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Trajectory tracking in rectangular billiards by unfolding the billiard table ⋆

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Abstract: In this paper, a position feedback controller is proposed to solve the tracking problem in rectangular billiards. Such a controller is obtained by transforming the billiard table into a surface on which the ball moves without experiencing any impact, *i.e.*, by reflecting the billiard table rather than the ball trajectory, and designing a position feedback controller based on such an unfolded billiard table. Furthermore, it is shown how such an infinite billiard table can be mapped to the surface of a torus, thus leading to bounded trajectories of the ball.

Keywords: Billiards, non-smooth impacts, trajectory tracking, feedback stabilization.

1. INTRODUCTION

State estimation and control of systems subject to impacts are problems of paramount importance in several practical applications (Brogliato et al., 2000; Brogliato and Zavala-Rio, 2000; Zavala-Rio and Brogliato, 2001; Brogliato, 2004; Galeani et al., 2008; Morris and Grizzle, 2009; Tanwani et al., 2014; Brogliato, 2016; de Carolis and Saccon, 2019). In this class of systems, planar billiards, *i.e.*, those modeling the motion of a mass in a region of the 2–dimensional plane, are among the most studied due to their chaotic behavior (Sinai, 1970; Lehman and White, 2002; Chernov and Markarian, 2006). In particular, a large research effort has been spent to solve the tracking problem in planar billiards, *i.e.*, to design a control law the steers the trajectory of the forced system to a given reference; see, *e.g.*, Tornambe (1999); Pagilla (2001); Menini and Tornambe (2001); Morarescu and Brogliato (2010). For instance, such a problem has been addressed, in Forni et al. (2011a,b, 2013), by using tools for the stability analysis of hybrid systems and the concept of mirrored image of the reference trajectory, in Rijnen et al. (2015), by introducing a new concept of distance, in Miranda-Villatoro et al. (2017, 2018), by using set-valued passivity-based and sliding mode controllers, and in Menini et al. (2018), by using augmented potential functions.

In this paper, a position feedback controller is proposed to track a reference trajectory in rectangular billiards. To pursue this objective, the trajectories of a mass moving within the billiard are first mapped into an infinite billiard table, which covers the whole Euclidean plane, wherein impacts never occur (Section 2.1). The idea of transforming the domain making the impacts disappear has been already used in Brogliato (2016); Pekarek and Murphey (2012); Pekarek and Murphey (2012); Oza et al. (2014); Pekarek (2014); Kim et al. (2016) (see, *e.g.*, the Zhuravlev-Ivanov method given in Brogliato, 2016), although not for rectangular billiards. This construction is then used in Section 3 to design position feedback controllers to solve the tracking problem in rectangular billiards.

The technique given in this paper is similar in spirit to the one given in Forni et al. (2011a,b, 2013), but it is based on different constructions. The controller proposed in Forni et al. (2011a,b, 2013) is based on mirroring the reference trajectory and choosing whether to follow the actual or the mirrored reference trajectory on the basis of the state of an automaton that is triggered by the impacts of the reference and of the ball. On the other hand, the approach proposed here consists of a transformation of the original billiard table that maps both the reference and the ball trajectories into an infinite billiard table, thus obtaining a fully linear control problem apart from changes of sign of the input depending on the past impacts that have been occurred A slight drawback of the proposed procedure is that the virtual billiard used for control design is unbounded; this is not a serious obstacle due to the linearity of the obtained control problem. However, in Section 4, it is shown how such an issue can be overcome by mapping the billiard trajectories to a torus. An example of application of the proposed position feedback controller are given in Section 5. Conclusions are drawn in Section 6.

2. BILLIARD TABLES

An *impact* represents an interaction of bodies for a short time interval. In the case of *non-smooth* impacts (as

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assumed in this paper), the impacts occur instantaneously, so that the positions of the colliding bodies do not change at the time of impact, whereas their velocities may present finite instantaneous variations. A very important feature of the impact theory is the determination of the relationship among the velocities immediately before and after an impact, which (as in the elementary case here considered) can be represented in a purely geometric form.

A *mathematical billiard* consists of a domain in the plane (a billiard table) and a point-mass (a billiard ball) that moves within the domain (freely or, possibly, subject to external forces), with phases of *flow motion* (*i.e.*, without velocity jumps) interspersed with velocity jumps due to the impacts between the ball and the boundary of the billiard. In the considered mathematical billiard, it is assumed that the ball has unitary mass and that it is not subject to friction, so that there is no loss of energy during the flow motion. Hence, if there are no external forces acting on the ball, the ball moves along a straight line with a constant speed until it hits the boundary of the billiard. The reflection from the boundary is perfectly elastic (*i.e.*, the *coefficient of restitution* is equal to 1) and subject to the following well known law: *the angle of incidence is equal to the angle of reflection* (see Fig. 1). This is equivalent to decompose, at the impact point, the pre-impact velocity of the ball into its normal and tangential components. Upon reflection, the normal component instantaneously changes sign, whereas the tangential one remains unchanged, so that the velocity vector does not change its modulus, but only its direction.

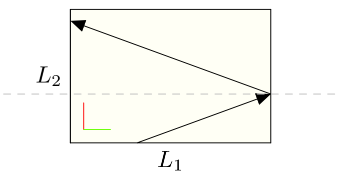


Fig. 1. The angles of incidence and of reflection are equal.

For the angles of incidence and reflection to be well defined, it is necessary that the barrier of the billiard does not have a corner point at the point of impact (all the trajectories that hit a corner point are ignored for the moment). Assume that the billiard table is *rectangular*, where L_1 is the length of the top and bottom edges of the table, and L_2 is the length of the left and right edges of the table; $q(t) = [\, q_1(t)\ \ q_2(t)\,]^\top$ is the position at time $t \in \mathbb{R}$ of the ball within the billiard table, subject to the unilateral constraints $0 \le q_j(t) \le L_j, i = 1, 2$, *i.e.*, $q(t) \in \mathcal{P}$, where $\mathcal{P} = [0, L_1] \times [0, L_2]$ is the billiard domain and

$$\mathcal{R} := \{q \in \mathcal{P} : q_1 = L_1\}, \qquad \mathcal{T} := \{q \in \mathcal{P} : q_2 = L_2\},$$
$$\mathcal{L} := \{q \in \mathcal{P} : q_1 = 0\}, \qquad \mathcal{B} := \{q \in \mathcal{P} : q_1 = 0\},$$

denote the four edges of the billiard.

If the trajectory hits either the top or the bottom edge of the table at the *impact time* t_i (hence, either $q_2(t_i) = 0$, $\dot{q}_2(t_i^-) < 0$ or $q_2(t_i) = L_2$, $\dot{q}_2(t_i^-) > 0$), then

$$\dot{q}_1(t_i^+) = \dot{q}_1(t_i^-), \qquad \dot{q}_2(t_i^+) = -\dot{q}_2(t_i^-), \tag{1a}$$

whereas if the trajectory hits either the left or the right edge of the table at the impact time t_i (hence, either $q_1(t_i) = 0$, $\dot{q}_1(t_i^-) < 0$ or $q_1(t_i) = L_1$, $\dot{q}_1(t_i^-) > 0$), then

$$\dot{q}_1(t_i^+) = -\dot{q}_1(t_i^-), \qquad \dot{q}_2(t_i^+) = \dot{q}_2(t_i^-). \tag{1b}$$

After the reflection, the ball continues its flow motion until it hits the boundary again, and so on. Let t_0 be the initial time and t_i, $i \in \mathbb{Z}$, $i \ge 1$, be the i-th impact time; (t_{i-1}, t_i) is called the i-th *flow interval*. *Degenerate impacts* (*i.e.*, those for which the normal component of the pre-impact velocity is zero) are excluded, as well as intervals of *persistent contact*.

During the i-th flow interval, the dynamics of the billiard are described by the following differential equations:

$$\ddot{q}_j(t) = u_j(t), \quad j = 1, 2, \tag{1c}$$

where $u = [\, u_1\ \ u_2\,]^\top$ is the external force acting on the ball (the control input). As well known (Gibbs, 2014), it is hard to predict the trajectory of the ball after its first few bounces with the edges of the billiard. There are initial velocities v_0 that yield a regular pattern to the ball motion (a periodic trajectory involving a certain number of impacts), but for almost all initial velocities, the resulting trajectory is a complicated and irregular path (in particular, since the domain is rectangular, the set of points touched by any non-periodic trajectory is dense in the domain of the billiard, *i.e.*, any non-periodic trajectory will pass arbitrarily close to any point of the billiard table; see also Chernov and Markarian, 2006; Brogliato, 2016).

2.1 The infinite billiard plane

For all trajectories to be analyzed, one possibility is to transform the table into a surface on which the ball is traveling in without ever bouncing: this is achieved by reflecting the billiard table, not the ball trajectory. This is equivalent to represent the billiard path as a beam of light. Instead of thinking each edge of the billiard as a mirror, it is considered as a pane of glass, and when the beam reaches it, the beam continues following a straight line, entering a reflected copy of the original billiard. The path continues and eventually reaches another edge of the current copy, so that an infinite number of successive copies of the billiard are introduced with the aim of continuing the path.

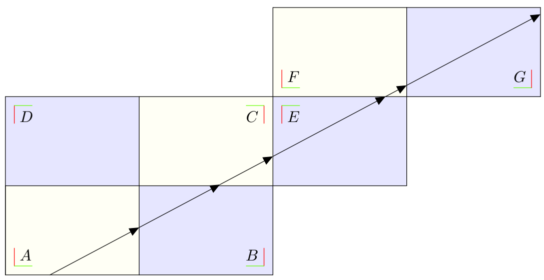


Fig. 2. The reflected billiard tables.

Consider the billiard table (the one labeled with the letter A in Fig. 2 and colored in yellow); define on the bottom-left part of the table a left-hand reference frame (in Fig. 2, it is draw in red), so to keep track of some transformations that will be carried out in the following. Assume that the initial position is on the bottom edge, and that the initial velocity points the top-right direction, so that the straight line trajectory hits the right edge of the billiard table A at the end of the first flow interval. Now, instead of reflecting the trajectory, reflect the table about its right edge and continue the trajectory of the ball in the reflected copy of the table (the one labelled with the letter B in Fig. 2 and colored in blue); note that also the reference frame has been reflected (the reflected frame is now a left-hand reference frame located on the bottom-right part of the billiard table B). In this way, the trajectory is still a straight line. So rather than drawing the piecewise linear trajectory of the ball inside a single rectangular domain, one can simply reflect the rectangular table about the edge where the ball hits, so to represent the trajectory as a straight line continuing in the reflected table. Repeat this at each impact, up to a certain number of impacts (in Fig. 2, five impacts have been considered, thus introducing five reflected billiard tables B, C, E, F, and G; the billiard table D is introduced for completeness). For some trajectories encountering an infinite number of impacts, the application of this trick may yield an infinite number of reflected billiards, whose whole is called the *infinite-billiards plane*. In particular, letting $\mathcal{P}_{j_1,j_2} = [j_1 L_1, (j_1+1)L_1] \times [j_2 L_2, (j_2+1)L_2]$ (so that $\mathcal{P}_{0,0}$, $\mathcal{P}_{0,1}$, $\mathcal{P}_{1,1}$, and $\mathcal{P}_{1,0}$ are the domains of the billiard tables A, B, C, and D), the domain of the infinite-billiards plane is defined as

$$\mathcal{P}^{(\infty)} := \bigcup_{j_1,j_2\in\mathbb{Z}} \mathcal{P}_{j_1,j_2} = \mathbb{R}^2.$$

Now the question is: "how does one reconstruct the actual trajectory in the billiard table A?" The answer is the *unfolding* of the trajectory. Consider the billiard table G; it contains only one segment of trajectory, starting from its left edge; reflect this segment about the left edge of the billiard table G, thus obtaining the fragmented trajectory in the billiard table F, consisting of two segments. The trajectory in the billiard table F starts from its bottom edge; reflect this trajectory about the bottom edge of the billiard table F, thus obtaining the fragmented trajectory in the billiard table E, consisting of three segments. Continuing in this way, one obtains a trajectory with 4 segments in the billiard table C, a trajectory with 5 segments in the billiard table B, and, finally, a trajectory with 6 segments in the billiard table A, which is the actual trajectory followed by the billiard ball.

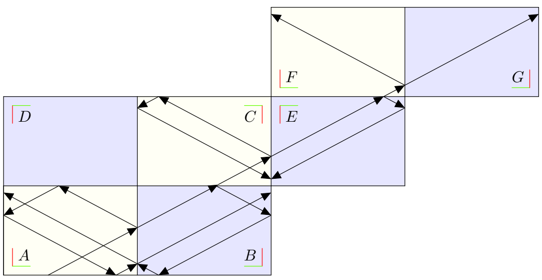


Fig. 3. Computation of the ball trajectory.

In view of the constructions detailed above, the following theorem has been proved.

Theorem 1. *There is a one–to–one correspondence between trajectories $q(t)$ in $\mathcal{P}$ and trajectories $q^{(\infty)}(t)$ in $\mathcal{P}^{(\infty)}$ such that $q^{(\infty)}(0) \in \mathcal{P}_{0,0}$.*

Given a control input $u(t)$ and letting $q(t)$ be the corresponding trajectory in $\mathcal{P}$ satisfying the dynamics (1), it is possible to determine the dynamics of the corresponding trajectory $q^{(\infty)}(t)$ in $\mathcal{P}^{(\infty)}$. In fact, defining the functions

$$\sigma_1(q_1^{(\infty)}) := \begin{cases} 1, & \text{if } q_1^{(\infty)} \in [2\,k\,L_1, (2\,k+1)L_1], \\ -1, & \text{if } q_1^{(\infty)} \in ((2\,k+1)\,L_1, (2\,k+2)L_1), \end{cases}$$

$$\sigma_2(q_2^{(\infty)}) := \begin{cases} 1, & \text{if } q_2^{(\infty)} \in [2\,k\,L_2, (2\,k+1)L_2], \\ -1, & \text{if } q_2^{(\infty)} \in ((2\,k+1)\,L_2, (2\,k+2)L_2), \end{cases}$$

$k \in \mathbb{Z}$, by (1), if u is such that there is no interval of persistent contact, then the dynamics of $q^{(\infty)}(t)$ in $\mathcal{P}^{(\infty)}$ are given for almost all $t \in \mathbb{R}$, $t \geq 0$, by

$$\ddot{q}_j^{(\infty)}(t) = \sigma_j(q_j^{(\infty)}(t))\, u_j(t), \tag{2}$$

whereas $\dot{q}_j^{(\infty)}(t)$ is not subject to any jump.

3. TRAJECTORY TRACKING IN RECTANGULAR BILLIARDS VIA UNFOLDING

In this section, by using the constructions made in Section 2, a technique to design a feedback controller to steer the trajectories of system (1) towards a reference trajectory y is proposed. Firstly, the concept of admissible reference trajectory is formalized.

Definition 1. A reference trajectory

$$y(t) = [\, y_1(t) \;\; y_2(t)\,]^\top \in \mathcal{P}$$

is said to be *admissible* if

(i) it is piecewise twice differentiable;
(ii) it does not have degenerate impacts or intervals of persistent contact;
(iii) it satisfies the impact relations (1a) and (1b) at all the impact times τ_i, $i \in \mathbb{N}$ (which are the times at which $y(t)$ is not differentiable), *i.e.*, it holds that

$$\dot{y}_1(\tau_i^+) = \dot{y}_1(\tau_i^-), \qquad \dot{y}_2(\tau_i^+) = -\dot{y}_2(\tau_i^-), \tag{3a}$$

if either $y_2(\tau_i) = 0$, $\dot{y}_2(\tau_i^-) < 0$ or $y_2(\tau_i) = L_2$, $\dot{y}_2(\tau_i^-) > 0$, and

$$\dot{y}_1(\tau_i^+) = -\dot{y}_1(\tau_i^-), \qquad \dot{y}_2(\tau_i^+) = \dot{y}_2(\tau_i^-). \tag{3b}$$

if either $y_1(\tau_i) = 0$, $\dot{y}_1(\tau_i^-) < 0$ or $y_1(\tau_i) = L_1$, $\dot{y}_1(\tau_i^-) > 0$.

Remark 1. If the reference $y(t)$ is a trajectory of a ball moving within the rectangular billiard that does not have degenerate impacts or intervals of persistent contact, *i.e.*, letting τ_i be the impact times of the reference trajectory, there are continuous functions $\upsilon_1(t)$ and $\upsilon_2(t)$ such that

$$\ddot{y}_j(t) = \upsilon_j(t), \quad j = 1, 2,$$

for all $t \in (\tau_i, \tau_i + 1)$, and the relations given in (3a) and in (3b) hold, then $y(t)$ is admissible.

The problem that is addressed in this section is formalized in the following statement.

Problem 1. Let an admissible reference trajectory $y(t)$ be given. Design a position feedback control law for system (1) such that $\lim_{t\to\infty} (q(t) - y(t)) = 0$.

To obtain $y^{(\infty)}(t)$ (a similar construction can be carried out to obtain $q^{(\infty)}(t)$), let τ_i be the i-th time at which $y(t)$ touches the boundary $\partial\mathcal{P}$ of $\mathcal{P}$, if any. If there does not exist any of such times, then let $y^{(\infty)}(t) = y(t)$, otherwise continue as follows.

Let $y^{(h)}(t) = [\, y^\top(t) \;\; 1\,]^\top$ be the reference trajectory expressed in *homogeneous coordinates*, and define the following basic transformation matrices in such coordinates

$$A_{\mathcal{R}} = \left[\begin{array}{cc|c} -1 & 0 & L_1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right], \qquad A_{\mathcal{T}} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & L_2 \\ \hline 0 & 0 & 1 \end{array}\right],$$

$$A_{\mathcal{L}} = \left[\begin{array}{cc|c} -1 & 0 & -L_1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right], \qquad A_{\mathcal{B}} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & -L_2 \\ \hline 0 & 0 & 1 \end{array}\right],$$

Thus, let $y^{(0)}(t) = y(t)$ and let $y^{(1,h)}(t) = [\,(y^{(1)}(t))^\top \ 1\,]^\top$,

$$y^{(1,h)}(t) = \begin{cases} A_{\mathcal{R}}\, y^{(h)}(t), & \text{if } y(t_1) \in \mathcal{R} \wedge \dot{y}_1(t_1) > 0, \\ A_{\mathcal{T}}\, y^{(h)}(t), & \text{if } y(t_1) \in \mathcal{T} \wedge \dot{y}_2(t_1) > 0, \\ A_{\mathcal{L}}\, y^{(h)}(t), & \text{if } y(t_1) \in \mathcal{L} \wedge \dot{y}_1(t_1) < 0, \\ A_{\mathcal{B}}\, y^{(h)}(t), & \text{if } y(t_1) \in \mathcal{B} \wedge \dot{y}_2(t_1) < 0, \\ y^{(h)}(t), & \text{otherwise.} \end{cases}$$

If τ_2 does not exist (*i.e.*, the reference trajectory has just a single impact with the billiard boundaries), then let

$$y^{(\infty)}(t) = \begin{cases} y^{(0)}(t), & \text{if } t \in [0, \tau_1], \\ y^{(1)}(t), & \text{if } t > \tau_1, \end{cases}$$

otherwise continue as follows.

Let $\bar{\mathcal{P}}$ be the billiard table in which $y^{(1)}(t)$ lies for $t \in (\tau_1, \tau_2)$ and let $\bar{\mathcal{R}}$, $\bar{\mathcal{T}}$, $\bar{\mathcal{L}}$, and $\bar{\mathcal{B}}$ be the right, top, left, and bottom edges of $\mathcal{P}$, respectively. Thus, let

$$y^{(2,h)}(t) = \begin{cases} A_{\mathcal{R}}\, y^{(1,h)}(t), & \text{if } y^{(1)}(t_1) \in \bar{\mathcal{R}} \wedge \dot{y}_1^{(1)}(t_1) > 0, \\ A_{\mathcal{T}}\, y^{(1,h)}(t), & \text{if } y^{(1)}(t_1) \in \bar{\mathcal{T}} \wedge \dot{y}_2^{(1)}(t_1) > 0, \\ A_{\mathcal{L}}\, y^{(1,h)}(t), & \text{if } y^{(1)}(t_1) \in \bar{\mathcal{L}} \wedge \dot{y}_1^{(1)}(t_1) < 0, \\ A_{\mathcal{B}}\, y^{(1,h)}(t), & \text{if } y^{(1)}(t_1) \in \bar{\mathcal{B}} \wedge \dot{y}_2^{(1)}(t_1) < 0, \\ y^{(1,h)}(t), & \text{otherwise.} \end{cases}$$

If τ_3 does not exist (*i.e.*, the reference trajectory has just two impacts with the billiard boundaries), then let

$$y^{(\infty)}(t) = \begin{cases} y^{(0)}(t), & \text{if } t \in [0, \tau_1], \\ y^{(1)}(t), & \text{if } t \in (\tau_1, \tau_2], \\ y^{(2)}(t), & \text{if } t > \tau_2, \end{cases}$$

otherwise iterate the procedure above, letting $\bar{\mathcal{P}}$ be the billiard table in which $y^{(2)}(t)$ is for $t \in (\tau_2, \tau_3)$, and so on.

Remark 2. By construction, if the original reference trajectory $y(t)$ in $\mathcal{P}$ is admissible, then the corresponding reference $y^{(\infty)}(t)$ in $\mathcal{P}^{(\infty)}$ is differentiable for all $t \geq 0$ and piecewise twice differentiable. Therefore, both $\dot{y}^{(\infty)}(t)$ and $\ddot{y}^{(\infty)}(t)$ exist for almost all $t \geq 0$.

By relying on the definition of the trajectory $y^{(\infty)}(t)$ in $\mathcal{P}^{(\infty)}$ corresponding to the trajectory $y(t)$ in $\mathcal{P}$, consider the following position feedback controller

$$\dot{\hat{q}}_1 = \hat{v}_1 + \ell_{1,1}(q_1^{(\infty)} - \hat{q}_1), \tag{4a}$$

$$\dot{\hat{v}}_1 = \sigma_1(q_1^{(\infty)})\, u_1 + \ell_{1,2}(q_1^{(\infty)} - \hat{q}_1), \tag{4b}$$

$$\dot{\hat{q}}_2 = \hat{v}_2 + \ell_{2,1}(q_2^{(\infty)} - \hat{q}_2), \tag{4c}$$

$$\dot{\hat{v}}_2 = \sigma_2(q_2^{(\infty)})\, u_2 + \ell_{2,2}(q_2^{(\infty)} - \hat{q}_2), \tag{4d}$$

$$u_1 = \sigma_1(q_1^{(\infty)}) \cdot \cdot \left(\ddot{y}_1^{(\infty)} + k_{1,1}(\dot{y}_1^{(\infty)} - \hat{v}_1) + k_{1,2}(y_1^{(\infty)} - \hat{q}_1)\right), \tag{4e}$$

$$u_2 = \sigma_2(q_2^{(\infty)}) \cdot \cdot \left(\ddot{y}_2^{(\infty)} + k_{2,1}(\dot{y}_2^{(\infty)} - \hat{v}_2) + k_{2,2}(y_2^{(\infty)} - \hat{q}_2)\right). \tag{4f}$$

where $\ell_{j,1}, \ell_{j,2}, k_{j,1}, k_{j,2} > 0$, $j = 1, 2$.

Theorem 2. *Assume that the reference trajectory $y(t)$ is admissible. Then the position feedback controller given in* (4) *solves Problem 1 and is such that*

$$\lim_{t\to\infty} \left(\dot{q}^{(\infty)}(t) - \dot{y}^{(\infty)}(t)\right) = 0.$$

Remark 3. The technique given in this section can be extended even to deal with non-elastic impacts, *i.e.*, when the restitution coefficient is strictly lower than 1. In such a case, the trajectory $q^{(\infty)}(t)$ need not be differentiable due to the fact that $\|[\,\dot{q}_1(t_i^+)\ \dot{q}_2(t_i^+)\,]^\top\| \leqslant \|[\,\dot{q}_1(t_i^-)\ \dot{q}_2(t_i^-)\,]^\top\|$. This implies that, at each impact, the closed loop system experiences a transient behavior that is required to let the estimates $\hat{v}_1$ and $\hat{v}_2$, which are continuous by construction, converge to the post-impact values of $\dot{q}_1$ and $\dot{q}_2$, respectively. Note that the length of this transient behavior can be shortened by suitably selecting the design parameters $\ell_{1,1}$ and $\ell_{1,2}$ in (4).

4. MAPPING THE BILLIARD INTO A TORUS

A drawback of the procedure given in Section 2.1 is that a new billiard table is introduced at each impact time, but this can be avoided with a simple trick. Referring to Fig. 2, consider the billiard table A; by reflection about its right edge, one obtains the billiard table B; by reflection about the top edge of the billiard table B, one obtains the billiard table C; by reflection about the left and the right edges of the billiard table C, one obtains the billiard tables D and E, respectively, and so on up to the billiard table G. From the initial position of the ball, draw a straight line with the same direction as the initial velocity of the ball, and continue such a line through the successive copies of the billiard table defined above, thus obtaining Fig. 3. The three billiard tables E, F and G contain one segment of the whole trajectory. Note that the billiard table E has the same color and orientation as the billiard table D, the billiard table F has the same color and orientation as the billiard table A, and the billiard table G has the same color and orientation as the billiard table B. Now, translate the billiard table E and the contained trajectory segment over the billiard table D, the billiard table F and the contained trajectory segment over the billiard table A, and, finally, the billiard table G and the contained trajectory segment over the billiard table B, thus obtaining Fig. 4, which is composed of only four billiard tables. The four billiard tables together form a bigger billiard table, called the *four-billiards table*, whose domain is

$$\mathcal{P}_{0,0} \cup \mathcal{P}_{0,1} \cup \mathcal{P}_{1,1} \cup \mathcal{P}_{1,0} =: \mathcal{P}^{(4)}.$$

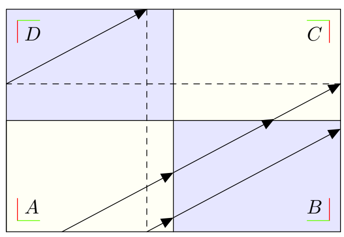


Fig. 4. The four-billiards table.

After these translations, when a trajectory leaves in some point the right (respectively, left) edge of the four-billiards table, it reappears exactly at the opposite point on the left (respectively, right) edge (to represent this, the two points are joined by a horizontal dashed segment). Similarly, when a trajectory leaves at its top (respectively, bottom) edge, it reappears exactly at the opposite point on the bottom (respectively, top) edge (to represent this, the two points are joined by a vertical dashed segment). Therefore, letting $q^{(4)}$ be the trajectory mapped into $\mathcal{P}^{(4)}$ as detailed above, the transformation form $\mathcal{P}^{(\infty)}$ to $\mathcal{P}^{(4)}$ is

$$q_1^{(4)} = \mathrm{mod}(q_1^{(\infty)}, 2L_1), \qquad q_2^{(4)} = \mathrm{mod}(q_2^{(\infty)}, 2L_2),$$

whereas the transformation from $\mathcal{P}^{(4)}$ to $\mathcal{P}$ is, in homogeneous coordinates,

$$q^{(h)} = \begin{cases} q^{(4,h)} & \text{if } q^{(4,h)} \in \mathcal{P}_{0,0}, \\ A_{\mathcal{R}}^{-1} q^{(4,h)} & \text{if } q^{(4,h)} \in \mathcal{P}_{1,0} \wedge q^{(4,h)} \notin \mathcal{P}_{0,0}, \\ A_{\mathcal{R}}^{-1} A_{\mathcal{T}}^{-1} q^{(4,h)} & \text{if } q^{(4,h)} \in \mathcal{P}_{1,1} \wedge q^{(4,h)} \notin \mathcal{P}_{0,0} \cup \mathcal{P}_{1,0}, \\ A_{\mathcal{R}}^{-1} A_{\mathcal{T}}^{-1} A_{\mathcal{L}}^{-1} q^{(4,h)} & \text{if } q^{(4,h)} \in \mathcal{P}_{1,0} \wedge q^{(4,h)} \notin \mathcal{P}_{0,0} \cup \mathcal{P}_{1,0} \cup \mathcal{P}_{1,1}. \end{cases}$$

Furthermore, by the construction given above, one can glue the top edge to the bottom edge and the right edge to the left one, thus obtaining the surface of a *torus*. Assuming that $L_1 > L_2$, the *parametric equations* describing such a surface with *inner radius* $r = \frac{L_2}{2\pi}$ and *revolving radius* $R = \frac{L_1}{2\pi}$, as functions of the cartesian coordinates of any point q either in $\mathcal{P}^{(4)}$ or in $\mathcal{P}^{(\infty)}$, are:

$$x = (R + r\,\cos(\tfrac{\pi}{L_2} q_2))\,\cos(\tfrac{\pi}{L_1} q_1), \tag{5a}$$
$$y = (R + r\,\cos(\tfrac{\pi}{L_2} q_2))\,\sin(\tfrac{\pi}{L_1} q_1), \tag{5b}$$
$$z = r\,\sin(\tfrac{\pi}{L_2} q_2); \tag{5c}$$

equations (5) are called the *torus map*. Hence, thanks to the above transformation, any unforced trajectory of the ball, which is *piecewise linear* in the rectangular domain of the billiard, becomes a collection of parallel lines on $\mathcal{P}^{(4)}$ (a straight line in $\mathcal{P}^{(\infty)}$), whence it is transformed into an *analytic* curve on the surface of the torus (see Fig. 5), which can be shown to be a *geodesic*.

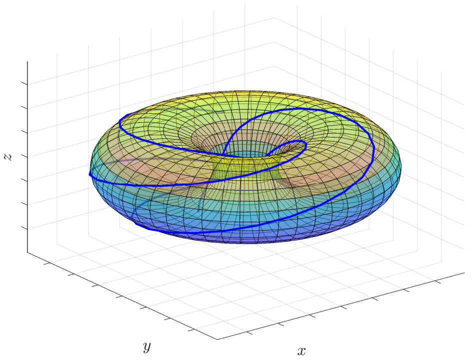


Fig. 5. The trajectory of the ball on the torus.

Lemma 1. *The torus surface is given by*

$$\mathcal{T} := \{[\,x\ y\ z\,]^\top \in \mathbb{R}^3 : g(x,y,z) = 0\},$$

where $g = x^4 + 2x^2y^2 + 2x^2z^2 + (-2R^2 - 2r^2)x^2 + y^4 + r^4 + 2y^2z^2 + (R^2 - 2r^2 - 2)y^2 + z^4 + (2R^2 - 2r^2)z^2 + R^4 - 2R^2r^2$.

Thus, consider the following lemma.

Lemma 2. *The map* (5) *is a local bijective from* $\mathcal{P}^{(\infty)}$ *to* $\mathcal{T}$ *in the neighborhood of each* $q \in \mathbb{R}^2$.

The next proposition shows how Lemma 2 can be strengthened by restricting the domain of the torus map to $\mathcal{P}^{(4)}$.

Proposition 1. *If* $R > r > 0$, *the torus map* (5) *is a* global *bijection from* $\mathcal{P}^{(4)} \setminus \{q \in \mathbb{R}^2 : q_1 = 2L_1 \vee q_2 = 2L_2\}$ *to* $\mathcal{T}$, *with the following inverse in closed-form:*

$$c_2 = \frac{-R^2 - r^2 + x^2 + y^2 + z^2}{2rR}, \qquad s_2 = \frac{z}{r}, \tag{6a}$$
$$c_1 = \frac{2Rx}{R^2 - r^2 + x^2 + y^2 + z^2}, \qquad s_1 = \frac{2Ry}{R^2 - r^2 + x^2 + y^2 + z^2}, \tag{6b}$$
$$q_1 = \frac{L_1}{\pi}\mathrm{atan2}(s_1, c_1), \qquad q_2 = \frac{L_2}{\pi}\mathrm{atan2}(s_2, c_2), \tag{6c}$$

where atan2 *is the 2-argument arctangent function.*

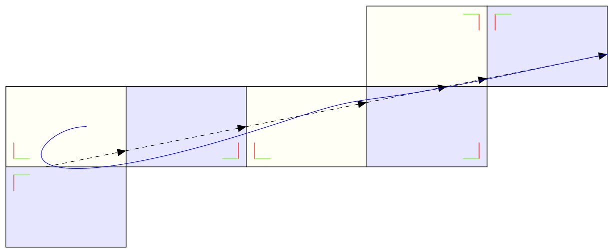
In view of Proposition 1 and of Theorem 1 there is a one-to-one correspondence between trajectories $q(t)$ in the billiard table $\mathcal{P}$ and trajectories on the torus $\mathcal{T}$.

5. EXAMPLE OF APPLICATION

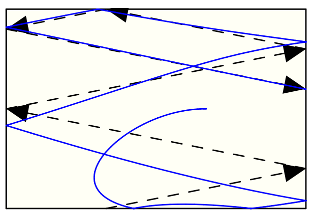
Let $L_1 = 3$, $L_2 = 1$, and assume that the objective is to let the ball track the trajectory of a mass moving unforced within the billiard table and starting at $q_1(0) = 1$, $q_2(0) = 0$, $\dot{q}_1(0) = 1$, and $\dot{q}_2(0) = 0.2$. Following the construction made in Section 2, the corresponding reference trajectory $y^{(\infty)}(t)$ in $\mathcal{P}^{(\infty)}$ is a straight line,

$$y^{(\infty)}(t) = \begin{bmatrix} t+1 \\ 0.2t \end{bmatrix}.$$

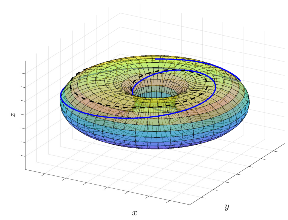
A numerical simulation has been carried out to test the position feedback controller given in (4) letting $\ell_{1,1} = \ell_{1,2} = \ell = 2, 1 = \ell_{2,2} = 1$, $k_{1,1} = k_{1,2} = k = 2, 1 = k_{2,2} = 1$, $q_1(0) = 1$, $q_2(0) = 2$, $\dot{q}_1(0) = -1$, $\dot{q}_2(0) = 1$, $\hat{q}_1(0) = 0$, $\hat{v}_1(0) = 0$, $\hat{q}_2(0) = 0$, and $\hat{v}_2(0) = 0$.



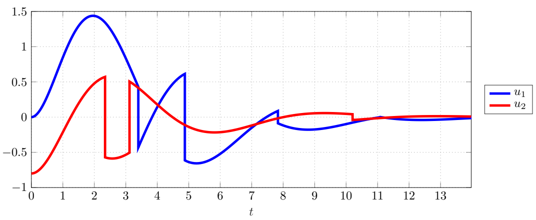
(a) Closed loop trajectory in $\mathcal{P}^{(\infty)}$.



(b) Closed loop trajectory in $\mathcal{P}$.



(c) Closed loop trajectory on $\mathcal{T}$.



(d) Control inputs.

Fig. 6. Results of the numerical simulation of the closed loop system with the position feedback controller (4).

Fig. 6 depicts the results of this simulation, showing, the trajectory of the closed loop system in the billiard table $\mathcal{P}$, the trajectory of the closed loop system in the infinite billiard plane $\mathcal{P}^{(\infty)}$, and the applied control inputs.

As shown by such a figure, the controller (4) solves Problem 1 for the considered reference trajectory.

6. CONCLUSIONS

In this paper, a position feedback controller has been proposed to solve the tracking problem in rectangular billiards. This controller has been designed by first mapping the trajectories in the billiard to an infinite surface wherein impacts never occur and, secondly, designing a position feedback controller with respect to trajectories evolving in this surface. A drawback of this procedure is that a new billiard table is introduced at every impact. In order to overcome such an issue, it has been further shown how trajectories in the infinite billiard can be mapped to the surface of a torus. An example of application of the proposed techniques has been given to illustrate and corroborate the theoretical results.

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