsky and Matveev, 2011; Matveev and Pogromsky, 2019; Hafstein and Kawan, 2019), where corresponding upper estimates were found by following up the ideas of the second Lyapunov method. Moreover, it was shown that for some particular prototypical chaotic systems of low dimensions, these upper estimates are exact in the sense that they coincide with the true value of the estimated quantity.

Whether these inspiring samples of precise calculations are mere incidents, or implications of specific traits of very special either systems or their classes, or, conversely, are particular manifestations of a comprehensive capacity inherent in the employed approach? Confidence in the last option would constitute a rationale for undertaking special efforts aimed to fully unleash the potential of this approach via its further elaboration.

The primary goal of the current paper is to answer the posed question; we show that among the above options, the last one is the true one. This is accomplished via a sort of a converse result that is similar in idea to celebrated converse Lyapunov theorems. An outcome is viewed as setting a theoretical benchmark and opening the perspective for the respective research direction. Among various descendants of TE, we pick the so-called *restoration entropy* (Matveev and Pogromsky, 2019) to deal with. Like the second Lyapunov method, this concept is much inspired by the concerns of control theory; this may be viewed as a "bedrock" of the fact that this method and concept appear to be closely linked by both direct and converse theorems. To complete the picture of the respective "direct" part, we also extend some core results of (Matveev and Pogromsky, 2019) on the case of a discrete-time plant. The tractability of the developed approach is confirmed by closed-form computation of the restoration entropy for the celebrated Landford system (see, e.g., (Belozyorov, 2015)). Meanwhile, computation or even fine estimation of TE and the likes has earned the reputation of an extremely complicated matter (Downarowicz, 2011).

The paper is organized as follows. Sect. 2 offers largely informal introduction to the issues raised in this paper. Sect. 3 injects the complete rigor into the problem setup and presents the main assumptions. Sections 4 and 5 contain the main results, which are illustrated by an example in Sect. 6.

2. STATE ESTIMATION VIA LIMITED COMMUNICATION, IN OUTLINE

To better highlight the incentive for this study, we first recall some relevant existing results on the remote state estimation via communication channels with limited bit-rate capacity. The goal of this overview is to shed a light on how a particular choice of the Riemannian metric in the state space can facilitate the solution of the state estimation problem. To this end, we consider the following discrete-time dynamical system

$$x(t+1) = \varphi(x(t)), \;\; t = 0, 1, 2, \dots, \quad x(0) \in K \subset \mathbb{R}^n. \quad (1)$$

Here φ is a continuously differentiable mapping on $\mathbb{R}^n$ and K is a given compact set of initial states that are of our interest. Throughout the text, φ^t stands for the t-th iterate of φ and $K(t)$ is the time-t image of the set K, i.e., $K(t) := \varphi^t(K)$.

We consider the case where state estimation involves two remote peers, say Alice and Bob. The both are aware of the dynamics φ and the set K. The difference between the peers is that Alice has full access to the current state $x(t)$ (either via direct measurement or computation from $x(0)$ and φ), whereas Bob has not, and his only option is to rely on data sent by Alice. The problem stems from the fact that the channel of communication between Alice and Bob has limited data rate capacity: on average, no more than c bits can be conveyed from Alice to Bob per unit time. So at best, only an approximate value of $x(t)$ is affordable to Bob, since the full knowledge about $x(t)$ is substantiated by infinitely many bits. The goal of Bob is to consecutively build an estimate of $x(t)$ at times $t = 0, 1, \dots$ with an acceptable accuracy. Alice should be cooperative and build the sequential messages $e(t), t = 0, 1, \dots$ to Bob in a way favorable for achieving that goal.

Overall success in this endeavor depends not only on operation of the peers but also on the channel data rate capacity c: there is a critical value c_{crit} of c below which any measures have no chance for success. To be definite in defining c_{crit}, we borrow the model of data rate capacity from (Matveev and Savkin, 2009, Sect. 3.4). In other words, we assume a uniform bound $b_+(\tau)$ on the total number of bits that can be transferred, in any way, via the channel within any time interval of duration τ. At the same time, it is not possible to transmit less than $b_-(\tau)$ bits. Finally, we assume that the average numbers of bits per unit time calculated from both b_+ and b_- converge to a common limit c as $\tau \to \infty$, which is called the *channel capacity*:

$$c := \lim_{\tau\to\infty} b_\pm(\tau)/\tau. \quad (2)$$

A detailed discussion of this definition and relevant examples are available in (Matveev and Savkin, 2009, Sect. 3.4).

The critical value c_{crit} might depend on the requested quality of estimation. Various grades of this quality are discussed in (Matveev and Pogromsky, 2016) and embodied in the definitions of various kinds of observability; to address the issue specifically, we shall focus on the *regular observability*. It assumes that both peers are given a common initial estimate $\widehat{x}(0)$ of the state and a common known bound $\delta \approx 0$ on its accuracy:

$$x(0) \in B_\delta(\widehat{x}(0)), \quad (3)$$

where $B_\delta(\xi)$ is the ball with a radius of δ centered at ξ. Suppose that the peers can act so that as time progresses, the current estimation error is kept proportional to its initial value δ:

$$x(t) \in B_\varepsilon(\widehat{x}(t)), \quad \varepsilon = G\delta, \qquad t \geq 0. \quad (4)$$

Here G does not depend on t, $\delta \approx 0$, and $x(0), \widehat{x}(0) \in K$ satisfying (3), but (4) should hold for all of them. The role of Alice is to compose messages to Bob that fit the channel capacity c. Bob converts the messages received until t into an estimate $\widehat{x}(t)$. If (4) can be achieved via the channel at hands, the system is said to be *regularly observable* via it. The infimum of c's over such channels is called the *regular observability rate* $\mathcal{R}_{\text{ro}}$ (Matveev and Pogromsky, 2016) and is fully defined by φ, K. Our interest is on how to compute $\mathcal{R}_{\text{ro}}$ from φ and K.

In order to respect the communication rate c, Alice and Bob can agree upon the communication protocol which is organized in *epochs* of duration τ, where the quantity τ is sufficiently large. The first step to proceed is to cover the set $\varphi^\tau[B_\delta(\widehat{x}(0)) \cap K]$ with δ-balls such that the size of the covering meets the given channel capacity: Alice can inform Bob of her choice from elements of the covering via sending a packet of messages $\pi(\tau) = [e(0), \dots, e(\tau)]$ via the channel. She uses this to inform Bob about the element (ball) that contains the current state $x(t)$, and Bob defines the estimate $\widehat{x}(t)$ as the center of this ball. As a result, (4) is ensured with $t = \tau$ and $G = 1$.

To extend (4) to all t, the above actions are repeated, first from τ to 2τ, then from 2τ to 3τ, and so on. Then (4) becomes true for $t = \tau k, k = 0, 1, \dots$. To fill the remaining gaps, the "trivial" observer $\widehat{x}(t+1) = \varphi[\widehat{x}(t)]$ is run from $\widehat{x}(\tau k)$ for $t = \tau k, \dots, \tau(k+1)-1$. Since the maps $\varphi^t(\cdot)$ are Lipschitz on $K(\tau k)$ in many cases (with a constant G that does not depend on $t = 0, \dots, \tau$), this ensures (4) for all t.

Being directly inspired by practical needs, the communication protocol operates with balls that are defined with respect to a common, Euclidean metric. So the idea to assign an individual Riemannian metric tensor to any point x and to modify the foregoing via dealing with the respective "balls" whose shapes critically change over the space may look as a complication, and nothing else. Meanwhile, it is shown in (Matveev and Pogromsky, 2016) that such an assignment may serve as a basis for constructive estimation, or even exact analytical computation of $\mathcal{R}_{\text{ro}}$.

Specifically, let $P(x)$ stand for the positive definite matrix associated with the Riemannian metric tensor at the point x (Chavel, 2006). According to (Matveev and Pogromsky, 2016),

$$\mathcal{R}_{\text{ro}} \leq \frac{1}{2} \max_{d=1,\ldots,n} \sup_{x\in K^\infty} \Big[\Delta v_d(x) + \sum_{i=1}^{d} \log_2 \lambda_i(x)\Big]. \tag{5}$$

Here $K^\infty := \bigcup_{t=0}^{\infty} K(t)$ and $\Delta v_d(x) := v_d[\varphi(x)] - v_d(x)$ is the one-step-increment of a d-dependent function $v_d : \mathbb{R}^n \to \mathbb{R}$, whereas $\lambda_1(x) \geq \ldots \geq \lambda_n(x)$ are the roots of the following algebraic equation:

$$\det[\mathrm{D}\varphi(x)^\top P(\varphi(x))\mathrm{D}\varphi(x) - \lambda P(x)] = 0 \tag{6}$$

and $\mathrm{D}\varphi(x)$ is the Jacobian matrix of φ at the point x. Finally, (5) is true for any set $v_1(\cdot), \ldots, v_n(\cdot)$ of bounded functions.

The formula (5) is largely constructive. This trait, intelligent choices of $\mathfrak{P} := [P(\cdot), v_1(\cdot), \ldots, v_n(\cdot)]$, and simple lower bounds on $\mathcal{R}_{\text{ro}}$ permitted to find a closed-form expression of $\mathcal{R}_{\text{ro}}$ in terms of the parameters of some celebrated chaotic systems, as is shown in (Matveev and Pogromsky, 2016).

Inspired by these precedents, this paper aims to explore the principal potentiality of the outlined "Riemannian metric" approach. In particular, this paper is to judge whether the aforementioned closed-form computations should be viewed as accidental successes or there are solid reasons to expect something like this in general and so to invest in developing technical tools needed to implement this approach. We shall show that the second option holds, and offer relevant technical developments.

In fact, the considered approach is similar in nature to the second (direct) Lyapunov method of stability study (Matveev and Pogromsky, 2016). These two techniques share a common challenge that stems from the reliance on wisely inventing an auxiliary object, i.e., $\mathfrak{P}$ or a Lyapunov function. In particular, a bad choice of $\mathfrak{P}$ results in an overly conservative estimate. Though the theory of stability has not yet provided general techniques for the construction of Lyapunov functions, the so-called inverse Lyapunov theorems guarantee (under certain technical assumptions) that the second Lyapunov method is in a sense comprehensive and thus justify its persistent developing. In this paper, we establish a similar in spirit "comprehensiveness" by showing that under a clever choice of $\mathfrak{P}$, the r.h.s. of (5) approaches its l.h.s. as close as desired. Moreover, we disclose and justify a way to simplify both (5) and search for a proper $\mathfrak{P}$ via showing that without any loss of generality, $\Delta v_d(\cdot)$ can be discarded by taking $v_d(\cdot) \equiv \text{const}$. In future technical developments, this prioritizes the design of $P(\cdot)$ over that of $v_d(\cdot)$.

3. RIGOROUS SETUP OF THE PROBLEM

In this paper, we adopt the following assumption.

Assumption 1. In (1), the map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1 and the set $K \subset \mathbb{R}^n$ is compact and forward-invariant.

The main idea behind the communication protocol described in Sec. 2 gave rise to a quantity called in (Matveev and Pogromsky, 2019) the *restoration entropy* $h_{\text{res}}(\varphi, K)$ of the system (1). Specifically, let a time duration τ, a state $a \in \mathbb{R}^n$, and a "tolerance level" $\delta > 0$ be given. The symbol $p(\tau, a, \delta)$ stands for the minimal number of open δ-balls required to cover the image $\varphi^\tau(B_\delta(a) \cap K)$. The restoration entropy is defined as

$$h_{\text{res}}(\varphi, K) := \lim_{\tau\to\infty} \frac{1}{\tau} \overline{\lim_{\delta\to 0}} \sup_{a\in K} \log_2 p(\tau, a, \delta). \tag{7}$$

Here, the existence of the limit is guaranteed by Fekete's lemma and is due to the easily verifiable subadditivity of the concerned quantity in τ (Matveev and Pogromsky, 2019). The interest in (7) is caused, inter alia, by the equality $h_{\text{res}}(\varphi, K) = \mathcal{R}_{\text{ro}}$, see Theorem 8 in (Matveev and Pogromsky (2019)).

It is also shown there that the restoration entropy is an upper bound for the classic topological entropy h_{top} of the system (1); for the definition of h_{top}, see, e.g., (Adler et al., 1965; Katok, 2007; Downarowicz, 2011). Meanwhile, these two concepts are not identical. For example, $h_{\text{top}} < h_{\text{res}}$ for the logistic map (Pogromsky and Matveev, 2016a, Ex. 5.1). In (Kawan, 2019), an exhaustive characterization of the systems with $h_{\text{top}} = h_{\text{res}}$ is obtained, and evidence is provided that $h_{\text{top}} = h_{\text{res}}$ is a relatively rare occurrence.

Unlike h_{top}, there are other tools of the classic theory of nonlinear dynamics that can be directly linked with the restoration entropy. These are the *finite-time Lyapunov exponents*

$$\Lambda_i(t, x) := \log_2 \alpha_i(t, x),$$

where $\alpha_1(t, x) \geq \ldots \geq \alpha_n(t, x)$ stand for the singular values of $\mathrm{D}\varphi^t(x)$, put in nonincreasing order. That linkage is fairly straightforward under the following.

Assumption 2. $K = \mathrm{cl}(\mathrm{int}(K))$.

Then (Matveev and Pogromsky, 2019)

$$h_{\text{res}}(\varphi, K) = \max_{x\in K} \overline{\lim_{t\to\infty}} \frac{1}{t} \sum_{i=1}^{n} \max\{0, \Lambda_i(t, x)\}, \tag{8}$$

whereas only the following inequality holds in the general case:

$$h_{\text{res}}(\varphi, K) \leq \max_{x\in K} \overline{\lim_{t\to\infty}} \frac{1}{t} \sum_{i=1}^{n} \max\{0, \Lambda_i(t, x)\}. \tag{9}$$

However, a constructive practical evaluation of the limit in (8) could be a challenging problem.

The Lyapunov exponents are also classically studied for dynamical systems on Riemannian manifolds where every state is assigned with an individual metric tensor. Though this paper addresses the property (4) with a homogeneous, state-independent metric, artificially transforming $\mathbb{R}^n$ into a Riemannian manifold with a spatially-varying metric tensor may much aid in computing the l.h.s. in (8) without taking any limits as $t \to \infty$.

To be specific, we denote by $\mathcal{S}$ the linear space of all symmetric $n \times n$ matrices, and by $\mathcal{S}^+ \subset \mathcal{S}$ the subset of the positive definite elements of $\mathcal{S}$. A continuous function $P : K \to \mathcal{S}^+$ gives rise to a Riemannian metric on K by defining the state-dependent inner product:

$$\langle v, w\rangle_{P,x} := \langle P(x)v, w\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product.

We first need the singular value equation for the matrix $A(x) := \mathrm{D}\varphi(x)$ in the metric $\langle\cdot,\cdot\rangle_P$. To obtain it, we observe that

$$\begin{aligned}
&\langle A(x)v, w\rangle_{P,\varphi(x)} = \langle P(\varphi(x))A(x)v, w\rangle \\
&= \langle v, A(x)^\top P(\varphi(x))w\rangle = \langle v, P(x)P(x)^{-1}A(x)^\top P(\varphi(x))w\rangle \\
&= \langle P(x)v, P(x)^{-1}A(x)^\top P(\varphi(x))w\rangle \\
&= \langle v, P(x)^{-1}A(x)^\top P(\varphi(x))w\rangle_{P,x}.
\end{aligned}$$

Hence, the adjoint of $A(x)$ w.r.t. $\langle\cdot,\cdot\rangle_P$ is given by

$$A(x)^* = P(x)^{-1}A(x)^\top P(\varphi(x))$$

and the associated singular value equation is

$$\det\big[P(x)^{-1}A(x)^\top P(\varphi(x))A(x) - \lambda I_n\big] = 0, \tag{10}$$

which is equivalent to

$$\det\left[A(x)^\top P(\varphi(x))A(x) - \lambda P(x)\right] = 0. \quad (11)$$

Let $\alpha_1^P(x) \geq \cdots \geq \alpha_n^P(x) \geq 0$ stand for the square roots of the solutions of this equation. These quantities are akin to the above $\alpha_i(t,x)$; in particular, $\alpha_i(t,x) = \alpha_i^I(t,x)$, where I is the constant function whose value is the $n \times n$ identity matrix. An analog of equation (11) in continuous-time is available in Pogromsky and Matveev (2011).

4. MAIN RESULT

In this section, we detail the afore-mentioned method for evaluating the restoration entropy $h_{\text{res}}(\varphi, K)$ without taking any limits as $t \to \infty$. Specifically, we first present an upper estimate on $h_{\text{res}}(\varphi, K)$ in terms of α_i^P for an *arbitrary* positive definite matrix function P. Second, we show that this estimate can be made *as tight as one wishes* by a proper choice of the function P. In other words, the proposed technique is exhaustive in the sense that it is enough to compute $h_{\text{res}}(\varphi, K)$ with as high exactness as desired. This is detailed by the following.

Theorem 3. Suppose that Assumption 1 holds. Then the following statements are true:

(i) Any map $P(\cdot) \in C^0(K, \mathcal{S}^+)$ gives rise to the following upper bound on the restoration entropy of the system (1):

$$h_{\text{res}}(\varphi, K) \leq \max_{x \in K} \sum_{i=1}^n \max\{0, \log_2 \alpha_i^P(x)\}.$$

(ii) Suppose that the set K satisfies Assumption 2 and the Jacobian matrix $\mathrm{D}\varphi(x)$ is invertible for every $x \in K$. Then for any $\varepsilon > 0$, there exists $P \in C^0(K, \mathcal{S}^+)$ so that

$$h_{\text{res}}(\varphi, K) \geq \max_{x \in K} \sum_{i=1}^n \max\{0, \log_2 \alpha_i^P(x)\} - \varepsilon.$$

In (i) and (ii), we assume that $\log_2 0 := -\infty$.

Remark 4. By (ii), the following new and exact formula for h_{res} holds under Assumption 2:

$$h_{\text{res}}(\varphi, K) = \inf_{P \in C^0(K, \mathcal{S}^+)} \max_{x \in K} \sum_{i=1}^n \max\{0, \log_2 \alpha_i^P(x)\}.$$

Corollary 5. Let the set K meet Assumption 2 and endow $\mathfrak{G} := \{\varphi \in C^1(\mathbb{R}^n, \mathbb{R}^n) : \varphi(K) \subset K\}$ with the C^1-topology. The restoration entropy $h_{\text{res}}(\varphi, K)$ is an upper semi-continuous function of the map $\varphi \in \mathfrak{G}$, i.e., $h_{\text{res}}(\varphi, K) \geq \overline{\lim}_{\phi \to \varphi} h_{\text{res}}(\phi, K)$ provided that $\phi, \varphi \in \mathfrak{G}$.

Indeed, for a fixed $P(\cdot)$, the quantities $\alpha_i^P(x)$ depend on the point x and map $\varphi \in \mathfrak{G}$ continuously, as easily follows from, e.g., the formulae at the beginning of Sect. 8 in (Pogromsky and Matveev, 2011). Hence, the quantity

$$\max_{x \in K} \sum_{i=1}^n \max\{0, \log_2 \alpha_i^P(x)\}$$

also depends on the map $\varphi \in \mathfrak{G}$ continuously. It remains to note that the infimum (over P's, in our case) of continuous functions (of the argument φ, in our case) is upper semi-continuous; see, e.g., (Young, 1910, Ch. 3, Sect. 6).

Our proof of (ii) in Theorem 3 is critically based on a recent deep result of J. Bochi that is reported in (Bochi, 2018). The proof of this statement is outlined in the Appendix.

Proof of (i) in Theorem 3: We write $\alpha_1^P(t,x) \geq \cdots \geq \alpha_n^P(t,x) \geq 0$ for the square roots of the solutions λ of

$$\det\left[(\mathrm{D}\varphi^t(x))^\top P(\varphi^t(x))\mathrm{D}\varphi^t(x) - \lambda P(x)\right] = 0. \quad (12)$$

We first show that there are constants C_-, C_+ for which

$$\begin{aligned} -\infty < C_- \leq \sum_{i=1}^n \max\{0, \log_2 \alpha_i^P(t,x)\} \\ - \sum_{i=1}^n \max\{0, \log_2 \alpha_i(t,x)\} \leq C_+ < \infty. \end{aligned} \quad (13)$$

Indeed, the solutions of (10) are the eigenvalues of the matrix

$$\begin{aligned} &P(x)^{-1/2}A(x)^\top P(\varphi(x))^{1/2}P(\varphi(x))^{1/2}A(x)P(x)^{-1/2} \\ &= B(x)^\top B(x), \end{aligned}$$

where $B(x) := P(\varphi(x))^{1/2}A(x)P(x)^{-1/2}$. Hence,

$$\sum_{i=1}^n \max\{0, \log_2 \alpha_i^P(t,x)\} - \sum_{i=1}^n \max\{0, \log_2 \alpha_i(t,x)\} = \log_2 \frac{\max_{0 \leq k \leq n} \omega_k(B(x))}{\max_{0 \leq k \leq n} \omega_k(A(x))}.$$

Here $\omega_k(C)$ stands for the product of k largest singular values of the square matrix C if $k \geq 1$; and $\omega_0(C) := 1$. Using Horn's inequality (Boichenko et al., 2005, Prop. 2.3.1), we see that

$$\omega_k(B(x)) \leq \omega_k(P(\varphi(x))^{1/2})\omega_k(A(x))\omega_k(P(x)^{-1/2}).$$

Since the singular values continuously depend on the matrix (Horn and Johnson, 2013, Thm. 2.6.4), the function

$$x \mapsto \omega_k(P(\varphi(x))^{1/2})\omega_k(P(x)^{-1/2})$$

is continuous as well. So its maximum on the compact set K is attained and finite. This observation yields the upper estimate in (13). The lower estimate is obtained likewise by applying Horn's inequality to $\omega_k(A(x))$. Thus we see that (13) does hold.

By combining (13) with (9), we get

$$h_{\text{res}}(\varphi, K) \leq \max_{x \in K} \limsup_{t \to \infty} \frac{1}{t} \sum_{i=1}^n \max\{0, \log_2 \alpha_i^P(t,x)\}.$$

Meanwhile, Assumption 1 and the generalized Horn's inequality (Boichenko et al., 2005, Prop. 7.4.3)) imply that $(t,x) \mapsto \sum_{i=1}^n \max\{0, \log_2 \alpha_i^P(t,x)\}$ is a continuous subadditive cocycle over φ. So by (Morris, 2013, Thm. A.3),

$$\begin{aligned} h_{\text{res}}(\varphi, K) &\leq \inf_{t > 0} \max_{x \in K} \frac{1}{t} \sum_{i=1}^n \max\{0, \log_2 \alpha_i^P(t,x)\} \\ &\leq \max_{x \in K} \sum_{i=1}^n \max\{0, \log_2 \alpha_i^P(x)\}, \end{aligned}$$

which completes the proof. ∎

5. A CONTINUOUS-TIME ANALOG OF THEOREM 3

Consider a continuous-time system

$$\dot{x} = f(x) \quad (14)$$

with the state $x \in \mathbb{R}^n$ and a continuously differentiable r.h.s. $f : \mathbb{R}^n \to \mathbb{R}^n$. As before, we assume that the admissible initial states are restricted to a compact forward-invariant set K. In this situation, the flow $\varphi^t(\xi) = x(t, \xi)$ plays the role of the map φ considered in the previous section. Here $x(t, \xi)$ is the solution of the Cauchy problem $x(0) = \xi$ for the ODE (14).

This section aims at presenting a continuous-time analog of Theorem 3. To this end, we consider a map $P(\cdot)$ that is defined and continuously differentiable in some open neighborhood of K and assumes values in $\mathcal{S}^+$. For any such a map, we put the entries of the matrix $\dot{P}$ as $\dot{P}_{*,*}(x) := \frac{\partial P_{*,*}}{\partial x}(x)f(x)$ and define the x-dependent quantities $\sigma_i^P(x)$, $i = 1, \ldots, n$ as solutions of the following algebraic equation:

$$\det\left[\mathrm{D}f(x)^\top P(x) + P(x)\mathrm{D}f(x) + \dot{P}(x) - \lambda P(x)\right] = 0. \quad (15)$$

In the case of the continuous-time system (14), the formal definition and discussion of the restoration entropy $h_{\mathrm{res}}(f, K)$ is available in (Matveev and Pogromsky, 2019).

Theorem 6. Let the r.h.s. of (14) be continuously differentiable and let the set K be compact and forward-invariant. Then the following statements are true:

(i) For any map $P(\cdot)$ with the described properties, the following inequality holds:

$$h_{\mathrm{res}}(f, K) \leq \frac{1}{2\ln 2} \max_{x \in K} \sum_{i=1}^{n} \max\{0, \sigma_i^P(x)\}.$$

(ii) Suppose that the set K satisfies Assumption 2. Then for any $\varepsilon > 0$, there exists a map $P(\cdot)$ with the described properties such that

$$h_{\mathrm{res}}(f, K) \geq \frac{1}{2\ln 2} \max_{x \in K} \sum_{i=1}^{n} \max\{0, \sigma_i^P(x)\} - \varepsilon.$$

Proof: Part (i) is immediate from Thm. 14 in (Matveev and Pogromsky, 2019), where $v_d(x) :\equiv 0$. The proof of (ii) goes along the lines of the proof of (ii) in Thm. 3. ∎

Like in the case of discrete time, analogs of Remark 4 and Corollary 5 follow from Theorem 6.

6. EXAMPLE: THE LANDFORD SYSTEM

Consider the following system:

$$\begin{aligned} \dot{x} &= (a-1)x - y + xz \\ \dot{y} &= x + (a-1)y + yz, \quad x, y, z \in \mathbb{R}, \; a > 0 \\ \dot{z} &= az - (x^2 + y^2 + z^2) \end{aligned} \quad (16)$$

This system is attributed to Landford and was studied in many publications, see, e.g. (Belozyorov, 2015). It is well known that the system (16) has only two equilibrium points:

$$O_1 = [0, 0, 0]^\top, \;\; O_2 = [0, 0, a]^\top.$$

The value $a = 2/3$ is of particular interest, since then there is a heteroclinic orbit connecting the equilibria (Belozyorov, 2015).

Let K be some compact forward-invariant set of (16).

Remark 7. In K, we necessarily have $z \geq 0$.

Indeed, if $z(0) < 0$, then the third equation of (16) implies that the corresponding solution escapes to $-\infty$ in finite time ($\dot{z} < -z^2$).

The Jacobian matrix is given as follows:

$$\mathrm{D}f(x, y, z) = \begin{bmatrix} a-1+z & -1 & x \\ 1 & a-1+z & y \\ -2x & -2y & a-2z \end{bmatrix}.$$

For equation (15), we take the positive definite matrix [1]

$$P(x, y, z) = P_0 e^{w(x,y,z)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}}_{:=P_0} \exp\underbrace{\left(\frac{2z}{a}\right)}_{:=w}. \quad (17)$$

Straightforward calculations yield that

$$P\mathrm{D}f(x) = \begin{bmatrix} a-1+z & -1 & x \\ 1 & a-1+z & y \\ -x & -y & \frac{1}{2}(a-2z) \end{bmatrix} e^w,$$

$$\mathrm{D}f(x)^\top P = \begin{bmatrix} a-1+z & 1 & -x \\ 1 & a-1+z & -y \\ x & y & \frac{1}{2}(a-2z) \end{bmatrix} e^w,$$

and therefore

$$\begin{aligned} &\mathrm{D}f(x)^\top P + P\mathrm{D}f(x) \\ &= e^w \begin{bmatrix} 2(a-1+z) & 0 & 0 \\ 0 & 2(a-1+z) & 0 \\ 0 & 0 & a-2z \end{bmatrix}. \end{aligned}$$

At the same time,

$$\begin{aligned} \dot{P} - \lambda P &= \left(\frac{2\dot{z}}{a} - \lambda\right) e^w P_0 \\ &\overset{(16)}{=} e^w \left(\frac{2}{a}(az - x^2 - y^2 - z^2) - \lambda\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}. \end{aligned}$$

Finally, the solutions of (15) can easily be found:

$$\begin{aligned} \lambda_1 &= 2(a-2z) + 2\frac{az - z^2 - x^2 - y^2}{a} \\ &\leq -\frac{2z^2}{a} - 2z + 2a \overset{(z \geq 0)}{\leq} 2a, \end{aligned} \quad (18)$$

$$\begin{aligned} \lambda_{2,3} &= 2(a-1+z) + 2\frac{az - z^2 - x^2 - y^2}{a} \\ &\leq -\frac{2z^2}{a} + 4z + 2(a-1) \leq 2(2a-1). \end{aligned} \quad (19)$$

By (i) of Theorem 6, the following upper estimate holds true:

$$\begin{aligned} h_{\mathrm{res}}(K) &\leq \frac{1}{2\ln 2} \max_K \left[\max\{0, \lambda_1\} + 2\max\{0, \lambda_{2,3}\}\right] \\ &= \frac{1}{2\ln 2} \max\{\max_K \lambda_1, 2\max_K \lambda_{2,3}, \max_K(\lambda_1 + 2\lambda_{2,3})\}. \end{aligned}$$

Maximizing $\lambda_1 + 2\lambda_{2,3}$ over $z \in \mathbb{R}$ yields

$$\max_{x,y,z}(\lambda_1 + 2\lambda_{2,3}) \leq \max_z \left(6a - 4 + 6z - \frac{6}{a}z^2\right) = \frac{15}{2}a - 4.$$

By using (18) and (19), we thus arrive at the following.

Theorem 8. Let K be a compact forward-invariant set of the system (16) with $a \geq 2/3$. Then

$$h_{\mathrm{res}}(K) \leq \frac{2(2a-1)}{\ln 2}.$$

Our next step is to derive a lower estimate for $h_{\mathrm{res}}(K)$ under an extra assumption imposed on the set K. We start with the calculation of the proximate entropy around the system equilibria (for the definition of the proximate entropy, see (Matveev and Pogromsky (2019))). Calculating the eigenvalues of $\mathrm{D}f(O_i)$, $i = 1, 2$ one can easily derive that

$$H_L(O_1) = \frac{1}{\ln 2} \begin{cases} a & \text{if } 0 < a \leq 1 \\ 3a - 2 & \text{if } a \geq 1 \end{cases},$$

[1] For a more detailed treatment of the metric in this form for related problems of stability of forced oscillations, see (Pogromsky and Matveev (2016b)).

$$H_L(O_2) = \frac{1}{\ln 2}\begin{cases} 0 & \text{if } 0 < a \le 1/2 \\ 2(2a-1) & \text{if } a \ge 1/2 \end{cases},$$

$$\max\{H_L(O_1), H_L(O_2)\} \overset{a \ge 2/3}{=} H_L(O_2) = \frac{2(2a-1)}{\ln 2}.$$

The last relation together with Corollary 12 in (Matveev and Pogromsky (2019)) and Theorem 8 gives the following result.

Theorem 9. Assume that $a \ge 2/3$. Let K be any compact forward-invariant set for system (16), which satisfies Assumption 2 and the inclusion $O_2 \in \mathrm{int}K$. Then

$$h_{\mathrm{res}}(K) = \frac{2(2a-1)}{\ln 2}.$$

At this point it is worth mentioning that the matrix P from (17) not only provides an upper estimate of the restoration entropy according to the statement (i) of Theorem 6, but also gives a Riemannian metric for which the lower estimate (see the statement (ii) of Theorem 6) holds true with $\varepsilon = 0$.

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