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Using delays for digital implementation of derivative-dependent control of stochastic multi-agents ⋆

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Abstract: In this paper, we study the digital implementation of derivative-dependent control for consensus of the nth-order stochastic multi-agent systems. The consensus controllers that depend on the output and its derivatives up to the order $n-1$ are approximated as delayed sampled-data controllers. For the consensus analysis, we propose novel Lyapunov-Krasovskii functionals to derive linear matrix inequalities (LMIs) that allow to find admissible sampling period. The efficiency of the presented approach is illustrated by numerical examples.

Keywords: Sampled-data control, stochastic multi-agent systems, consensus, LMIs.

1. INTRODUCTION

During the last decade, consensus of multi-agent systems has received much attention due to its wide applications (Olfati-Saber et al., 2007). Consensus requires all agents to achieve a desired objective via neighbors' information. For example, the second-order consensus problem was studied by the position and velocity information (Yu et al., 2010; Gao and Wang, 2011). If the velocity (i.e. the derivative of the position) is not available, it can be approximated by finite differences leading to a delayed feedback (see e.g. Niculescu and Michiels (2004); Kharitonov et al. (2005); Fridman and Shaikhet (2016); Ramírez et al. (2016); Fridman and Shaikhet (2017) and reference therein). This idea has been employed for the second-order deterministic multi-agent systems in Yu et al. (2013); Ma et al. (2014); Cui et al. (2019).

In many areas of applications, e.g. aircraft engineering, process control, population dynamics, multiplicative noises that occur due to the parameter uncertainties and nonlinearities cannot be avoided (Mao, 2007; Shaikhet, 2013). Consensus of stochastic multi-agent systems was studied in Li and Zhang (2010); Ding et al. (2015); Ma et al. (2017). In our recent paper (Zhang and Fridman, 2019), the delayed implementation of derivative-dependent control for stochastic systems was considered. However, the idea of using the delayed position information has not been studied yet for the nth-order deterministic multi-agents ($n \geq 3$) or stochastic multi-agents ($n \geq 2$).

In this paper, we study digital implementation of derivative-dependent control by using delays for the nth-order stochastic multi-agent systems. Following the improved approximation method by using consecutive sampled outputs (Selivanov and Fridman, 2018), we approximate the consensus controllers that depend on the output and its derivatives up to the order $n-1$ as delayed sampled-data controllers. Note that results of Zhang and Fridman (2019) cannot be directly applied to the multi-agent case because of an additional term in the closed-loop system. To compensate the additional term, we propose novel Lyapunov-Krasovskii functionals that lead to LMIs. Finally, we present numerical examples to illustrate the efficiency of the presented approach.

Notations: Throughout this paper, $\mathbf{1}_n = [1, \dots, 1]^T \in \mathbb{R}^n$, $\mathbf{0}_n = [0, \dots, 0]^T \in \mathbb{R}^n$, I_n is the identity $n \times n$ matrix, $\otimes$ stands for the Kronecker product, the superscript T stands for matrix transposition. $\mathbb{R}^n$ denotes the n dimensional Euclidean space with Euclidean norm $|\cdot|$, $\mathbb{R}^{n\times m}$ denotes the set of all $n \times m$ real matrices. Denote by diag$\{\dots\}$ and col$\{\dots\}$ block-diagonal matrix and block-column vector, respectively. $X > 0$ implies that X is a positive definite symmetric matrix, $|X|^2_S$ denotes the expression $X^T S X$ with matrix S and vector X of appropriate dimensions, $\mathbf{E}X$ denotes the mathematical expectation of stochastic variable X.

2. PROBLEM FORMULATION AND PRELIMINARIES

2.1 Graph theory

The communication topology among N agents is represented by a directed weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix with $a_{ij} \geq 0$, $\forall i, j \in \mathcal{V}$. It is assumed that $a_{ii} = 0$, $\forall i \in \mathcal{V}$. Notice that $(i, j) \in \mathcal{E}$ when $a_{ij} > 0$. An edge $(i, j) \in \mathcal{E}$ implies that node i can receive information from node j. Correspondingly, the Laplacian matrix $L = [L_{ij}] \in \mathbb{R}^{N\times N}$ of graph $\mathcal{G}$ is defined by $L_{ii} = \sum_{j=1}^N a_{ij}$ and $L_{ij} = -a_{ij}$ when $i \neq j$. Graph $\mathcal{G}$ is said to have a spanning tree if there exists node $i \in \mathcal{V}$ such that node i is reachable from any other nodes.

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2.2 Derivative-dependent control of stochastic multi-agents

Consider the nth-order stochastic dynamic for each agent $i \in \mathcal{V}$ governed by

$$y_i^{(n)}(t) = \sum_{j=0}^{n-1} \big(a_j + c_j \dot{w}(t)\big) y_i^{(j)}(t) + b u_i(t), \tag{1}$$

where $y_i(t) = y_i^{(0)}(t) \in \mathbb{R}^k$ is the output, $y_i^{(j)}(t)$ is the jth derivative of $y_i(t)$, $u_i(t) \in \mathbb{R}^m$ is the control input, $w(t)$ is the scalar standard Wiener process (Mao, 2007; Shaikhet, 2013), and $a_i,\ c_i \in \mathbb{R}^{k\times k}$, $b \in \mathbb{R}^{k\times m}$ are constant matrices. Denoting

$$\begin{aligned}
x_i(t) &= \mathrm{col}\{y_i^{(0)}(t), \dots, y_i^{(n-1)}(t)\} \\
&= \mathrm{col}\{x_{i,0}(t), \dots, x_{i,n-1}(t)\} \in \mathbb{R}^{nk}, \\
A &= \begin{bmatrix} 0 & I_k & 0 & \cdots & 0 \\ 0 & 0 & I_k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I_k \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix} \in \mathbb{R}^{nk\times nk}, \\
B &= \mathrm{col}\{0, b\} \in \mathbb{R}^{nk\times m}, \\
C &= \mathrm{col}\{0, \bar{C}\} \in \mathbb{R}^{nk\times nk}
\end{aligned}$$

with $\bar{C} = [c_0, \dots, c_{n-1}]$, we present (1) as

$$dx_i(t) = \big(Ax_i(t) + Bu_i(t)\big)dt + Cx_i(t)dw(t), \tag{2}$$

where the initial condition is given by $x_i(0) = x_i^0$.

For multi-agent system (2), it is common to look for a consensus controller of the form (see e.g. Sun and Wang (2009))

$$u_i(t) = -\bar{K}\sum_{j=1}^{N} L_{ij} x_j(t) = -\sum_{j=1}^{N}\sum_{l=0}^{n-1} L_{ij}\bar{K}_l x_{j,l}(t) \tag{3}$$

with $\bar{K} = [\bar{K}_0, \dots, \bar{K}_{n-1}] \in \mathbb{R}^{m\times nk}$ such that for any initial condition, the consensus of (2) can be exponentially mean-square achieved, i.e.

$$\mathbf{E}|x_i(t) - x_j(t)| \le c e^{-\alpha t}\mathbf{E}|x_i(0) - x_j(0)|, \quad \forall i, j \in \mathcal{V},$$

where $c > 0$ is a constant and $\alpha > 0$ is called the decay rate.

Differently from the state-feedback case with the full knowledge of the system state, we consider the output-feedback control where the derivatives $x_{j,l}(t)$ ($l = 1, \dots, n-1$) in (3) are not available. As in Selivanov and Fridman (2018), we employ their finite-difference approximations:

$$\begin{aligned}
x_{j,0}(t) &= \bar{x}_{j,0}(t), \\
x_{j,l}(t) \approx \bar{x}_{j,l}(t) &= \frac{\bar{x}_{j,l-1}(t) - \bar{x}_{j,l-1}(t-h)}{h} \\
&= \frac{1}{h^l}\sum_{p=0}^{l}\binom{l}{p}(-1)^p x_{j,0}(t - ph), \\
l &= 1, ..., n-1
\end{aligned} \tag{4}$$

with a constant delay $h > 0$ and the binomial coefficients $\binom{l}{p} = \frac{l!}{p!(l-p)!}$. By replacing $x_{j,l}(t)$ in (3) with their approximations, we have the following delay-dependent controller

$$\begin{aligned}
u_i(t) &= -\sum_{j=1}^{N}\sum_{l=0}^{n-1} L_{ij}\bar{K}_l \bar{x}_{j,l}(t) \\
&= -\sum_{j=1}^{N}\sum_{l=0}^{n-1} L_{ij} K_l x_{j,0}(t - lh),
\end{aligned} \tag{5}$$

where $x_{j,0}(t) = x_{j,0}(0)$ for $t < 0$ and

$$K_l = (-1)^l \sum_{m=l}^{n-1}\binom{m}{l}\frac{1}{h^m}\bar{K}_m. \tag{6}$$

For practical application of the delayed feedback (5), we suggest its sampled-data implementation. We suppose that $x_{j,0}(t)$ is only available at the time instants $t_k = kh$, $k \in \mathbb{N}_0$, where $h > 0$ is the sampling period. Then we have the following sampled-data controller

$$\begin{aligned}
u_i(t) &= -\sum_{j=1}^{N}\sum_{l=0}^{n-1} L_{ij}\bar{K}_l \bar{x}_{j,l}(t_k) \\
&= -\sum_{j=1}^{N}\sum_{l=0}^{n-1} L_{ij} K_l x_{j,0}(t_{k-l}), \\
t &\in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0,
\end{aligned} \tag{7}$$

where $\bar{x}_{j,l}(t)$ and K_l are given by (4) and (6), respectively. For the sampled-data controller (7), we introduce the errors due to sampling

$$\begin{aligned}
\bar{x}_{j,0}(t_k) &= x_{j,0}(t) - \int_{t_k}^{t} \dot{x}_{j,0}(s)ds, \\
\bar{x}_{j,i}(t_k) &= \bar{x}_{j,i}(t) - \int_{t_k}^{t} \dot{\bar{x}}_{j,i}(s)ds, \quad i = 1, \dots, n-1.
\end{aligned} \tag{8}$$

Then we follow the idea of Selivanov and Fridman (2018) to present the approximation errors $x_{j,i}(t) - \bar{x}_{j,i}(t)$ ($i = 1, \dots, n-1$) as

$$\bar{x}_{j,l}(t) = x_{j,l}(t) - \int_{t-lh}^{t} \varphi_l(t-s)\dot{x}_{j,l}(s)ds, \tag{9}$$

where $\varphi_1(v) = \frac{h-v}{h}$, $v \in [0, h]$ and for $i = 1, \dots, n-2$

$$\varphi_{i+1}(v) = \begin{cases} \dfrac{1}{h}\displaystyle\int_0^v \varphi_i(\lambda)d\lambda + \dfrac{h-v}{h}, & v \in [0, h] \\ \dfrac{1}{h}\displaystyle\int_{v-h}^{v} \varphi_i(\lambda)d\lambda, & v \in (h, ih). \\ \dfrac{1}{h}\displaystyle\int_{v-h}^{ih} \varphi_i(\lambda)d\lambda, & v \in [ih, ih+h]. \end{cases}$$

For further proceeding, we present several properties of the functions $\varphi_i(s)$ ($i = 1, \dots, n-1$).

Proposition 1. The functions $\varphi_i(s)$ ($i = 1, \dots, n-1$) have the following properties (Zhang and Fridman, 2019; Selivanov and Fridman, 2018):

(1) $0 \le \varphi_i(v) \le 1$, $v \in [0, ih]$,
(2) $\varphi_i(0) = 1$, $\varphi_i(ih) = 0$,
(3) $\int_0^{ih} \varphi_i(v)dv = \frac{ih}{2}$,
(4) $\frac{d}{dv}\varphi_i(v) \in [-\frac{1}{h}, 0]$, $v \in [0, ih]$.

Based on these properties, it follows from (9) that

$$\dot{\bar{x}}_{j,i}(t) = \int_{t-ih}^{t} \psi_i(t-s)\dot{x}_{j,i}(s)ds, \quad i = 1, \dots, n-1 \tag{10}$$

where $\psi_i(t-s) = \frac{d}{ds}\varphi_i(t-s)$.

Denote

$$\begin{aligned}
x(t) &= \mathrm{col}\{x_1(t), \dots, x_N(t)\}, \ \chi(t) = \mathrm{col}\{\chi_2(t), \dots, \chi_N(t)\}, \\
\bar{x}(t) &= \mathrm{col}\{\bar{x}_1(t), \dots, \bar{x}_N(t)\}, \ \bar{\chi}(t) = \mathrm{col}\{\bar{\chi}_2(t), \dots, \bar{\chi}_N(t)\}
\end{aligned}$$

with

$$\begin{aligned}
\chi_i(t) &= x_1(t) - x_i(t) = \mathrm{col}\{\chi_{i,0}(t), \dots, \chi_{i,n-1}(t)\}, \\
\bar{\chi}_i(t) &= \bar{x}_1(t) - \bar{x}_i(t) = \mathrm{col}\{\bar{\chi}_{i,0}(t), \dots, \bar{\chi}_{i,n-1}(t)\}.
\end{aligned}$$

From Sun and Wang (2009), it follows that $\chi(t) = (E_1 \otimes I_{nk})x(t)$, $x(t) = (E_2 \otimes I_{nk})\chi(t) + (\mathbf{1}_N \otimes I_{nk})x_1(t)$ and $\bar{x}(t) = (E_2 \otimes I_{nk})\bar{\chi}(t) + (\mathbf{1}_N \otimes I_{nk})\bar{x}_1(t)$, where $E_1 = [\mathbf{1}_{N-1}, -I_{N-1}]$ and $E_2 = [\mathbf{0}_{N-1}, -I_{N-1}]^T$. Then the system (2), (3) takes the form

$$d\chi(t) = D\chi(t)dt + g(t)dw(t), \tag{11}$$

where

$$\begin{aligned} &D = I_{N-1} \otimes A - \mathcal{L} \otimes B\bar{K}, \quad \mathcal{L} = E_1 L E_2, \\ &g(t) = (I_{N-1} \otimes C)\chi(t). \end{aligned} \tag{12}$$

Using (8) and (9), the system (2), (7) takes the form

$$d\chi(t) = f(t)dt + g(t)dw(t), \tag{13}$$

where

$$\begin{aligned} &f(t) = D\chi(t) + \sum_{i=1}^{n-1} (\mathcal{L} \otimes B\bar{K}_i)\kappa_i(t) + (\mathcal{L} \otimes B\bar{K})\delta(t), \\ &\kappa_i(t) = \int_{t-ih}^{t} \varphi_i(t-s) H_i \dot{\chi}(s) ds, \quad \delta(t) = \int_{t_k}^{t} \dot{\bar{\chi}}(s) ds \end{aligned} \tag{14}$$

with D, $g(t)$ given by (12) and

$$H_i = I_{N-1} \otimes \varepsilon_i, \quad \varepsilon_i = [0_{k\times ik}, I_k, 0_{k\times(n-i-1)k}].$$

As in the determinstic case (see e.g. French et al. (2009)), we will show in the next section that for small enough stochastic perturbations (i.e. small enough $|C|$), if the system (11) is exponentially mean-square stable with a decay rate $\bar{\alpha} > 0$, then for any $\alpha \in (0, \bar{\alpha})$ the system (13) is exponentially mean-square stable with a decay rate α and small enough $h > 0$.

Remark 1. Comparatively to the single-agent system in Zhang and Fridman (2019), the multi-agent system (13) contains additional term $\delta(t)$ due to the sampling. This term will be further compensated by additional terms in Lyapunov functionals (see e.g. (24), (26), (28), (30)).

3. CONSENSUS ANALYSIS

In this section, we will analyze the exponential stability of system (13). First, we follow arguments of Sun and Wang (2009) to present following result:

Proposition 2. Assume that directed graph $\mathcal{G}$ has a spanning tree. Consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) can be exponentially mean-square achieved if and only if system (13) is exponentially mean-square stable.

It is clear from Proposition 2 that consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) is converted into stability of (13).

We now present the following LMI conditions:

Theorem 1. Given $\bar{K} = [\bar{K}_0, \dots, \bar{K}_{n-1}]$, let system (11) with $C = 0$ be exponentially mean-square stable with a decay rate $\bar{\alpha} > 0$.

(i) Given tuning parameters $h > 0$ and $\alpha \in (0, \bar{\alpha})$, let there exist matrices $P > 0$, $W_0 > 0$, $R_i > 0$, $W_i > 0$ $(i = 1, \dots, n-1)$, $Q > 0$, $F_1 > 0$ and $F_2 > 0$ of appropriate dimensions that satisfy

$$\Phi < 0, \quad \Omega < 0, \tag{15}$$

where Φ and Ω are, respectively, the symmetric matrices composed from

$$\begin{aligned}
\Phi_{11} &= PD + D^T P + 2\alpha P + |\mathcal{C}|_P^2 + \sum_{i=1}^{n-2} \frac{(ih)^2}{4} |H_{i+1}|_{R_i}^2 \\
&\quad + \sum_{i=0}^{n-2} h^2 e^{2\alpha ih} |H_{i+1}|_{\mathcal{W}_i}^2 + \frac{(n-1)h}{2} |H_{n-1}\mathcal{C}|_{F_1+F_2}^2, \\
\Phi_{12} &= P[(\mathcal{L} \otimes B\bar{K}_1), \dots, (\mathcal{L} \otimes B\bar{K}_{n-1})], \\
\Phi_{14} &= P(\mathcal{L} \otimes B\bar{K}), \\
\Phi_{15} &= D^T H_{n-1}^T (R_{n-1} + Q), \\
\Phi_{22} &= -\mathrm{diag}\{e^{-2\alpha h} R_1, \dots, e^{-2\alpha(n-1)h} R_{n-1}\}, \\
\Phi_{23} &= [0_{k\times(n-2)k}, e^{-2\alpha(n-1)h} R_{n-1}]^T, \\
\Phi_{25} &= [(\mathcal{L} \otimes B\bar{K}_1), \dots, (\mathcal{L} \otimes B\bar{K}_{n-1})]^T H_{n-1}^T (R_{n-1} + Q), \\
\Phi_{33} &= -e^{-2\alpha(n-1)h}(R_{n-1} + F_1), \\
\Phi_{44} &= -\frac{\pi^2}{4} e^{-2\alpha h} I_{N-1} \otimes \mathrm{diag}\{W_0, \dots, W_{n-1}\}, \\
\Phi_{45} &= (\mathcal{L} \otimes \bar{K}^T B^T) H_{n-1}^T (R_{n-1} + Q), \\
\Phi_{55} &= -\frac{4}{(nh-h)^2}(R_{n-1} + Q), \\
\Omega_{11} &= \mathcal{W}_{n-1} - \frac{(n-1)^2}{4} e^{-2\alpha(n-1)h} Q, \\
\Omega_{12} &= \mathcal{W}_{n-1}, \\
\Omega_{22} &= \mathcal{W}_{n-1} - \frac{n-1}{2} e^{-2\alpha(n-1)h} F_2
\end{aligned}$$

with $\mathcal{C} = I_{N-1} \otimes C$ and $\mathcal{W}_i = I_{N-1} \otimes W_i$. Then consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) is exponentially mean-square achieved with a decay rate $\alpha > 0$.

(ii) Given any $\alpha \in (0, \bar{\alpha})$, the LMI $\Phi < 0$ is always feasible for small enough stochastic perturbations and $h > 0$ (meaning that consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) is exponentially mean-square achieved with a decay rate $\alpha > 0$).

Proof: (i) Let $\mathfrak{L}$ be the generator of the system (13) (Mao, 2007; Fridman and Shaikhet, 2019). For the term

$$V_P = |\chi(t)|_P^2, \quad P > 0, \tag{16}$$

we have

$$\mathfrak{L}V_P + 2\alpha V_P = \chi^T(t) P f(t) + 2\alpha |\chi(t)|_P^2 + |g(t)|_P^2. \tag{17}$$

To compensate the terms $\kappa_i(t)$ $(i = 1, \dots, n-2)$, we consider

$$\begin{aligned} &V_{R_i} = \int_{t-ih}^{t} \int_{t-s}^{ih} e^{-2\alpha(t-s)} \varphi_i(v) |H_{i+1}\chi(s)|_{R_i}^2 dv ds, \\ &R_i > 0, \quad i = 1, \dots, n-2. \end{aligned} \tag{18}$$

Using Jensen's inequality (Gu et al., 2003; Solomon and Fridman, 2013) and Proposition 1 with $H_i\dot{\chi}(t) = H_{i+1}\chi(t)$ $(i = 1, \dots, n-2)$, we have

$$\begin{aligned} \mathfrak{L}V_{R_i} + 2\alpha V_{R_i} &= \frac{ih}{2} |H_{i+1}\chi(t)|_{R_i}^2 \\ &\quad - \int_{t-ih}^{t} e^{-2\alpha(t-s)} \varphi_i(t-s) |H_{i+1}\chi(s)|_{R_i}^2 ds \\ &\leq \frac{ih}{2} |H_{i+1}\chi(t)|_{R_i}^2 - \frac{2}{ih} e^{-2\alpha ih} |\kappa_i(t)|_{R_i}^2, \\ & i = 1, \dots, n-2. \end{aligned} \tag{19}$$

For the term $\kappa_{n-1}(t)$, we consider

$$\begin{aligned} V_{R_{n-1}} = &\int_{t-(n-1)h}^{t} \int_{t-s}^{(n-1)h} e^{-2\alpha(t-s)} \varphi_{n-1}(v) \\ &\times |H_{n-1} f(s)|_{R_{n-1}}^2 dv ds, \quad R_{n-1} > 0. \end{aligned} \tag{20}$$

Then we have

$$\begin{aligned}\mathfrak{L}V_{R_{n-1}}+2\alpha V_{R_{n-1}} \le \frac{(n-1)h}{2}|H_{n-1}f(t)|^2_{R_{n-1}}\\ -\frac{2}{(n-1)h}e^{-2\alpha(n-1)h}|\kappa_{n-1}(t)-\rho_1(t)|^2_{R_{n-1}},\end{aligned}\tag{21}$$

where

$$\rho_1(t)=\int_{t-(n-1)h}^{t}\varphi_{n-1}(t-s)H_{n-1}g(s)dw(s).$$

By using Itô integral properties (see e.g. Mao (2007); Fridman and Shaikhet (2019)) and Proposition 1, we have for any matrix $F_1>0$

$$\begin{aligned}&\mathbf{E}e^{-2\alpha(n-1)h}|\rho_1(t)|^2_{F_1}\\ &=\mathbf{E}e^{-2\alpha(n-1)h}\int_{t-(n-1)h}^{t}\varphi^2_{n-1}(t-s)|H_{n-1}g(s)|^2_{F_1}ds\\ &\le \mathbf{E}\int_{t-(n-1)h}^{t}e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)|H_{n-1}g(s)|^2_{F_1}ds.\end{aligned}$$

For the term

$$\begin{aligned}V_{F_1}=&\int_{t-(n-1)h}^{t}\int_{t-s}^{(n-1)h}e^{-2\alpha(t-s)}\varphi_{n-1}(v)\\ &\times|H_{n-1}g(s)|^2_{F_1}dvds,\quad F_1>0,\end{aligned}\tag{22}$$

we have

$$\begin{aligned}&\mathbf{E}\mathfrak{L}V_{F_1}+\mathbf{E}2\alpha V_{F_1}\\ &\le \mathbf{E}\frac{(n-1)h}{2}|H_{n-1}g(t)|^2_{F_1}-\mathbf{E}e^{-2\alpha(n-1)h}|\rho_1(t)|^2_{F_1}.\end{aligned}\tag{23}$$

Denote $0<W=I_{N-1}\otimes\operatorname{diag}\{W_0,\ldots,W_{n-1}\}$. Let us consider

$$\begin{aligned}V_W=&h^2\int_{t_k}^{t}e^{-2\alpha(t-s)}|\dot{\hat{\chi}}(s)|^2_W ds\\ &-\frac{\pi^2}{4}e^{-2\alpha h}\int_{t_k}^{t}e^{-2\alpha(t-s)}|\delta(s)|^2_W ds,\\ &t\in[t_k,t_{k+1}),\quad k\in\mathbb{N}_0\end{aligned}\tag{24}$$

to compensate $\delta(t)$. The exponential Wirtinger's inequality (Selivanov and Fridman, 2016) implies $V_W\ge 0$ since $\dot\delta(t)=\dot{\hat{\chi}}(t)$, $\delta(t_k)=0$ and $W>0$. Then one can easily arrive at

$$\mathfrak{L}V_W+2\alpha V_W=h^2|\dot{\hat{\chi}}(t)|^2_W-\frac{\pi^2}{4}e^{-2\alpha h}|\delta(t)|^2_W.$$

From (4) and (10), it follows that

$$\begin{aligned}\dot{\hat{\chi}}_{j,0}(t)&=\dot\chi_{j,0}(t)\\ \dot{\hat{\chi}}_{j,i}(t)&=\dot{\hat{x}}_{1,i}(t)-\dot{\hat{x}}_{j,i}(t)=\int_{t-ih}^{t}\psi_i(t-s)\dot\chi_{j,i}(s)ds,\\ &\quad i=1,\ldots,n-1.\end{aligned}$$

which yields

$$\begin{aligned}|\dot{\hat{\chi}}(t)|^2_W=&|H_1\chi(t)|^2_{\mathcal{W}_0}+|(\rho_2(t)+\rho_3(t))|^2_{\mathcal{W}_{n-1}}\\ &+\sum_{i=1}^{n-2}|\int_{t-ih}^{t}\psi_i(t-s)H_{i+1}\chi(s)ds|^2_{\mathcal{W}_i},\end{aligned}$$

where

$$\begin{aligned}\rho_2(t)=&\int_{t-(n-1)h}^{t}\psi_{n-1}(t-s)H_{n-1}f(s)ds,\\ \rho_3(t)=&\int_{t-(n-1)h}^{t}\psi_{n-1}(t-s)H_{n-1}g(s)dw(s).\end{aligned}$$

Thus

$$\begin{aligned}&\mathfrak{L}V_W+2\alpha V_W=h^2[|H_1\chi(t)|^2_{\mathcal{W}_0}+|(\rho_2(t)+\rho_3(t))|^2_{\mathcal{W}_{n-1}}\\ &+\sum_{i=1}^{n-2}|\int_{t-ih}^{t}\psi_i(t-s)H_{i+1}\chi(s)ds|^2_{\mathcal{W}_i}]-\frac{\pi^2}{4}e^{-2\alpha h}|\delta(t)|^2_W.\end{aligned}\tag{25}$$

To compensate the term $\rho_2(t)$, we consider

$$V_Q=\int_{t-(n-1)h}^{t}e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)|H_{n-1}f(s)|^2_Q ds\tag{26}$$

with $Q>0$. Using Jensen's inequality and Proposition 1 yields

$$\mathfrak{L}V_Q+2\alpha V_Q\le|H_{n-1}f(t)|^2_Q-e^{-2\alpha(n-1)h}|\rho_2(t)|^2_Q.\tag{27}$$

Similarly, for the term $\rho_3(t)$, we consider

$$V_{F_2}=\int_{t-(n-1)h}^{t}e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)|H_{n-1}g(s)|^2_{F_2}ds\tag{28}$$

with $F_2>0$. Via

$$\begin{aligned}&\mathbf{E}he^{-2\alpha(n-1)h}|\rho_3(t)|^2_{F_2}\\ &=\mathbf{E}e^{-2\alpha(n-1)h}h\int_{t-(n-1)h}^{t}\psi^2_{n-1}(t-s)|H_{n-1}g(s)|^2_{F_2}ds\\ &\le\mathbf{E}\int_{t-(n-1)h}^{t}e^{-2\alpha(t-s)}\psi_{n-1}(t-s)|H_{n-1}g(s)|^2_{F_2}ds,\end{aligned}$$

we have

$$\begin{aligned}&\mathbf{E}\mathfrak{L}V_{F_2}+\mathbf{E}2\alpha V_{F_2}\\ &\le\mathbf{E}|H_{n-1}g(t)|^2_{F_2}-\mathbf{E}he^{-2\alpha(n-1)h}|\rho_3(t)|^2_{F_2}.\end{aligned}\tag{29}$$

Finally, to cancel the third positive term on the right-hand side of (25), we consider

$$\begin{aligned}&V_{\mathcal{W}_i}=\int_{t-ih}^{t}e^{-2\alpha(t-s)}\varphi_i(t-s)|H_{i+1}\chi(s)|^2_{\mathcal{W}_i}ds,\\ &i=1,\ldots,n-2.\end{aligned}\tag{30}$$

Via Jensen's inequality and Proposition 1, we have

$$\begin{aligned}&\mathfrak{L}V_{\mathcal{W}_i}+2\alpha V_{\mathcal{W}_i}\le|H_{i+1}\chi(t)|^2_{\mathcal{W}_i}\\ &\quad -e^{-2\alpha ih}|\int_{t-ih}^{t}\psi_i(t-s)H_{i+1}\chi(s)ds|^2_{\mathcal{W}_i}.\end{aligned}\tag{31}$$

We now consider the functional

$$\begin{aligned}V=&V_P+\sum_{i=1}^{n-1}\frac{ih}{2}V_{R_i}+V_W+\sum_{i=1}^{n-2}h^2e^{2\alpha ih}V_{\mathcal{W}_i}\\ &+\frac{(nh-h)^2}{4}V_Q+V_{F_1}+\frac{(n-1)h}{2}V_{F_2}.\end{aligned}\tag{32}$$

Then in view of (17), (19), (21), (23), (25), (27), (29) and (31), we have

$$\begin{aligned}&\mathbf{E}\mathfrak{L}V+\mathbf{E}2\alpha V\le\mathbf{E}\xi^T(t)\bar\Phi\xi(t)+\mathbf{E}h^2\zeta^T(t)\Omega\zeta(t)\\ &\quad+\mathbf{E}\frac{(nh-h)^2}{4}|H_{n-1}f(t)|^2_{R_{n-1}+Q},\end{aligned}\tag{33}$$

where

$$\begin{aligned}\xi(t)&=\operatorname{col}\{\chi(t),\bar\xi(t),\rho_1(t),\delta(t)\},\\ \bar\xi(t)&=\operatorname{col}\{\kappa_1(t),\ldots,\kappa_{n-1}(t)\},\\ \zeta(t)&=\operatorname{col}\{\rho_2(t),\rho_3(t)\},\end{aligned}$$

and $\bar\Phi$ is obtained from Φ by taking away the last block-column and block-row. By substituting $f(t)$ given by (14) and applying Schur's complement, it follows from $\Phi<0$ and $\Omega<0$ that $\mathbf{E}\mathfrak{L}V+\mathbf{E}2\alpha V\le 0$ implying the exponential mean-square stability of the system (13) with a decay rate $\alpha>0$. Then following arguments of Proposition 2, consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) is thus exponentially mean-square achieved with a decay rate $\alpha>0$.

(ii) If the system (11) with $C=0$ is exponentially mean-square stable with a decay rate $\bar\alpha>0$, then for any

$\alpha \in (0, \bar{\alpha})$ there exists matrix $P > 0$ of appropriate dimension such that $PD + D^T P + 2\alpha P < 0$. Thus

$$PD + D^T P + 2\alpha P + |\mathcal{C}|_P^2 < 0 \tag{34}$$

for small enough $|C|$. We choose R_i $(i = 1, \dots, n)$, Q, F_1, F_2, as $\frac{1}{\sqrt{h}} I_{(N-1)k}$ and W_i $(0 = 1, \dots, n)$ as $\frac{1}{\sqrt{h}} I_k$. By using Schur's complement, $\bar{\Phi} < 0$ given by (33) is equivalent to

$$PD + D^T P + 2\alpha P + |\mathcal{C}|_P^2 + \sqrt{h}(G_1 + hG_2) < 0 \tag{35}$$

where

$$G_1 = \frac{n-1}{2}|H_{n-1}\mathcal{C}|^2 + \frac{4}{\pi^2}e^{2\alpha h}|P(\mathcal{L} \otimes B\bar{K})|$$
$$+ \sum_{i=1}^{n-2} e^{2\alpha i h}|P(\mathcal{L} \otimes B\bar{K}_i)| + 2e^{2\alpha(n-1)h}|P(\mathcal{L} \otimes B\bar{K}_{n-1})|,$$
$$G_2 = \sum_{i=1}^{n-2} \frac{i^2}{4}|H_{i+1}|^2 + \sum_{i=0}^{n-2} e^{2\alpha i h}|H_{i+1}|^2.$$

It is clear that (34) implies (35) for small enough $h > 0$ since $\sqrt{h}(G_1 + hG_2) \to 0$ for $h \to 0$, implying the feasibility of $\bar{\Phi} < 0$ for small enough $h > 0$. Finally, applying Schur's complement to the last block-column and block-row of Φ given by (15), we find that $\Phi < 0$ for small enough $h > 0$ if $\bar{\Phi} < 0$ is feasible. Therefore, the LMI $\Phi < 0$ is always feasible for small enough $h > 0$ and $|C|$.

From (33), it follows that for small enough $h > 0$, $\mathbf{E}\mathfrak{L}V + \mathbf{E}2\alpha V \leq 0$ always holds provided (34) holds. This implies that for small enough $h > 0$ and $|C|$, consensus of multi-agent system (2) under sampled-data controller (7) with controller gains (6) is exponentially mean-square achieved with a decay rate $\alpha > 0$.

□

Remark 2. As in Selivanov and Fridman (2018), we consider the scenario that multi-agent system (2) is under continuous-time control (i.e. delay-dependent controller (5)). Based on Theorem 1, new LMIs are obtained by setting $W_i = 0$ $(i = 0, \dots, n-1)$ and $Q = F_2 = 0$. Thus, consensus of multi-agent system (2) under delay-dependent controller (5) with controller gains (6) is exponentially mean-square achieved with a decay rate $\alpha > 0$.

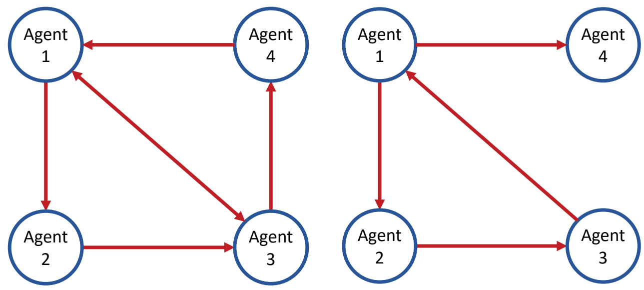
4. NUMERICAL EXAMPLE

We consider each agents described by (1) with

$$a_j = 0, \quad b = 1, \quad c_j = \sigma \in \mathbb{R}, \quad j = 0, \dots, n-1. \tag{36}$$

Two communication topologies are given as directed graphs with a spanning tree in Fig. 1. Without loss of generality, all the weights are assumed to be 1.

Case I: $n = 2$, $\sigma = 0$. First, we choose $K_0 = 1$, $K_1 = -0.5$, and consider the communication topology Fig. 1(a). Via the frequency-domain approach in Yu et al. (2013), the maximum value of h is obtained as 1.8137. From (6), one obtains $\bar{K}_0 = 0.5$, $\bar{K}_1 = 0.9069$. LMIs in Remark 2 with $\alpha = 0$ lead to the maximum value of h as 1.1025. Second, as in Ma et al. (2014), we choose $\bar{K}_0 = 2.5$, $\bar{K}_1 = 2$. Under the communication topology Fig. 1(b), the direct discretization approach in Ma et al. (2014) leads to the maximum value of h as 0.2, whereas LMIs in Theorem 1 with $\alpha = 0$ give the maximum value of h as 0.13. It should be pointed out that the approaches in Yu et al. (2013); Ma et al. (2014) are only applicable to the second-order deterministic multi-agent systems (i.e. $n = 2$, $\sigma = 0$). Instead, our method allows to cope with high-order stochastic multi-agent systems (i.e. $n \geq 3$, $\sigma \neq 0$).



(a) (b)

Fig. 1. Directed graphs with a spanning tree.

Table 1. Maximum values of h for different σ and $\alpha = 0.1$

σ	0.02	0.1	0.2	0.5	1
Re. 2	0.145	0.141	0.135	0.085	0.012
Th. 1	0.079	0.076	0.070	0.037	0.004

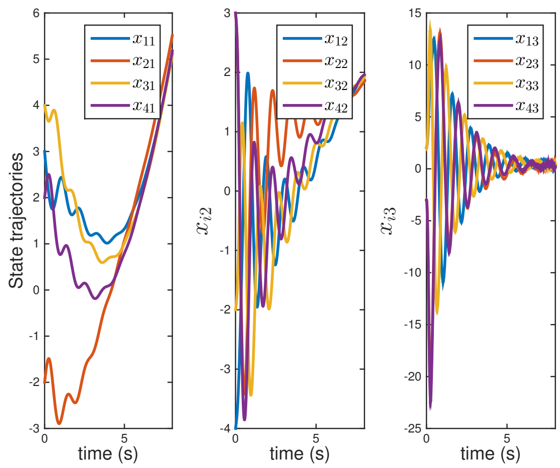


Fig. 2. State trajectories under delay-dependent controller (5) with $h = 0.145$.

Case II: $n = 3$, $\sigma \neq 0$. We choose $\bar{K}_0 = 1$, $\bar{K}_1 = 2$, $\bar{K}_2 = 3$, and consider the communication topology Fig. 1(b). For different values of σ and $\alpha = 0.1$, Table 1 presents the maximum values of h via LMIs in Remark 2 and Theorem 1. For the further simulation, we choose $\sigma = 0.02$ and the initial conditions of the four agents as $x_1(0) = [3, -4, 2]^T$, $x_2(0) = [-2, 3, -3]^T$, $x_3(0) = [4, -2, 2]^T$ and $x_4(0) = [2, 3, -3]^T$. By using the Euler-Maruyama method (Higham, 2001) with step size 0.001, Figs. 2 and 3 depict the state trajectories under delay-dependent controller (5) with $h = 0.145$ and sampled-data controller (7) with $h = 0.079$, respectively, showing that the consensus is achieved in the presence of stochastic perturbations.

5. CONCLUSION

In this paper, the digital implementation of derivative-dependent control by using delays has been investigated

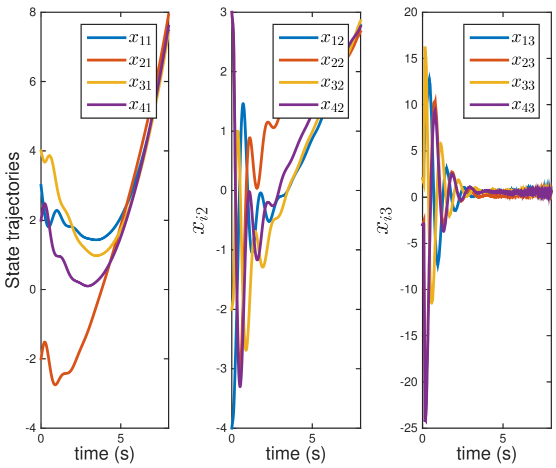


Fig. 3. State trajectories under sampled-data controller (7) with $h = 0.079$.

for consensus of stochastic multi-agent systems. Simple LMIs that allow to find admissible sampling period have been presented by using appropriate Lyapunov-Krasovskii functionals. The efficiency of the presented approach has been illustrated by numerical example. The presented method can be further applied to digital implementation of PID control for the second-order stochastic multi-agent systems. This may be a topic for future research.

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