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State estimation for a locally unobservable parameter-varying system: one gradient-based and one switched solutions

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Abstract: This work is motivated by a case study of a mechanical system where a sensor bias yields loose of observability for certain values of time-varying parameters. Two solutions are proposed: a nonlinear gradient-based observer that requires the persistency of excitation of the system trajectories and a switched observer that imposes an average dwell-time requirement. For both observers, asymptotic convergence of the estimates is proven. The theoretical results are supported by illustrative numerical simulations.

Keywords: parameter-varying systems, switched observer, nonlinear observer, switched Lyapunov function

1. INTRODUCTION

State estimation for parameter-varying and time-varying systems is a long-standing problem arising in numerous applications. Whereas a solution based on Kalman filtering has been proposed by H. Cox in 1964, see Cox (1964), nowadays this problem still attracts researchers' attention; particularly, it is due to its importance to the state estimation of nonlinear systems, see Bastin and Gevers (1988) and Nijmeijer and Fossen (1999).

Many results are available under the assumption that the system to be estimated is always observable, *e.g.*, the adaptive estimation for a parametrically uncertain system proposed by Zhang (2002), the set-membership approach by Jaulin (2002) and interval observers by Efimov et al. (2013), and the finite-/fixed-time estimation studied by Rios et al. (2017). However, this assumption is violated in some applications: for example, Liu et al. (2014) have shown that a class of power systems is not observable for some combinations of input signals. Nevertheless, if uniform complete observability may be assumed (see Rugh (1996); Jazwinski (2007); Besançon (2007)), then observers with time-varying or parameter-varying gains can be designed. Particularly, these gains can be chosen based on the observability Gramian of the system ensuring exponential convergence, see Rugh (1996). The drawback of this approach is that the gains have to be computed online as a solution of a nonlinear matrix differential equation that can be undesirable for embedded real-time applications.

One class of parameter-varying observers, where the gains can be precomputed off-line, is the switched observers. The switched observers are typically applied for state estimation of systems operating in a finite number of modes, where the commutation among these modes obeys a supervisory signal. The conventional approach to the switched observer design is the common Lyapunov function as described, for example, by Alessandri and Coletta (2001). However, the existence of a common Lyapunov function is a restrictive assumption that does not hold if some of the operation modes are not observable. To relax this assumption, one can utilize the idea of the average dwell time (ADT), where the stability of the switched system under the ADT assumption has been studied by Hespanha and Morse (1999). Examples of switched observers with the ADT assumptions can be found, *e.g.*,

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in the works by Nouailletas et al. (2007) and Pettersson (2006). In addition, Tanwani et al. (2013) have formulated necessary and sufficient conditions for observability of switched systems with unobservable modes.

Novelty and Contribution. In this paper, we consider the state estimation problem for a linear parameter-varying system that is unobservable for certain values of the varying parameters. Our research is motivated by the recent work of Ryadchikov et al. (2019) where it has been shown that a simple mechanical system becomes unobservable in some operating modes if the position sensor has a constant bias. Drawing on that result as a motivating example, we propose two novel observers that do not require to compute the matrix gains online. The first solution is to construct (with a nonlinear procedure) a novel instrumental output that can be then used to design a gradient-based observer. The convergence of this observer can be shown under the persistency of excitation assumption for the system trajectories. The second solution is the switched observer, where the operation mode is changed when the system looses observability. It is worth noting that the considered system is not switched, and the operation modes are introduced instrumentally allowing us for a switched observer design. The gains of the switched observer can be obtained off-line as a solution to an LMI problem. We also derive the dwell-time condition to be imposed on the switching signal to establish the exponential convergence.

The rest of the paper is organized as follows. In Section 2 we present an illustrative example to motivate the considered problem and show that some straightforward solutions do not apply. The formal problem statement is given in Section 3. The nonlinear gradient-based observer is proposed in Section 4, and the switched observer is presented in Section 5. Section 6 provides illustrative simulations, and the conclusion of the paper is given in Section 7. Finally, an auxiliary proof can be found in the Appendix.

Used notations. For integers n, m, we define I_n as the $n \times n$ unit matrix, and $0_{n\times m}$ and $1_{n\times m}$ as $n \times m$ matrices of zeros and ones, respectively.

2. MOTIVATING EXAMPLE

This section presents a motivating example, which, as it has been shown by Ryadchikov et al. (2019), arises in robotics applications when a position sensor has a constant bias. Consider the linear parameter-varying (LPV) single-input single-output (SISO) system

$$\begin{aligned} \dot{x}(t) &= A(z(t))x(t) + \beta(y(t), u(t)), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where [1] $x \in \mathbb{R}^3$ is the state, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the known input and output signals, respectively, the function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ is known,

$$A(z) := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \cos(z) \\ 0 & 0 & 0 \end{bmatrix}, \; C := [1 \;\; 0 \;\; 1], \tag{2}$$

and z is a known piecewise continuous time-varying signal. We also assume that the input signal u is such that trajectories of the system (1) are bounded. The goal is to design an observer of the unmeasurable state x.

[1] When clear from the context, in the sequel the argument of time is omitted.

It is worth noting that the observability matrix of the system (1) is given by

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \cos(z) \end{bmatrix},$$

and the system is not observable for $\cos(z) = 0$. This obviously implies that some constrains should be imposed for the signal z, *e.g.*, the states x_1 and x_3 cannot be reconstructed for $z(t) \equiv \pm\frac{\pi}{2}$. To simplify the motivating example, assume that $z \in \mathcal{Q}_z := \left\{ z \mid \cos^2(z) \geq \delta_z^2 \right\}$ for some $\delta_z \in (0, 1)$, *i.e.*, the system is always observable but $\cos(z)$ may change its sign at some isolated instants of time.

One classic solution for linear time-varying systems is to construct a linear time-varying observer in the form

$$\dot{\hat{x}} = A(z)\hat{x} + \beta(y, u) - HC^\top (C\hat{x} - y), \tag{3}$$

where the time-varying symmetric gain matrix $H(t) \in \mathbb{R}^{3\times 3}$ is the solution of the matrix differential equation

$$\dot{H} = HA^\top(z) + A(z)H - HC^\top CH + Q$$

for some $H(0) = H_0 > 0$, and $Q > 0$ is the design parameter. It is known (see, *e.g.*, Rugh (1996); Rueda-Escobedo et al. (2019)) that the observer (3) ensures exponential convergence if the system is uniformly observable, that is there exist T_O, δ_1, and δ_2, all positive, such that for all t

$$\delta_1 I_3 \leq \int_t^{t+T_O} \Phi^\top(\tau, t) C^\top C \Phi(\tau, t) d\tau \leq \delta_2 I_3,$$

where $\Phi(\cdot, \cdot)$ is the state-transition matrix of the system (1). The uniform observability can be connected with the assumption that the system (1) does not stuck in the domain where $\cos(z) \approx 0$. However, the implementation of the observer (3) in embedded systems has certain drawbacks since computation of the gain matrix $H(t)$ requires to solve online 6 differential equations with quadratic terms that may be sensitive to numerical methods. Thus, in what follows we aim to designs that are less demanding for *online* computations than the observer (3), *e.g.*, by the means of off-line gains precalculation.

Let us now show that some straightforward constant-gain and parameter-varying solutions do not apply to this problem. To this end, consider the observer

$$\dot{\hat{x}} = A(z)\hat{x} + \beta(y, u) - L(z)(C\hat{x} - y), \tag{4}$$

where the parameter-varying gain vector $L(z)$ is to be defined, and $\hat{x}$ is the estimate of x. Define the estimation error as $e := \hat{x} - x$, then

$$\dot{e} = \left(A(z) - L(z)C\right) e.$$

A simple solution would be to find a constant vector L stabilizing the system for all values of $z \in \mathcal{Q}_z$, *e.g.*, with a common Lyapunov function. However, it can be shown that such a solution does not exist; hence, we have to calculate a vector function $L(z)$. Methods of design of LPV observers typically consider the quadratic Lyapunov function $V = e^\top Pe$, where the matrix P can be constant or parameter-varying, $P = P(z)$. The main drawback of the (continuous in z) parameter-varying matrix $P(z)$ is that the time derivative of the Lyapunov function will depend on the time derivative of the signal z implying some probably restrictive assumptions on its continuity and differentiability. Concerning the constant matrix P, it can be shown that the system (1) does not admit such a solution, or more precisely, there do not exist a parameter-varying gain vector $L(z)$ and a positive-definite constant matrix P, such that for all $z \in \mathcal{Q}_z$ the linear matrix inequality (LMI)

$$(A(z) - L(z)C)^\top P + P\left(A(z) - L(z)C\right) < 0$$

holds. The proof of this claim is straightforward and we omit it for brevity.

Motivated by the discussed shortcomings, we consider designs of two possible observers. First, we propose a nonlinear gradient-like observer that is based on a novel instrumental output of the system and requires the persistency of excitation of the system's trajectory. Next, we design a switched observer, where we consider the Lyapunov function $V = e^\top P(z)e$ with $P(z)$ being piecewise constant. Thus, we do not impose assumption on the time derivative of z but study possible jumps of the value of the Lyapunov function when switches occur and $\cos(z)$ changes the sign.

In next section, we formulate the problem statement, for which the system (1) is a particular case.

3. PROBLEM STATEMENT

Consider an LPV SISO system in the following form

$$\dot{x} = A(q)x + \beta(u, y, t),\ y = Cx, \tag{5}$$

where $x \in \mathbb{R}^3$ is the state, $u \in \mathbb{R}$ is the known input, $y \in \mathbb{R}$ is the measured output signal,

$$A(q) := \begin{bmatrix} 0 & q_1 & 0 \\ 0 & 0 & q_2 \\ 0 & 0 & 0 \end{bmatrix},\ C := [1\ 0\ 0],$$

with $q = (q_1, q_2)$, $q_i \in [-1,\ 1]$, $i \in \bar{i} := \{1, 2\}$, being a time-varying parametric variable available for measurements. The function β is known. The goal is to design an observer for x.

The main issue for solution of this problem is that the system looses its observability for zero values of varying parameters q. Indeed, the observability matrix of the system is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & q_1 & 0 \\ 0 & 0 & q_1 q_2 \end{bmatrix}.$$

As it can be seen from the observability matrix, the system is observable when all elements of q are not zero, the rank of the observability matrix equals 2 and the state x_3 is not observable when $q_1 \neq 0$ and $q_2 = 0$, and for $q_1 = 0$ only the state x_1 is observable. Thus, a hypothesis has to be introduced that the system does not spend much time in the mode where an element of q equals zero. This hypothesis will be formulated as a sufficient condition for the signal q ensuring observer convergence.

4. NONLINEAR OBSERVER

In this section we present a nonlinear observer that is based on construction of an instrumental output signal and a gradient-based estimate update that is widely used in adaptive control. To this end, we define a new instrumental output $y^\dagger(t)$ as described in the following proposition.

Proposition 1. Consider system (5). Define the signals $\phi_1(t)$, $\phi_2(t)$, and $\xi(t)$ as solutions of

$$\begin{aligned} \dot{\phi}_1(t) &= -a\phi_1(t) + aq_1(t), \\ \dot{\phi}_2(t) &= -a\phi_2(t) + \phi_2(t)q_2(t), \\ \dot{\xi}(t) &= -a\xi(t) + a^2 y(t) + ah(t), \end{aligned} \tag{6}$$

with zero initial conditions, where $a > 0$ is the tuning parameter and

$$h(t) := \beta_1(t) - \frac{1}{a}\phi_1(t)\beta_2(t) + \frac{1}{a}\phi_2(t)\beta_3(t),$$

where $\beta_i(t)$ is the ith element of $\beta(u(t), y(t), t)$. Define the signal

$$y^\dagger(t) := (a+1)y(t) - \xi(t). \tag{7}$$

Then it holds

$$y^\dagger(t) = w^\top(t)x(t) + \varepsilon(t), \tag{8}$$

where

$$w(t) := [1\ \phi_1(t)\ -\phi_2(t)]^\top \tag{9}$$

and $\varepsilon(t)$ is a (generic) exponentially decaying term.

Proof. The proof is based on iterative use of the Swapping Lemma, see Ioannou and Sun (1996). Let $p = \frac{d}{dt}$ be the differential operator, then from (6) we get

$$\phi_1(t) = \frac{a}{p+a}q_1(t), \quad \phi_2(t) = \frac{1}{p+a}\left[\phi_1(t)q_2(t)\right].$$

For brevity, here and below we omit exponentially decaying terms caused by initial conditions. From (5),

$$\frac{ap}{p+a}y(t) = \frac{a}{p+a}\left[q_1(t)x_2(t)\right] + \frac{a}{p+a}\beta_1(t). \tag{10}$$

Applying the Swapping Lemma,

$$\begin{aligned} \frac{a}{p+a}\left[q_1(t)x_2(t)\right] &= x_2(t)\phi_1(t) - \frac{1}{p+a}\left[\phi_1(t)\dot{x}_2(t)\right] \\ &= x_2(t)\phi_1(t) - \frac{1}{p+a}\left[\phi_1(t)q_2(t)x_3(t)\right] \\ &\quad - \frac{1}{p+a}\left[\phi_1(t)\beta_2(t)\right]. \end{aligned} \tag{11}$$

Applying the Swapping Lemma again,

$$\begin{aligned} \frac{1}{p+a}\left[\phi_1(t)q_2(t)x_3(t)\right] &= x_3(t)\phi_2(t) \\ &\quad - \frac{1}{p+a}\left[\phi_2(t)\beta_3(t)\right]. \end{aligned} \tag{12}$$

Substituting (11) and (12) in (10), we obtain

$$\frac{ap}{p+a}y(t) - \frac{a}{p+a}h(t) = \phi_1(t)x_2(t) - \phi_2(t)x_3(t).$$

Finally, it is straightforward to verify that

$$\frac{ap}{p+a}y(t) - \frac{a}{p+a}h(t) = ay(t) - \xi(t),$$

where $\xi(t)$ obeys (6), that completes the proof. □

To design the observer, we have to assume that the trajectory $q(t)$ sufficiently excites the system and does not stay at zero. More formally, it can be formulates as the following assumption.

Assumption 2. The trajectory $q(t)$ is such that the vector signal $w(t)$ defined in (9) is persistently excited, i.e., there exist T_w and δ_w such that for all $t \geq 0$ it holds

$$\int_t^{t+T_w} w(s)w^\top(s)ds \geq \delta_w I_3.$$

The nonlinear observer and its applicability are stated in the following proposition.

Proposition 3. Consider the system (5) with the new instrumental output $y^{\dagger}(t)$ defined in (7). Consider the nonlinear observer

$$\begin{aligned}\dot{\hat{x}}(t) =& A(q(t))\hat{x}(t) + \beta(u, y, t) \\ &+ \Gamma w(t) \left(y^{\dagger}(t) - w^{\top}(t)\hat{x}(t)\right),\end{aligned} \tag{13}$$

where $\Gamma > 0$ is the tuning parameter. If the trajectory $q(t)$ satisfies Assumption 2, then there exists $\Gamma_0 > 0$ such that $|\hat{x}(t) - x(t)| \to 0$ for all $\Gamma > \Gamma_0$.

Proof. Define the estimation error $e(t) := \hat{x}(t) - x(t)$. Recalling (8) and neglecting the exponentially decaying term $\varepsilon(t)$, the error dynamics is given by

$$\dot{e} = A(q)e - \Gamma w w^{\top} e.$$

Define the Lyapunov function candidate $V(e) := \frac{1}{2}e^{\top}\Gamma^{-1}e$. Then

$$\dot{V} = e^{\top}\left(\Gamma^{-1}A_{sym}(q) - ww^{\top}\right)e,$$

where $A_{sym}(q) := \frac{1}{2}\left(A(q) + A^{\top}(q)\right)$ is the symmetric part of $A(q)$. Since the matrix $A(q)$ is bounded and $w(t)$ satisfies Assumption 2 for some T_w and δ_w, there exist $\Gamma_0 > 0$ sufficiently large such that for any $\Gamma > \Gamma_0$ there is $T \geq T_w$ and $\delta \in (0, \delta_w]$ such that

$$\int_t^{t+T} \left(\Gamma^{-1}A_{sym}(q(s)) - w(s)w^{\top}(s)\right) ds \geq \delta I_3$$

for all $t \geq 0$. Thus, the same arguments as used for the proof of asymptotic stability of persistently exited systems can be applied, see Ioannou and Sun (1996). □

The observer (13) provides exponential convergence under the PE assumption on $w(t)$, which is related to the excitation of the trajectory $q(t)$, and cannot be applied if, for example, $q(t)$ converges to a constant vector. Note that analysis and verification of the convergence conditions remain a challenging problem. In the following Section, we present a switched observer that does not impose the PE assumption.

5. SWITCHED OBSERVER

5.1 Observer design

To present the switched observer, note that the vector q belongs to a square in $\mathbb{R}^2$ that has $(\pm 1, \pm 1)$ as its vertices. Let us enumerate these vertices in any particular order, v^k, $k \in \bar{k} := \{1, 2, 3, 4\}$. Let us say that $q \in \mathcal{Q}_k$, $k \in \bar{k}$, if q belongs to a smaller square that has the vertices at the origin and v^k. For example, one possible enumeration is

$$\begin{aligned}&v^1 := (1, 1),\ v^2 := (1, -1), \\ &v^3 := (-1, -1),\ v^4 := (-1, 1), \\ &\mathcal{Q}_1 := \{(q_1, q_2) \mid 1 \geq q_1 \geq 0,\ 1 \geq q_2 \geq 0\}, \\ &\mathcal{Q}_2 := \{(q_1, q_2) \mid 1 \geq q_1 \geq 0,\ -1 \leq q_2 < 0\}, \\ &\mathcal{Q}_3 := \{(q_1, q_2) \mid -1 \leq q_1 < 0,\ -1 \leq q_2 < 0\}, \\ &\mathcal{Q}_4 := \{(q_1, q_2) \mid -1 \leq q_1 < 0,\ 1 \geq q_2 \geq 0\}.\end{aligned} \tag{14}$$

Define the sign function as

$$\operatorname{sgn}(q_i) := \begin{cases} 1 \text{ if } q_i \geq 0, \\ -1 \text{ if } q_i < 0, \end{cases}$$

and the set of matrices

$$A_k := \begin{bmatrix} 0 & \operatorname{sgn}\left(v_1^k\right) & 0 \\ 0 & 0 & \operatorname{sgn}\left(v_2^k\right) \\ 0 & 0 & 0 \end{bmatrix}.$$

In other words, the matrix A_k has the same structure as the matrix $A(q)$ where the parameters q_1, q_2 are replaced by their extremes in the set $\mathcal{Q}_k$, $k \in \bar{k}$, and the matrices A_k form a convex polytope where the matrix function $A(q)$ is embedded. Note that all pairs (C, A_k) are observable. For the previously considered example, for the sets (14) we have

$$\begin{aligned}A_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.\end{aligned} \tag{15}$$

Define the function $f_s : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$,

$$f_s(q, x) = \begin{bmatrix} (q_1 - \operatorname{sgn}(q_1))\, x_2 \\ (q_2 - \operatorname{sgn}(q_2))\, x_3 \end{bmatrix},$$

and consider the switched observer

$$\begin{aligned}\dot{\hat{x}} &= A_s(q)\hat{x} + \beta(u, y, t) + Bf_s(q, \hat{x}) \\ &\quad - L_s(q)\left(C\hat{x} - y\right),\end{aligned} \tag{16}$$

where $A_s(q)$ is the switched matrix, $A_s(q) = A_k$ if $q \in \mathcal{Q}_k$,

$$B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{17}$$

and $L_s(q)$ is the switched gain defined as $L_s(q) := L_k$ when $q \in \mathcal{Q}_k$, $k \in \bar{k}$, and the gains L_k are computed as

$$L_k = \mathcal{O}_k^{-1}\mathcal{O}_1 L_1,$$

where $\mathcal{O}_k$ is the observability matrix of the pair (C, A_k) and L_1 is to be defined.

In what follows, we show that if a certain LMI is feasible, then there exists L_1 such that under some assumptions on the signal q the observer (16) ensures exponential convergence of the estimate $\hat{x}$ to the state vector x of the system (5).

5.2 Convergence of the observer

To present the convergence result, we define the function $\gamma : \mathbb{R} \to \mathbb{R}$ as

$$\gamma(q_i) := |q_i|\left(2 - |q_i|\right).$$

Note that $\gamma(0) = 0$, $\gamma(1) = 1$, and $\gamma(q_i)$ is monotonically increasing as $|q_i|$ varies from 0 to 1.

Next choose $\delta_q \in (0, 1)$ and compute $\delta_\gamma := \gamma(\delta_q)$, $0 < \delta_\gamma < 1$. Suppose that for some δ_γ there exist matrices $P_1 = P_1^{\top} > 0$ and L_1 being a solution of the matrix inequality $M_1 \leq 0$, where

$$M_1 := \begin{bmatrix} F_1 + (1 - \delta_\gamma) I_3 & P_1 B \\ B^{\top} P_1 & -I_2 \end{bmatrix},$$

and

$$F_1 := (A_1 - L_1 C)^{\top} P_1 + P_1 (A_1 - L_1 C). \tag{18}$$

Let λ_M and λ_m be the maximum and the minimum eigenvalues of P_1, respectively. Then define

$$\mu := \frac{\lambda_M}{\lambda_m} - 1, \tag{19}$$

and

$$\eta(q) := \frac{\min(\gamma(q_1), \gamma(q_2))}{\lambda_M}, \; \eta_0 := \frac{\delta_\gamma}{\lambda_M}. \tag{20}$$

Now we are in the position to impose an assumption on the signal q implying that the trajectory should not cross the borders between the sets $\mathcal{Q}$ too often and should not remain for a long time in small vicinities of these borders having $|q_i| \leq \delta_q$ for some i yielding $\min(\gamma(q_1), \gamma(q_2)) \leq \delta_\gamma$. More formally, this assumption can be formulated as follows.

Assumption 4. For the trajectory $q(t)$ there exist $T_q > 0$ and $\kappa > 0$ such that for all t_0 it holds:

- during the time interval $[t_0,\ t_0 + T_q]$ the trajectory $q(t)$ crosses the borders between the sets $\mathcal{Q}$ not more then $n_{\mathcal{Q}} \geq 0$ times;
- during the time interval $[t_0,\ t_0 + T_q]$ the trajectory $q(t)$ satisfies

$$\int_{t_0}^{t_0+T_q} \eta(q(\tau))d\tau \geq n_{\mathcal{Q}} \ln(1+\mu) + (\eta_0 + \kappa)\, T_q.$$

Finally, applicability of the proposed switched observer is summarized in the following Theorem.

Theorem 5. Consider the system (5). Choose $\delta_\gamma \in (0,\, 1)$ such that there exist $P_1 > 0$ and L_1 satisfying the matrix inequality

$$\left[\begin{array}{c|c} (A_1 - L_1 C)^\top P_1 + P_1 (A_1 - L_1 C) & \\ + (1-\delta_\gamma)\, I_3 & P_1 B \\ \hline B^\top P_1 & -I_2 \end{array}\right] \leq 0, \tag{21}$$

where A_1 is defined in (15), and B is defined in (17). Then if the trajectory $q(t)$ satisfies Assumption 4 for $\eta(q)$ and η_0 defined in (20) and μ defined in (19), then the observer (16) ensures exponential convergence of the estimate $\hat{x}(t)$ to the state vector $x(t)$ of the system (5).

The proof of Theorem 5 is omitted due to the lack of space.

Remark 6. Using standard methods for matrix inequalities, see Boyd et al. (1994), and defining $H_1 := L_1^\top P_1$, the matrix inequality (21) can be rewritten as the LMI

$$\left[\begin{array}{c|c} A_1^\top P_1 + P_1 A_1 - C^\top H_1 - H_1^\top C & \\ + (1-\delta_\gamma)\, I_3 & P_1 B \\ \hline B^\top P_1 & -I_2 \end{array}\right] \leq 0,$$

which can be efficiently solved for P_1 and H_1, see Lofberg (2005). Then $L_1 = P_1^{-1} H_1^\top$.

6. ILLUSTRATIVE EXAMPLE

In this section, we present an illustrative example of the system (5), where

$$\beta(u,y,t) := -\begin{bmatrix}9\\4\\1\end{bmatrix} y(t) - \begin{bmatrix}0\\0\\1\end{bmatrix} u(t)$$

and $u(t) = 0.15 + \sin(2\pi t)$. The trajectory $q = (q_1,\, q_2)$ is constructed as $q_1(t) := \cos(t)$, $q_2(t) := \cos(t + \frac{\pi}{4})$, and the vertices v^k and sets $\mathcal{Q}_k$, $k \in \bar{k}$, are chosen as (14). The trajectories $x(t)$ of the system are depicted in Figure 1, where $x(0) = [1\ 2\ -1]^\top$.

For the nonlinear observer (13) we choose $a = 1$ and $\Gamma = \mathrm{diag}\,(1,\, 8,\, 10)$. For the chosen $q_1(t)$, $q_2(t)$, and a it is

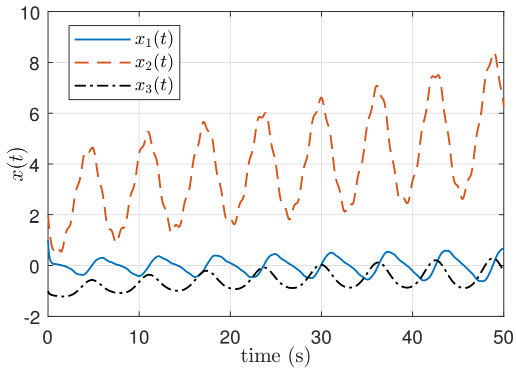

Fig. 1. The states $x(t)$ in the illustrative example.

straightforward to verify that (in steady state) the signals $\phi_1(t)$ and $\phi_2(t)$ defined in (6) satisfy

$$\phi_1(t) = 2^{-\frac{1}{2}} \cos(t + \frac{\pi}{4}),$$
$$\phi_2 = A_2 \sin(2t + \psi_2)$$

for some $A_2 > 0$ and $\psi_2 \in [0, 2\pi)$; thus $w(t)$ defined in (9) satisfies Assumption 2 for $T_w = 2\pi$.

For the switched observer (16) we choose $\delta_\gamma = 0.1$ and compute a feasible solution to the matrix inequality (21) yielding

$$L_1 = [73\ 442\ 68]^\top. \tag{22}$$

The estimation errors $e(t)$ for both observers are depicted in Figure 2 and illustrate convergence of the state estimation errors. The Lyapunov function curve for the switched observer is shown in Figure 3, where the level jumps can be observed at the switch instances, i.e., where $q_1(t)$ or $q_2(t)$ change their signs.

7. CONCLUSION

We have considered the problem of state estimation for a parameter-varying system that is not observable for some values of the time-varying parameters. To this end, two observers have been proposed. The nonlinear observer (13) is based on construction of a new instrumental output that allows to apply a gradient-like estimate update and yield exponential convergence under the persistency of excitation condition. The switched observer (16) is a linear observer with switched gains and ensure exponential convergence under the dwell-time conditions. Applicability of the observers is illustrated with numerical simulations.

For further researches, we intend to extend the presented observers to a more general case of systems with dimension greater than 3 and also consider other classes of observers, such as observers with finite/fixed-time convergence.

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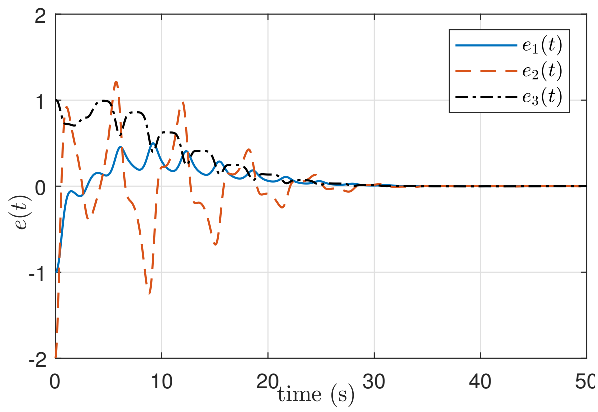
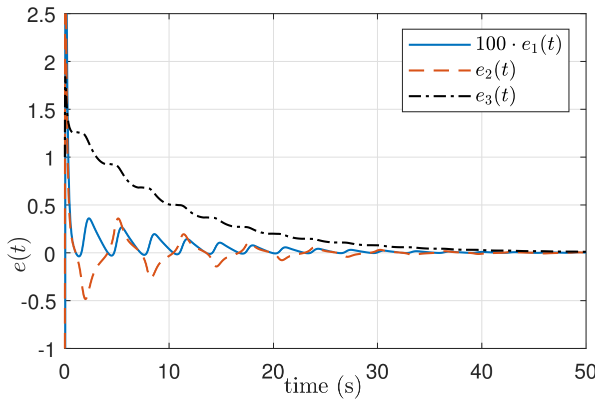


(a) Nonlinear observer (13).



(b) Switched observer (16).

Fig. 2. Estimation errors $e(t)$ in the illustrative example.

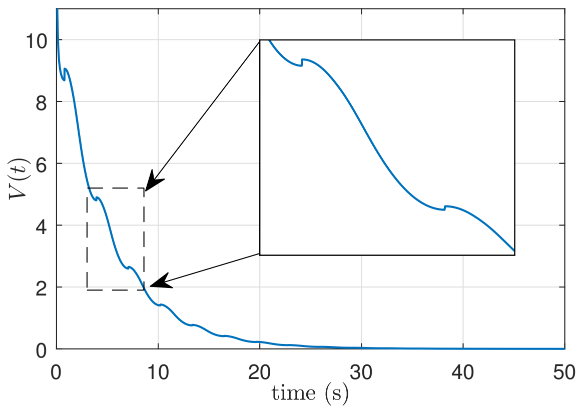


Fig. 3. The Lyapunov function $V(t)$ for the switched observer in the illustrative example.

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