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Control of Flexible Euler-Bernoulli Beam with Input/Output Delay and Stochastic Disturbances

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Abstract: We consider the problem of stability enhancement of an undamped flexible beam with a tip mass in presence of input delay and random disturbances. In absence of delay this problem is classically solved through output feedback based on a suitable approximation of an infinite-dimensional Kalman filter. To cope with the presence of input or output delays we derive and compare two solutions, one based on a predictor from estimates in the past and the other one based on a filter with delayed measurements. An identical delay bound in closed form is derived for both solutions and we show that by an appropriate choice of the control gain it is possible to stabilize the system in presence of arbitrarily large delays. A modular structure is proposed for the case of arbitrary gain and delay bound. Finally, we consider the problem of deriving a finite-dimensional approximation of the predictor.

Keywords: Distributed parameter systems; Stochastic systems; Input delay

1. INTRODUCTION

Flexible 1D structures such as flexible robot arms, antennas, etc. are widely used in flexible manipulators, aerospace industry and many others industrial applications. For this reason the vibration control of such structures has been widely investigated in the last decades (Balakrishnan, 1991; Conrad and Morgül, 1998; Ge et al., 2011; Guo et al., 2008; Guo and Jin, 2013; Jin and Guo, 2019; Smyshlyaev et al., 2009). In recent years this control problem has been studied also for distributed systems affected by sensor or actuator time delay. It has been known for long time that control systems for hyperbolic PDE can be destabilized even by small delays (Datko, 1988; Fridman and Orlov, 2009), and this makes the extension of techniques developed for finite-dimensional systems particularly challenging. In our context Guo and Yang (2009) proposed a solution for the case of delay in observations by means of an infinite-dimensional observer used to estimate the delayed beam configuration and a predictor to compensate the delay. In practice, the computation of the predictor is equivalent to solving at each time point a PDE having the observer estimate as the initial value. This is reminiscent of the reduction approach used in the context of finite-dimensional systems (Artstein, 1982; Manitius and Olbrot, 1979). The same approach has been recently proposed for general distributed parameter linear systems with output delay in Mei and Guo (2019). In Shang and Xu (2012) a similar approach is used in the case of input delay when the input contains both delayed and un-delayed feedback and the state (*i.e.* the configuration of the beam) is available. Finally, in Liu et al. (2018) a backstepping method is proposed for the case of input delay and known state.

In contrast with these approaches we recast the control problem in presence of input delay for a Eulero-Bernoulli beam as a linear stochastic PDE with delays by fully considering the presence of disturbances on the measurements and actuators. The proposed control law differs from previous proposals because it is based on the closed-loop predictor, in analogy with a method used for the finite-dimensional case (Cacace et al., 2016). Our aim is to obtain a control law amenable of a finite-dimensional implementation and simple to compute, and therefore more suited for practical applications. In Section 2 we summarize the results of Balakrishnan (1991) for the control problem without delay. Two related versions of the output-feedback control based on the use of the closed-loop predictor are presented in Section 3. The finite-dimensional implementation is described in Section 4 and conclusions follow.

2. PROBLEM DESCRIPTION

This section describes the continuum infinite-dimensional model of the flexible structure and the solution of the corresponding optimal control problem in absence of delays developed by Balakrishnan (1991). For simplicity of presentation only planar beam deflections are considered. This model has been extensively studied in the literature (see for example Benchimol (1978); Guo (2001); Gupta (1980); Germani et al. (2006)) and is well suited for the description of large flexible space systems characterized by the practical absence of damping and gravity. Consider a homogeneous, uniform, undamped Euler-Bernoulli beam of length $2l$, clamped at $-l$ and with a free end at l where a tip mass m is concentrated. Let $w(t,s)$ be the deflection, $t \geq 0$, $s \in [-l,l]$, and w', w'', ... its space derivatives.

$2l$	beam length
ρ	mass density
σ	cross-sectional area
E	Young modulus
I	beam cross section moment of inertia
m	tip mass

Table 1. Parameters of the flexible beam PDE

The dynamics of the structure and the sensed data are described by a stochastic PDE with boundary conditions and by a measurement equation of the boundary rates affected by a measurement error,

$$\rho\sigma\ddot{w}(t,s) + EIw''''(t,s) = 0, \quad s \in [-l,l] \tag{1}$$
$$w(t,-l) = w'(t,-l) = w''(t,l) = 0 \tag{2}$$
$$m\ddot{w}(t,l) + u(t) + N_s(t) = EIw(t,l) \tag{3}$$
$$y(t) = \dot{w}(t,l) + N_o(t), \tag{4}$$

where the meaning of the parameters is summarized in Table 1, the boundary control $u(t) \in \mathbb{R}$ is applied at the free end and affected by an actuator disturbance $N_s(t)$ modeled as Gaussian white noise with constant spectral density d_s, while the measurement $y(t) \in \mathbb{R}$ is affected by a measurement noise $N_o(t)$ with spectral density d_o. Since the system is linear and the noise finite-dimensional it seems appropriate to recast the problem in the context of white noise theory, that provides a more uniform conceptual framework. In this context the set of finite energy signals $L_2([0,T],\mathcal{H}_e)$ on a separable Hilbert space $\mathcal{H}_e$ (that for our case will be defined below) is endowed with a finitely additive Gaussian measure with identity covariance defined on the algebra of cylinder sets of $L_2([0,T],\mathcal{H}_e)$ (see Balakrishnan (2012) for a rigorous formulation of the white noise theory in the context of linear systems). To represent a scalar white noise process $N(t)$ with spectral density D it suffices to take the sample paths in $L_2([0,T],\mathbb{R})$, and require that for every $h \in L_2([0,T],\mathbb{R})$ the variable $[N,h](T) = \int_0^T N(t)h(t)\,dt$ is Gaussian with zero mean and variance

$$\mathbb{E}\left[\left(\int_0^T N(t)h(t)\,dt\right)^2\right] = D\int_0^T h^2(t)\,dt, \tag{5}$$

where the expected value $\mathbb{E}[\cdot]$ is defined over $L_2([0,T],\mathbb{R})$ according to the probability measure induced by $[N,h]$. Clearly, when $h(t) \equiv 1$ then $\mathbb{E}[(\int_0^T N(t)\,dt)^2] = DT$.

Model (1)–(4) can be represented as an abstract wave equation over a Hilbert space, where the main feature is the inclusion of the boundary variable $w(l)$ in the beam "state" (see Balakrishnan (1991) for details). Let $\mathcal{H} = L_2([-l,l],\mathbb{R}) \times \mathbb{R}$ be the Hilbert space of elements $x = [f,b]^\top$ endowed with the inner product

$$[x,y]_{\mathcal{H}} = \int_{-l}^{l} f_x(s)f_y(s)\,ds + b_x b_y, \tag{6}$$

and consider the class $\mathcal{S}$ of L_2 functions in $[-l,l]$ compatible with the boundary conditions (2) and having the first four derivatives in L_2,

$$\mathcal{S} = \{w \in L_2([-l,l],\mathbb{R}) : \ w', w'', w''', w'''' \in L_2([-l,l],\mathbb{R}), \\ w(-l) = w'(-l) = w''(l) = 0\}. \tag{7}$$

The boundary values must be interpreted in the sense of the Sobolev spaces, and actually $\mathcal{S} = W_0^{4,2}(-l,l)$ in the usual notation for such spaces (Evans and Gariepy, 2015). Clearly, for $w \in \mathcal{S}$ the element $x = [w, w(l)] \in \mathcal{H}$, thus with a slight abuse of notation we can write $\mathcal{S} \subset \mathcal{H}$. Moreover, $\mathcal{S}$ is dense in $\mathcal{H}$ since the completion of $\mathcal{S}$ under the inner product (6) yields $\mathcal{H}$. System (1) with the measurement equation (4) can be rewritten as

$$M\ddot{x}(t) + Ax(t) + Bu(t) + BN_s(t) = 0 \tag{8}$$
$$y(t) = B^*\dot{x}(t) + N_o(t), \tag{9}$$

with the following definition of the operators:

$$A : \mathcal{S} \to \mathcal{H},\ z = \begin{bmatrix} w \\ w(l) \end{bmatrix} \in \mathcal{S},\ Az = \begin{bmatrix} EIw'''' \\ -EIw'''(l) \end{bmatrix} \in \mathcal{H} \tag{10}$$

$$M : \mathcal{H} \to \mathcal{H},\ z = \begin{bmatrix} w \\ b \end{bmatrix} \in \mathcal{H},\ Mz = \begin{bmatrix} \rho\sigma w \\ mb \end{bmatrix} \in \mathcal{H} \tag{11}$$

$$B : \mathbb{R} \to \mathcal{H},\ z \in \mathbb{R},\ Bz = \begin{bmatrix} 0 \\ z \end{bmatrix} \in \mathcal{H} \tag{12}$$

$$B^* : \mathcal{H} \to \mathbb{R},\ z = \begin{bmatrix} w \\ b \end{bmatrix} \in \mathcal{H},\ B^*z = b \in \mathbb{R}. \tag{13}$$

A is self-adjoint, non-negative definite, with bounded compact inverse A^{-1} (Balakrishnan, 1991). B and its adjoint B^* are bounded and M is nonsingular, self-adjoint and positive definite. Moreover, for $x \in \mathcal{S}$,

$$[Ax,x]_{\mathcal{H}} = \int_{-l}^{l} w''(s)^2\,ds. \tag{14}$$

A first-order equation can be obtained from (8) by introducing the Hilbert space $\mathcal{H}_e = \mathcal{D}(\sqrt{A}) \times \mathcal{H}$, dense in $\mathcal{H} \times \mathcal{H}$, with the inner product for $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$,

$$[X,Y]_{\mathcal{H}_e} = [\sqrt{A}X_1, \sqrt{A}X_2]_{\mathcal{H}} + [MX_2, Y_2]_{\mathcal{H}}. \tag{15}$$

If $X_2 = \dot{X}_1$ and $x_1 \in \mathcal{S}$, that is, X_1 is an admissible deflection and X_2 its time derivative, then $\|X\|^2_{\mathcal{H}_e} = [X,X]_{\mathcal{H}_e}$ is the double of the mechanical energy. System (8)–(9) can be written as

$$\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{B}u(t) + \mathcal{B}N_s(t) \tag{16}$$
$$y(t) = -\mathcal{B}^*\mathcal{X}(t) + N_o(t). \tag{17}$$

with the operators

$$\mathcal{A} : \mathcal{S} \times \mathcal{D}(\sqrt{A}) \to \mathcal{H}_e, \quad \mathcal{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -M^{-1}Ax_1 \end{bmatrix} \tag{18}$$

$$\mathcal{B} : \mathbb{R} \to \mathcal{H}_e, \quad \mathcal{B}u = \begin{bmatrix} 0 \\ -M^{-1}Bu \end{bmatrix} \tag{19}$$

$$\mathcal{B}^* : \mathcal{H}_e \to \mathbb{R}, \quad \mathcal{B}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -B^*x_2 \tag{20}$$

Since $[\mathcal{A}\mathcal{X}, \mathcal{Y}] = -[\mathcal{X}, \mathcal{A}\mathcal{Y}]$, $\mathcal{A}$ is skew-adjoint, generates a contraction semigroup and has a compact resolvent. System (16) has the mild solution

$$\mathcal{X}(t) = S(t)\mathcal{X}(0) + \int_0^t S(t-\sigma)\mathcal{B}(u(\sigma) + N_s(\sigma))\,d\sigma \tag{21}$$

where $S(\cdot)$ is the semigroup generated by $\mathcal{A}$. The pair $(\mathcal{A},\mathcal{B})$ is "controllable" (see Balakrishnan (1991) for the definition and details). In particular, this implies that if $\forall t$, $\mathcal{B}^*S^*(t)Y = 0$, then $Y = 0$.

$\mathcal{A}_\gamma = \mathcal{A} - \gamma\mathcal{B}\mathcal{B}^*$ generates a strongly continuous strongly stable semigroup $S_\gamma(t)$ for all $\gamma > 0$, that is, $\|S_\gamma(t)x\|_{\mathcal{H}_e} \to 0$ for any x. A stabilizing control is therefore

$$u(t) = -\gamma\mathcal{B}^*\mathcal{X}(t) = -\gamma\dot{w}(t,l). \tag{22}$$

Since $\dot{w}(t,l)$ is not available due to the measurement noise, the objective of reducing the boundary rates while minimizing the control energy can be formulated as a LQG problem with the objective function

$$J = \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T |\mathcal{B}\mathcal{X}(t)|^2\,dt + \frac{\lambda}{T}\int_0^T |u(t)|^2\,dt\right], \tag{23}$$

where λ is the weight of the control law. The optimal solution is

$$u^*(t) = -\frac{1}{\lambda}\mathcal{B}^* P_c\hat{\mathcal{X}}(t), \tag{24}$$

where $\hat{\mathcal{X}}(t)$ is the estimate of $\mathcal{X}(t)$ provided by the infinite-dimensional Kalman filter (Balakrishnan, 1991)

$$\dot{\hat{\mathcal{X}}}(t) = \left(\mathcal{A} - \left(\frac{P_f}{d_o} + \frac{P_c}{\lambda}\right)\mathcal{B}\mathcal{B}^*\right)\hat{\mathcal{X}}(t) - P_f d_o^{-1}\mathcal{B}y(t). \tag{25}$$

The operators P_c, P_f are the unique self-adjoint, non-negative definite solutions of the steady state infinite-dimensional Riccati equations of the filter and the control, that can be determined in closed-form as

$$P_c = \sqrt{\lambda}I, \quad P_f = \sqrt{d_s d_o}I. \tag{26}$$

Thus, $\gamma = 1/\sqrt{\lambda}$ and

$$u^*(t) = -\frac{1}{\sqrt{\lambda}}\mathcal{B}^*\hat{\mathcal{X}}(t). \tag{27}$$

With the control law (27) the processes $\mathcal{X}$ and $\hat{\mathcal{X}}$ are asymptotically stationary and the optimal cost is $J^* = \frac{\sqrt{d_s d_o}+\sqrt{\lambda}d_s}{m}$. For a generic $u(t)$ the Kalman filter (25) is

$$\dot{\hat{\mathcal{X}}}(t) = \mathcal{A}\hat{\mathcal{X}}(t) + \mathcal{B}u(t) - \frac{d_s}{d_0}\mathcal{B}\left(y(t) + \mathcal{B}^*\hat{\mathcal{X}}(t)\right) \tag{28}$$

and the estimation error $\mathcal{E}(t) = \mathcal{X}(t) - \hat{\mathcal{X}}(t)$ is asymptotically stationary with bounded covariance operator $R(t) = \mathbb{E}[\mathcal{E}(t)\mathcal{E}^*(t)]$ such that for any $\mathcal{X} \in \mathcal{H}_e$,

$$\lim_{t\to\infty}[R(t)\mathcal{X},\mathcal{X}] = [P_f\mathcal{X},\mathcal{X}] = \sqrt{d_s d_o}\|\mathcal{X}\|^2. \tag{29}$$

In particular, the asymptotic variance of the estimation error at the boundary is $\lim_{t\to\infty}\mathbb{E}[|\mathcal{B}^*\mathcal{E}(t)|^2] = \sqrt{d_s d_o}$.

Remark 1. It must be noticed that in the deterministic case the feedback law (22) ensures asymptotic stability but not *uniform* (*i.e. exponential*) stability. It was proved by Conrad and Morgül (1998) that the feedback

$$u(t) = -\gamma\dot{w}(t,l) + \beta\dot{w}'''(t,l) \tag{30}$$

where γ and β are positive constants is sufficient to obtain exponential stability. Moreover, Guo (2001) proved that with (30) the eigenfunctions of the closed-loop generator form a Riesz basis for $\mathcal{H}_e$ and provided an asymptotic expression of closed-loop eigenvalues and eigenfunctions. The approach that we present in the next section can be extended to the feedback law (30) with the additional advantage that the implementation is greatly simplified by the presence of a Riesz basis. However, the implementation of (30) is challenging in practical applications because $\dot{w}'''(t,l)$ is difficult to measure and it is not part of the state $\mathcal{X}(t)$. For this reason we study the simpler and more realistic controller (22).

3. PREDICTOR-BASED OUTPUT FEEDBACK

In presence of a constant input delay $\delta > 0$ (16)–(17) becomes

$$\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{B}u(t-\delta) + \mathcal{B}N_s(t) \tag{31}$$

$$y(t) = -\mathcal{B}^*\mathcal{X}(t) + N_o(t). \tag{32}$$

We compare two alternative strategies. The first one is to use the filter introduced above to compute $\hat{\mathcal{X}}(t)$ from $y(t)$ and then a state predictor to compute the prediction $\hat{\mathcal{X}}(t+\delta)$. The second one is to modify the filter to compute $\hat{\mathcal{X}}(t+\delta)$ from $y(t)$. We shall prove that both approaches stabilize the system in the same delay range characterized by the inequality $\Omega(\gamma,\delta) < 1$,

$$\Omega(\gamma,\delta) = \int_0^\delta \gamma|\mathcal{B}^*S_\gamma(s)\mathcal{B}|\,ds, \tag{33}$$

where $S_\gamma(t)$ is the semigroup generated by $\mathcal{A}_\gamma$. Notice that $\mathcal{B}^*S_\gamma(s)\mathcal{B}:\ \mathbb{R}\to\mathbb{R}$ is a scalar operator that goes to 0 for $s\to\infty$ and (see (5.2) in Balakrishnan (1991)),

$$\int_0^\infty |\mathcal{B}^*S_\gamma(s)\mathcal{B}|^2\,ds < \infty. \tag{34}$$

Clearly, $\Omega(\gamma,0) = 0$ and Ω is monotonically increasing with δ. A crucial property of Ω is that for any fixed δ,

$$\lim_{\gamma\to 0}\Omega(\gamma,\delta) = 0, \tag{35}$$

that stems from the fact that for $\gamma = 0$ $S_0(t) = S(t)$ is a contraction semigroup, *i.e.* $\forall t$ $\|S(t)\| \le 1$.

3.1 State predictor

Assume that an estimate $\hat{\mathcal{X}}(t)$ is available for $t \ge -\delta$.

Theorem 1. Consider system (31)–(32) with the control law

$$u(t) = -\gamma\mathcal{B}^*S_\gamma(\delta)\hat{\mathcal{X}}(t), \quad t \ge -\delta \tag{36}$$

where $\mathcal{S}_\gamma(t)$ is the semigroup generated by $\mathcal{A}_\gamma = \mathcal{A} - \gamma\mathcal{B}\mathcal{B}^*$ and $\hat{\mathcal{X}}(t)$ is computed by (28) for $t \ge -\delta$. If $\Omega(\gamma,\delta) < 1$ then the closed-loop system is asymptotically stationary with bounded covariance and such that $\mathbb{E}[\mathcal{X}(t)] \to 0$.

Proof. Replacing (36) in (31) we obtain for $t \ge 0$

$$\begin{aligned}\dot{\mathcal{X}}(t) =& \mathcal{A}\mathcal{X}(t) - \gamma\mathcal{B}\mathcal{B}^*S_\gamma(\delta)\hat{\mathcal{X}}(t-\delta) + \mathcal{B}N_s(t)\\ =& \mathcal{A}_\gamma\mathcal{X}(t) + \gamma\mathcal{B}v(t) + \mathcal{B}\widetilde{N}(t)\end{aligned} \tag{37}$$

$$v(t) = \mathcal{B}^*\left(\mathcal{X}(t) - S_\gamma(\delta)\mathcal{X}(t-\delta)\right) \tag{38}$$

$$\widetilde{N}(t) = \gamma\mathcal{B}^*S_\gamma(\delta)\mathcal{E}(t-\delta) + N_s(t). \tag{39}$$

The mild solution of (37) between $t-\delta$ and $t > \delta$ is

$$\mathcal{X}(t) = \mathcal{S}_\gamma(\delta)\mathcal{X}(t-\delta) + \int_{t-\delta}^t S_\gamma(t-s)\mathcal{B}(\gamma v(s) + \widetilde{N}(s))\,ds. \tag{40}$$

Thus, for $t > \delta$,

$$v(t) = \int_0^\delta \mathcal{B}^*S_\gamma(\theta)\mathcal{B}\left(\gamma v(t-\theta) + \widetilde{N}(t-\theta)\right)d\theta. \tag{41}$$

Clearly, $v \in L_2([0,T],\mathbb{R})$ and $\mathcal{X} \in L_2([0,T],\mathcal{H}_e)$ for any T. By exploiting the Minkowski inequality and the isometry (5) we obtain for the expected value $\mathbb{E}[v^2(t)]^{\frac{1}{2}}$

$$\begin{aligned}\mathbb{E}[v^2(t)]^{\frac{1}{2}} \leq\ & \Omega(\gamma,\delta) \sup_{\tau\in[t-\delta,t]} \mathbb{E}[v^2(\tau)]^{\frac{1}{2}} \\ & + \gamma \left(\int_0^\delta (\mathcal{B}^* S_\gamma(\theta)\mathcal{B})^2 (\mathcal{B}^* S_\gamma(\delta)\mathcal{E}(t-\theta-\delta))^2 \, d\theta \right)^{\frac{1}{2}} \\ & + \sqrt{d_s} \left(\int_0^\delta (\mathcal{B}^* S_\gamma(\theta)\mathcal{B})^2 \, d\theta \right)^{\frac{1}{2}} \\ \leq & \, \Omega(\gamma,\delta) \sup_{\tau\in[t-\delta,t]} \mathbb{E}[v(\tau)^2]^{\frac{1}{2}} + \nu(\gamma,\delta), \end{aligned} \tag{42}$$

where $\nu(\gamma,\delta)$ is finite, because for all s $\mathbb{E}[(\mathcal{B}^* S_\gamma(\delta)\mathcal{E}(s))^2]$ is bounded and (34) holds. By taking the supremum of (42) in $[t-\delta,t]$ and by denoting $w_k = \sup_{s\in[k\delta,(k+1)\delta]} \mathbb{E}[v^2(s)]^{\frac{1}{2}}$ we obtain for $k \geq 0$

$$w_{k+1} \leq \Omega(\gamma,\delta) \max\{w_k, w_{k+1}\} + \nu(\gamma,\delta) \tag{43}$$

from which it follows that

$$w_k \leq \bar{w} = \max\left\{ w_0, \frac{\nu(\gamma,\delta)}{1-\Omega(\gamma,\delta)} \right\}, \quad \forall k, \tag{44}$$

and since $\Omega(\gamma,\delta) < 1$ we conclude that $\mathbb{E}[v^2(t)]$ is uniformly bounded. Since $\mathcal{A}_\gamma$ generates a strongly stable semigroup, it follows with standard arguments from (37) that $\mathcal{X}(t)$ has bounded covariance operator for any t. In order to prove that v is asymptotically stationary we start by observing that both $\widetilde{N}$ and $\bar{N}(t) = \int_0^\delta \gamma\mathcal{B}^* S_\gamma(\theta)\mathcal{B}\widetilde{N}(t-\theta)d\theta$ are asymptotically stationary processes. Define the projectors $\Pi_{>\tau}, \Pi_{<\tau}$ such that for $x \in L_2([0,T],\mathbb{R})$, $(\Pi_{>\tau}x)(t) = x(t)$ if $t \geq \tau$ and 0 elsewhere (resp. $(\Pi_{<\tau}x)(t) = x(t)$ if $t \leq \tau$ and 0 elsewhere). Define moreover the processes $v_s, v_0 \in L_2([0,T],\mathbb{R})$ as

$$v_0(t) = \Pi_{<\delta}\mathcal{B}^* \left(\mathcal{X}(t) - S_\gamma(\delta)\mathcal{X}(t-\delta)\right) \tag{45}$$

$$v_s(t) = \Pi_{>\delta}\left((Lv)(t) + \bar{N}(t)\right) \tag{46}$$

$$(Lv)(t) = \int_0^\delta \gamma\mathcal{B}^* S_\gamma(\theta)\mathcal{B}v(t-\theta)d\theta. \tag{47}$$

Clearly, $v = v_0 + v_s$, $v_s = \Pi_{>\delta}(Lv + \bar{N})$ and

$$v = \Pi_{>\delta}Lv + v_0 + \Pi_{>\delta}\bar{N}. \tag{48}$$

Next, 0 is not an eigenvalue of the operator $I - \Pi_{>\delta}L$ because $v = \Pi_{>\delta}Lv$ is possible only for $v = 0$. Therefore, $(I - \Pi_{>\delta}L)^{-1}$ exists and v, that can be represented as

$$v = (I - \Pi_{>\delta}L)^{-1}(v_0 + \Pi_{>\delta}\bar{N}), \tag{49}$$

is asymptotically stationary because both v_0 and $\bar{N}$ are asymptotically stationary. Again, by considering (37) is easy to see that $\mathcal{X}$ is asymptotically stationary as well. Finally, the exponential stability of $\mathbb{E}[v(t)]$ is obtained by taking expectations in (41) and by using the same steps as above. Since $\mathbb{E}[\widetilde{N}(t)] = 0$, $\mathbb{E}[\mathcal{X}(t)]$ is asymptotically stable. □

Remark 2. Thanks to (35) the hypothesis $\Omega(\gamma,\delta) < 1$ can always be satisfied by taking γ sufficiently small. Or, a stabilizing control exists for any δ. However, reducing γ makes $\mathcal{A}_\gamma$ "less stable" since the damping ratio of disturbances is reduced as well.

Remark 3. The implementation of (36) requires to predict the whole state $\mathcal{X}(t+\delta)$ as $S_\gamma(\delta)\hat{\mathcal{X}}(t)$ from the filter estimate $\hat{\mathcal{X}}(t)$ and then to use its boundary value at l in the feedback.

The result of Theorem 1 can be used to prove the stability of a related stochastic PDE with delay.

Corollary 1. Consider the process $\mathcal{W}(t) \in \mathcal{H}_e$ evolving for $t \geq \delta$ with the equation

$$\begin{aligned}\dot{\mathcal{W}}(t) =& \mathcal{A}\mathcal{W}(t) - \gamma S_\gamma(\delta)\mathcal{B}\mathcal{B}^*\mathcal{W}(t-\delta) \\ & + \mathcal{B}\left(\widetilde{N}(t) - S_\gamma(\delta)\widetilde{N}(t-\delta)\right)\end{aligned} \tag{50}$$

If $\Omega(\gamma,\delta) < 1$ then $\mathcal{W}$ is asymptotically stationary with bounded covariance and such that $\mathbb{E}[\mathcal{W}(t)] \to 0$.

Proof. Consider system (37) and let $\mathcal{W}(t) = \mathcal{X}(t) - S_\gamma(\delta)\mathcal{X}(t-\delta)$ so that $v(t) = \mathcal{B}^*\mathcal{W}(t)$. From (40) it follows

$$\mathcal{W}(t) = \int_0^\delta \gamma S_\gamma(s)\mathcal{B}\left(v(t-s) + \widetilde{N}(t-s)\right) ds. \tag{51}$$

In the hypothesis, v is asymptotically stationary with bounded variance and $|\mathbb{E}[v(t)]|$ is exponentially stable, thus $\mathcal{W}$ is asymptotically stationary with bounded covariance and $\mathbb{E}[\mathcal{W}(t)] \to 0$. But $\mathcal{W}$ satisfies

$$\begin{aligned}\dot{\mathcal{W}}(t) =& \dot{\mathcal{X}}(t) - \dot{\mathcal{X}}(t-\delta) \\ =& \mathcal{A}_\gamma\mathcal{X}(t) + \gamma\mathcal{B}\mathcal{B}^*\mathcal{W}(t) + \mathcal{B}\widetilde{N}(t) - S_\gamma(\delta)\mathcal{A}_\gamma\mathcal{X}(t-\delta) \\ & - \gamma S_\gamma\mathcal{B}\mathcal{B}^*\mathcal{W}(t-\delta) - S_\gamma(\delta)\mathcal{B}\widetilde{N}(t-\delta) \\ =& \mathcal{A}\mathcal{W}(t) - \gamma S_\gamma(\delta)\mathcal{B}\mathcal{B}^*\mathcal{W}(t-\delta) \\ & + \mathcal{B}\left(\widetilde{N}(t) - S_\gamma(\delta)\widetilde{N}(t-\delta)\right),\end{aligned} \tag{52}$$

where we have used $\mathcal{A}_\gamma S_\gamma(t) = S_\gamma(t)\mathcal{A}_\gamma$, and the thesis follows. □

Remark 4. It is not difficult to prove that Corollary 1 holds for the generic equation

$$\dot{\mathcal{W}}(t) = \mathcal{A}\mathcal{W}(t) - \gamma S_\gamma(\delta)\mathcal{B}\mathcal{B}^*\mathcal{W}(t-\delta) + \mathcal{B}N(t) \tag{53}$$

where $N \in L_2(\Omega \times \mathbb{R}_+, \mathbb{R})$ is an asymptotically stationary scalar stochastic process with bounded variance.

3.2 Output-based predictor

The main drawback of the control law (36) is the need of computing $S_\gamma(\delta)\hat{\mathcal{X}}(t)$, a prediction of the distributed state from the estimate. Clearly, this entails a significant computational cost that must be added to the implementation of the filter (28). In this section we describe an output-feedback control law based on an auxiliary system that merges filtering and prediction. In this case the semigroup S_γ occurs in the gain term of the predictor, but is applied to a finite dimensional correction term that contains the boundary value. The new control law is

$$u(t) = -\gamma\mathcal{B}^*\Theta(t), \quad t \geq -\delta \tag{54}$$

$$\dot{\Theta}(t) = \mathcal{A}_\gamma\Theta(t) - \gamma S_\gamma(\delta)\mathcal{B}\left(y(t) + \mathcal{B}^*\Theta(t-\delta)\right), \ t \geq 0 \tag{55}$$

with $\Theta(\tau) = S_\gamma(\tau+\delta)\Theta(-\delta)$ as initial condition for $\tau \in [-\delta, 0]$, $\Theta(-\delta) \in \mathcal{H}_e$ being an arbitrary element. The value $\Theta(t)$ of the controller is an estimate of $\mathcal{X}(t+\delta)$.

Theorem 2. Consider system (31)–(32) with the control law (54)–(55). If $\Omega(\gamma,\delta) < 1$ then the closed-loop system is asymptotically stationary with bounded covariance and such that $\mathbb{E}[\mathcal{X}(t)] \to 0$.

Proof. Let us denote $\mathcal{W}(t) = \mathcal{X}(t) - \Theta(t-\delta)$ the prediction error. System (31)–(32) with the control law (54)–(55) can be written for $t \geq 0$ as

$$\begin{aligned}\dot{\mathcal{X}}(t) =& \mathcal{A}\mathcal{X}(t) - \gamma\mathcal{B}\mathcal{B}^*\Theta(t-\delta) + \mathcal{B}N_s(t) \\ =& \mathcal{A}_\gamma\mathcal{X}(t) + \gamma\mathcal{B}\mathcal{B}^*\mathcal{W}(t) + \mathcal{B}N_s(t).\end{aligned} \tag{56}$$

The dynamics of $\mathcal{W}(t)$ is, for $t \geq \delta$,

$$\begin{aligned}
&\dot{\mathcal{W}}(t) = \dot{X}(t) - \dot{\Theta}(t-\delta)\\
=&\mathcal{A}_\gamma \mathcal{X}(t) + \gamma \mathcal{B}\mathcal{B}^* \mathcal{W}(t) + \mathcal{B} N_s(t) - \mathcal{A}_\gamma \Theta(t-\delta)\\
&+ \gamma S_\gamma(\delta)\mathcal{B}\left(-\mathcal{B}^*\mathcal{X}(t-\delta) + N_o(t-\delta) + \mathcal{B}^*\Theta(t-2\delta)\right)\\
=&\mathcal{A}\mathcal{W}(t) - \gamma S_\gamma(\delta)\mathcal{B}\mathcal{B}^*\mathcal{W}(t-\delta)\\
&+ \mathcal{B}N_s(t) + \gamma S_\gamma(\delta)\mathcal{B}N_o(t-\delta), \qquad (57)
\end{aligned}$$

and $\mathcal{W}$ is asymptotically stationary with bounded covariance and $\mathbb{E}[\mathcal{W}(t)] \to 0$ in virtue of Corollary 1 (see Remark 4). From (56) descends that the same holds for $\mathcal{X}$. □

Remark 5. When the delay affects the output rather than the input the predictor Θ in (55) can be used to estimate $\mathcal{X}(t)$ from $y(t-\delta)$. The only difference is that in the correction term of (55) $y(t)$ is replaced by $y(t-\delta)$. Notice that the control law remains the same. Thus the approach proposed in this section solves both the input delay and the output delay problems. The presence of both delays can be easily solved in the same way.

Remark 6. The main difference between (55) and the Kalman filter (28), is the presence of the delayed term and that $\mathcal{B}$ is replaced by $S_\gamma(\delta)\mathcal{B}$. The implementation of the term containing the semigroup predictor $S_\gamma(\delta)$ in (55) is however meaningfully simpler than in (36). In the latter case, $S_\gamma(\delta)$ is applied to the whole distributed state $\hat{\mathcal{X}}(t)$. In contrast, in (55) $S_\gamma(\delta)$ is applied to a vector having 0 in the distributed component and the scalar correction term $y(t) - \mathcal{B}^*\Theta(t-\delta)$ in the boundary condition $\dot{w}(t,l)$. In other words, $S_\gamma(\delta)\mathcal{B}y$ is the solution in δ of the closed-loop dynamics starting from a null initial condition except for the speed at the end boundary which is y.

3.3 Modular predictor

Theorem 2 states that for any delay is always possible to find a stabilizing $\gamma > 0$. However, as highlighted in Remark 2, large delays require small gains and consequently a limited attenuation of the noise effect. To overcome this effect it is possible to resort to a modular, or cascaded, predictor, as in the finite-dimensional case (Cacace et al., 2016). The idea is to partition the delay obtaining a sequence $\{\delta_i\}$, $i = 0, \ldots, m$, such that $\sum_i \delta_i = \delta$ and $\Omega(\gamma, \delta_i) < 1$ for any i. The elements of the partition are not necessarily identical, but we adopt the obvious choice $\delta_i = \delta/m$. With this choice it is always possible to find m such that $\Omega(\gamma, \delta/m) < 1$ for any γ, δ, that is, the choice of the gain can be decoupled from the delay. The i-th predictor $\Theta_i(t)$ aims at predicting $\mathcal{X}(t + \frac{m-i+1}{m}\delta)$, thus $\Theta_1(t)$ predicts $\mathcal{X}(t+\delta)$. The structure of the predictors is

$$\begin{aligned}
\dot{\Theta}_i(t) =& \mathcal{A}\Theta_i - \gamma \mathcal{B}^*\Theta_1\left(t - \frac{i-1}{m}\delta\right)\\
&- \gamma S_\gamma\left(\frac{\delta}{m}\right)\mathcal{B}\left(y_i(t) + \mathcal{B}^*\Theta_i\left(t - \frac{\delta}{m}\right)\right), \quad (58)
\end{aligned}$$

where $y_i(t) = -\mathcal{B}^*\Theta_{i+1}(t)$ for $i < m$ and $y_m(t) = y(t)$. The proof of the following result is similar to the proof of Theorem 2 and is omitted for space reasons.

Theorem 3. Consider system (31)–(32) with the control law $u(t) = -\gamma \mathcal{B}^*\Theta_1(t)$, where Θ_1 is the first element of the modular predictor (58), $i = 1, \ldots, m$. If $\Omega(\gamma, \delta/m) < 1$ then the closed-loop system is asymptotically stable in mean and asymptotically stationary with bounded covariance.

4. FINITE-DIMENSIONAL IMPLEMENTATION

In this section we study how to implement the controllers introduced in the previous section, and in particular we describe a Galerkin approximation of the output-based predictor introduced in Section 3.2. For the case $\delta = 0$ of Section 2 a finite-dimensional implementation of the filter (25) is described in Germani et al. (2006), where it shown that the closed-loop system is stable with any order of approximation of the filter. We follow the same approach, based on the subspaces generated by the natural modes of vibration of the structure that are obtained as the solutions of the following generalized eigenvalues-eigenfunctions problem,

$$A\phi_i = \omega_i^2 M\phi_i, \qquad (59)$$

that form an M-orthogonal basis for $\mathcal{H}$, i.e. $[\phi_i, M\phi_j]_{\mathcal{H}} = 1$ iff $i = j$. The solutions of (59) can be obtained by standard computations (see for example Erturk and Inman (2011), Appendix C). Let $V_n = \text{span}\{\phi_2, \ldots, \phi_n\}$, and $\Pi_n : \mathcal{H} \to V_n$ be the operator

$$\Pi_n x = \sum_{i=1}^{n} [x, M\phi_i]_{\mathcal{H}} \phi_i. \qquad (60)$$

Let moreover

$$W_n = \text{span}\left\{ \begin{bmatrix}\phi_i\\0\end{bmatrix}, \begin{bmatrix}0\\\phi_j\end{bmatrix}, \ i, j = 1, \ldots, n \right\} \qquad (61)$$

$$\Pi_n^e : \mathcal{H}_e \to W_n, \quad \Pi_n^e \mathcal{X} = \begin{bmatrix}\Pi_n x_1\\ \Pi_n x_2\end{bmatrix}, \ \mathcal{X} = \begin{bmatrix}x_1\\x_2\end{bmatrix}. \qquad (62)$$

With these definitions (i) Π_n is an idempotent operator; (ii) $x - \Pi_n x$ is M-orthogonal to V_n; (iii) $\Pi_n^* x = \sum_{i=1}^n [x, \phi_i]_{\mathcal{H}} M\phi_i$; (iv) Π_n^e is an orthogonal projector on W_n; (v) the sequence $\{\Pi_n^e\}$ converges strongly to the identity operator in $\mathcal{H}_e$, that is

$$\lim_{n\to\infty} \|\Pi_n^e \mathcal{X} - \mathcal{X}\|_{\mathcal{H}_e} = 0, \qquad (63)$$

see Germani et al. (2006). The control law (54)–(55) can be approximated by the following finite-dimensional control law obtained by projecting on the space W_n,

$$u(t) = -\gamma \mathcal{B}^* \Pi_n^e \Theta_n(t), \quad t \geq -\delta \qquad (64)$$

$$\begin{aligned}
\dot{\Theta}_n(t) =& \Pi_n^e \mathcal{A}_\gamma \Pi_n^e \Theta_n(t) - \gamma \Pi_n^e S_\gamma(\delta)\mathcal{B}\\
&\cdot \left(y(t) + \mathcal{B}^*\Pi_n^e\Theta_n(t-\delta)\right), \quad t \geq 0. \qquad (65)
\end{aligned}$$

An element $\mathcal{X} = [x_1, x_2]^\top \in \mathcal{H}_e$ and its projection $\Pi_n^e \mathcal{X}$ can be represented in the orthonormal basis $\{\phi_i\}$ as

$$\mathcal{X} = \begin{bmatrix} \sum_{i=1}^{\infty} [x_1, M\phi_i]_{\mathcal{H}}\phi_i \\ \sum_{i=1}^{\infty} [x_2, M\phi_i]_{\mathcal{H}}\phi_i \end{bmatrix}, \quad \Pi_n^e \mathcal{X} = \begin{bmatrix} \sum_{i=1}^{n} [x_1, M\phi_i]_{\mathcal{H}}\phi_i \\ \sum_{i=1}^{n} [x_2, M\phi_i]_{\mathcal{H}}\phi_i \end{bmatrix}. \qquad (66)$$

Thus, from $A\phi_i = \omega_i^2 M\phi_i$ we get

$$\Pi_n^e \mathcal{A}_\gamma \Pi_n^e \mathcal{X} = \begin{bmatrix} \Pi_n x_2 \\ -\sum_{i=1}^{n} \omega_i^2 [x_1, M\phi_i]_{\mathcal{H}}\phi_i - \gamma \Pi_n B x_2(l) \end{bmatrix}, \qquad (67)$$

and $[Bx_2(l), M\phi_i]_{\mathcal{H}} = \sum_{k=1}^n \phi_i(l)\phi_k(l)[x_2, M\phi_k]_{\mathcal{H}}$. Analogously, let $c(t) = y(t) + \mathcal{B}^*\Pi_n^e\Theta_n(t-\delta) \in \mathbb{R}$ be the correction term in (65). We get the representation

$$\Pi_n^e S_\gamma(\delta)\mathcal{B}c(t) = \begin{bmatrix} \sum_{i=1}^{n} \left[(S_\gamma(\delta)\mathcal{B}c(t))_1 , M\phi_i \right]_{\mathcal{H}} \phi_i \\ \sum_{i=1}^{n} \left[(S_\gamma(\delta)\mathcal{B}c(t))_2 , M\phi_i \right]_{\mathcal{H}} \phi_i \end{bmatrix}. \quad (68)$$

Given n, a finite-dimensional representation of $\Theta_n = [\theta_1, \theta_2]^\top$ in (65) is obtained with a vector $Z_n \in \mathbb{R}^{2n}$,

$$Z_n(t) = \begin{bmatrix} [\theta_1(t), M\phi_i]_{\mathcal{H}} \\ [\theta_2(t), M\phi_i]_{\mathcal{H}} \end{bmatrix} = \begin{bmatrix} Z_{n,1}(t) \\ Z_{n,2}(t) \end{bmatrix}. \quad (69)$$

A dynamical equation for $Z(t)$ can be obtained from (65), (67), (68) in the form

$$\dot{Z}_n(t) = A_n^z Z_n(t) + B_n^z c(t), \quad (70)$$

where $A_n^z \in \mathbb{R}^{2n\times 2n}$, $B_n^z \in \mathbb{R}^{2n\times 1}$ are given by

$$A_n^z = \begin{bmatrix} 0 & I_n \\ -\Omega_2 & -\gamma\xi_n\xi_n^\top \end{bmatrix}, \; B_n^z = \begin{bmatrix} \mathrm{col}_i \left[(S_\gamma(\delta)\mathcal{B})_1 , M\phi_i \right]_{\mathcal{H}} \\ \mathrm{col}_i \left[(S_\gamma(\delta)\mathcal{B})_2 , M\phi_i \right]_{\mathcal{H}} \end{bmatrix}. \quad (71)$$

Here, $\Omega_2 = \mathrm{diag}\{\omega_i^2\}$, and $\xi_n = \mathrm{col}_i\{\phi_i(l)\} \in \mathbb{R}^n$. Notice that for any $\gamma > 0$, $n \geq 1$, the eigenvalues of A_n^z have negative real parts (see Theorem 5.3 in Germani et al. (2006)). The control input (64) can be computed in this scheme as $u(t) = -\gamma[0, \xi_n^\top]Z_n(t)$. The constant matrices A_n^z, B_n^z can be computed off-line, thus making the finite-dimensional controller easy to compute.

5. CONCLUSIONS

We have described a predictor-based controller for the Euler-Bernoulli beam with input delay with nice formal properties and endowed with a computationally cheap finite-dimensional implementation. Further work is needed to prove the stability of the prediction error and of the closed-loop system under the finite-dimensional approximation of the controller.

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