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Robust stability of a perturbed nonlinear wave equation ⋆

Sergey Dashkovskiy * **Vitalii Slyn'ko** *

* *Institute of mathematics, University of Würzburg, Germany (e-mail: {sergey.dashkovskiy, vitalii.slynko}@mathematik.uni-wuerzburg.de).*

Abstract: In this work we consider a nonlinear wave equation subject to both distributed as well as boundary perturbations and derive several ISS-like estimates for solutions for such equations by means of Lyaponov and Faedo-Galerkin methods. Depending on the regularity of the boundary input signals different types of estimates are derived.

Keywords: Stability, robustness, hyperbolic PDEs, nonlinear systems, Lyapunov method, input-to-state stability.

1. INTRODUCTION

The notion of input-to-state stability (ISS) was introduced by Sontag (1989) for systems given by ordinary differential equations as an extension of the asymptotic stability property to systems with external inputs. The theory of ISS is well established for finite dimensional systems and is known to be fruitful in many applications, see, e.g., Sontag (2008), Karafyllis and Jiang (2011). During last few years the ISS framework was rapidly developed to the case of infinite dimensional systems and in particular to the case of systems given by partial differential equations, see Dashkovskiy and Mironchenko (2013a), Dashkovskiy and Mironchenko (2013b), Mironchenko and Wirth (2018), Mironchenko (2019), Karafyllis and Krstic (2019).

The ISS-like estimates are useful in order to quantify the influence of disturbances, as well they are useful in studying interconnected systems applying small gain theorems. Many works are devoted to derivations of explicit estimations of the ISS type. For example, Karafyllis et al. (2019) consider a linear wave equation with Kelvin-Voigt and viscous damping and disturbances acting at the boundary. Estimation of the ISS-type are derived in this paper under the assumption of rather smooth disturbances, namely of the class C^2.

In this work we investigate under which conditions a class of nonlinear second order hyperbolic equations possess an ISS-like property with respect to inputs entering not only in domain but also at the boundary. As well we derive explicit ISS estimates for the solutions. Theses derivations are based on an ISS-Lyapunov function and suitable choice of the function spaces and norms. In contrary to Karafyllis et al. (2019) the Kelvin-Voigt damping is assumed to be zero in our work, however instead we have a nonlinear term in our equation, moreover the disturbances are assumed to be of class C^1 only.

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A special attention is paid to the measure of deviation of the initial "state" from the equilibrium. It turns out that taking a norm of the initial function and of its time derivative is not sufficient in order to say how far this state is away from the equilibrium. Additionally we need to consider the norm of its spatial derivative, which in particular describes the elastic energy stored in the system initially.

Also we would like to emphasize that the regularity of input signals at the boundary is decisive for the ISS estimates. For a weaker restriction on the disturbance smoothness a partial ISS property is derived. In this case the combination of the Lyapunov and the Faedo-Galerkin methods, see Lions (1969), are applied. In case of a higher regularity (boundedness of the second time derivatives) we obtain a uniform estimate for the solution and an L_2-estimate for its time derivative by means of the Lyapunov method only.

2. NOTATION

For $(a, b) \subset \mathbb{R}$ by $L^2(a, b)$ we denote the Lebesgue space of measurable square integrable functions with scalar product $(f, g) := \int_a^b f(s)g(s)\,ds$ and norm $\|f\|_{L^2(0,1)} = (\int\limits_0^1 |f(s)|^2\,ds)^{1/2}$.

For a Banach space $\mathcal{X}$ we denote by $C(\mathbb{R}, \mathcal{X})$ the space of continuous functions and by $\mathbb{R} \to \mathcal{X}$, $C^1(\mathbb{R}, \mathcal{X})$ of continuously differentialble functions with values in $\mathcal{X}$. $L^\infty(\mathbb{R}_+)$ denotes the space of measurable and essentially bounded functions equipped with the norm $\|f\|_\infty = ess\sup_{t\geq 0} |f(t)|$.

Let $H^k(0, 1)$ be the space of functions $g \in L^2(0, 1)$ such that their generalized derivatives up to the order k belong to $L^2(0, 1)$. By $C^k(0, 1)$ we denote the space of functions having continuous derivatives up to the order k on $[0, 1]$.

$C_0^1[0, 1]$ is the subspace of $C^1[0, 1]$ of function with a compact support in $(0, 1)$ and $H_0^1(0, 1)$ is the closure of $C_0^1(0,1)$ with respect to the $H_0^1(0,1)$-norm given by $\|f\|_{H_0^1(0,1)} = \sqrt{\|f\|^2_{L^2(0,1)} + \|f'\|^2_{L^2(0,1)}}$.

$\mathcal{K}$ is the class of continuous, strictly increasing functions on $\mathbb{R}_+ = [0,\infty)$ vanishing at the origin. $\mathcal{L}$ denotes the set of continuous, strictly decreasing functions, vanishing at infinity. $\mathcal{KL} = \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ :$ continuous with $\beta(\cdot,t) \in \mathcal{K},\ \beta(t,\cdot) \in \mathcal{L}\}$. For $x \in \mathbb{R}$ we denote by $[x]$ the integer part of x, that is $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$.

3. PROBLEM STATEMENT

Consider the linear positive definite operator $\mathcal{A} = -\frac{d^2}{dz^2}$, $\mathcal{D}(\mathcal{A}) = \{f \in C^2(0,1) : f(0) = f(1) = 0\}$, and its energetic space $H_0^1(0,1)$ equipped with the scalar product

$$[u,v]_{H_0^1(0,1)} = \int_0^1 (u'(z)v'(z) + u(z)v(z))\, dz.$$

For any $T > 0$ we define the space of test functions as the following set, cf. Mihlin (1977),

$$K_T = \{\eta : \eta \in C(\mathbb{R}_+; H(0,1)) \cap C^1(\mathbb{R}_+, L^2(0,1)),\ \eta(T) = 0\}.$$

For $\alpha \in (0,\infty)$ we consider the following nonlinear differential equation

$$\begin{gathered} u_{tt}(z,t) + 2\alpha u_t(z,t) - u_{zz}(t,z) = f(u(z,t)) + D(z,t) \\ (z,t) \in (0,1) \times (0,+\infty) \end{gathered} \tag{1}$$

with initial conditions

$$\begin{gathered} u(z,0) = \varphi_0(z), \quad u_t(z,0) = \varphi_1(z), \\ \varphi_0 \in H_0^1(0,1), \varphi_1 \in L^2(0,1), \end{gathered} \tag{2}$$

and with boundary conditions

$$u(0,t) = d_0(t), \quad u(1,t) = d_1(t),\ d_i \in C^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+). \tag{3}$$

conditions. The nonlinear function $f \in C(\mathbb{R};\mathbb{R})$ is assumed to satisfy $f(0) = 0,\ sf(s) < 0,\ s \neq 0$, and to be globally Lipschitz, that is for some $L > 0$ it holds that

$$|f(x) - f(y)| \leq L|x-y|, \quad x,y \in \mathbb{R}.$$

$D \in C([0,\infty); L^2(0,1))$ denotes the distributed perturbation and d_i with $d_i(0) = 0$ denotes the perturbation acting at the boundary points. Furthermore, we assume that the first derivatives of boundary disturbances are essentially bounded $\dot{d}_0, \dot{d}_1 \in L^\infty(\mathbb{R}_+)$ and that the distributed disturbance $D(z,t)$ satisfies $\|D(\cdot,t)\|_{L^2(0,1)} \in L^\infty(\mathbb{R}_+)$.

To define the weak solutions of the problem (1)-(3) we denote

$$U(z,t) := u(z,t) - (zd_1(t) + (1-z)d_0(t)).$$

Definition 1. *The function $u(\cdot,t) \in L^2(0,1)$ is called a weak solution to (1)—(3), if the function $U(\cdot,t)$ satisfies the following conditions*

1) $U(\cdot,t) \in C([0,\infty); H_0^1(0,1)) \cap C^1([0,\infty); L^2(0,1))$

2) $\lim_{t\to 0} \|u(\cdot,t) - \varphi_0\|_{H_0^1(0,1)} = 0$.

3) for any test function $\eta \in K_T$ the following equality holds

$$-\int_0^T (\dot{U}(\cdot,t), \dot{\eta}(t))dt + \int_0^T [U(\cdot,t), \eta(t)]_{H_0^1(0,1)}dt - (\varphi_1, \eta(0))$$
$$= \int_0^T (f(U(\cdot,t)) + H(\cdot,t), \eta(t))\, dt$$

where

$$\begin{gathered} H(z,t) := \\ -(z\ddot{d}_1(t) + (1-z)\ddot{d}_0(t)) - 2\alpha(z\dot{d}_1(t) + (1-z)\dot{d}_0(t)) \\ +f(U(z,t) + zd_1(t) + (1-z)d_0(t)) - f(U(z,t)) + D(z,t). \end{gathered}$$

The question of existence and uniqueness of weak solutions to the problem (1)-(3) was considered in Evans (2010), Lions (1969).

In this paper we are going to investigate the ISS-like properties of (1)-(3).

Definition 2. *The system defined by (1) and boundary conditions (3) with input $\mathbf{D} := (d_0, d_1, D)$ is called partially ISS, if there exist functions $\beta \in \mathcal{KL},\ \gamma \in \mathcal{K}$ such that for any initial condition (2) and any disturbance d_0, d_1, D from the spaces specified above the corresponding solution satisfies*

$$\|u(\cdot,t)\|_{\mathcal{X}} \leq \beta(\rho_0, t) + \gamma(\|\mathbf{D}\|_{\mathcal{D}}), \quad t \geq 0.$$

where $\rho_0 \in C(H_0^1(0,1) \times L^2(0,1); \mathbb{R}_+)$ is the norm of the initial state, $\rho_0(0,0) = 0$ and the linear normed spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ are given by $\mathcal{X} \subset C([0,\infty); H_0^1(0,1)) \cap C^1([0,\infty); L^2(0,1))$, $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ and $\mathcal{D} \subset L^\infty(\mathbb{R}_+) \times L^\infty(\mathbb{R}_+) \times L^\infty(\mathbb{R}_+; L^2(0,1))$, $\mathbf{D} \in \mathcal{D}$. The latter one is called the input space.

Definition 3. *The systems (1), (3) is called ISS if there are functions $\beta_i \in \mathcal{KL},\ \gamma_i \in \mathcal{K}$ such that for any initial condition and any disturbance $\mathbf{D}$ the following estimates hold*

$$\begin{gathered} \|u(\cdot,t)\|_{\mathcal{X}_1} \leq \beta_1(\rho_0,t) + \gamma_1(\|\mathbf{D}\|_{\mathcal{D}}), \quad t \geq 0 \\ \|u_t(\cdot,t)\|_{\mathcal{X}_2} \leq \beta_2(\rho_0,t) + \gamma_2(\|\mathbf{D}\|_{\mathcal{D}}), \quad t \geq 0. \end{gathered}$$

Here $(\mathcal{X}_i, \|\cdot\|_{\mathcal{X}_i})$ are linear normed spaces given by $\mathcal{X}_1 \subset C([0,\infty); H_0^1(0,1)) \cap C^1([0,\infty); L^2(0,1))$, $\mathcal{X}_2 \subset C^1([0,\infty); L^2(0,1))$.

Let us recall that stability of the unperturbed problem (1)-(3), i.e., in the case of $d_0 = d_1 = D = 0$ was studied in Caughey and Ellison (1975). Here we are interested in an extensions of these results to the case of non-vanishing perturbations and the corresponding estimates as in the last two definitions.

4. MAIN RESULTS

Here we are going to verify whether the properties defined above are satisfied by the initial value problem (1)-(3) under different assumptions imposed on the boundary disturbances d_i. Namely the following two cases will be concidered:

(1) $d_i \in L^\infty(\mathbb{R}_+),\ \dot{d}_i \in L^\infty(\mathbb{R}_+),\ i = 0,1$;

(2) $d_i \in L^\infty(\mathbb{R}_+),\ \dot{d}_i \in L^\infty(\mathbb{R}_+),\ \ddot{d}_i \in L^\infty(\mathbb{R}_+),\ i = 0,1$.

In the first case we will check the partial ISS property with respect to $\mathcal{X} = L^2(0,1)$, $\mathcal{D} = \{\mathbf{D} = (d_0, d_1, D)\}$, where $d_i \in L^\infty(\mathbb{R}_+) \cap C^2(\mathbb{R}_+)$, $\dot{d}_i \in L^\infty(\mathbb{R}_+)$ for any $t \geq 0$

$D(\cdot,t) \in L^2(0,1)$, $\|D(\cdot,t)\|_{L^2(0,1)} \in L^\infty(\mathbb{R}_+)$. The norm in $\mathcal{D}$ is defined by $\|\mathbf{D}\|_{\mathcal{D}} = \max_{i=0,1}(\|d_i\|_\infty, \|\dot{d}_i\|_\infty, D_\infty)$, where $D_\infty = \|\|D(\cdot,t)\|_{L^2(0,1)}\|_\infty$.

In the second case we will check the ISS property with respect to $\mathcal{X}_1 = \mathcal{X}_2 = C[0,1]$, $\mathcal{D} = \{\mathbf{D} = (d_0, d_1, D)\}$, where $d_i \in L^\infty(\mathbb{R}_+) \cap C^2(\mathbb{R}_+)$, $\dot{d}_i \in L^\infty(\mathbb{R}_+)$, $\ddot{d}_i \in L^\infty(\mathbb{R}_+)$ for all $t \geq 0$ $D(\cdot,t) \in L^2(0,1)$, $\|D(\cdot,t)\|_{L^2(0,1)} \in L^\infty(\mathbb{R}_+)$.

4.1 Partial ISS

Let us write the solution to (1)-(3) as the sum

$$u(z,t) = v(z,t) + w(z,t), \tag{4}$$

where $v(z,t)$ is the solution of following equation

$$\begin{aligned} v_{tt}(z,t) + 2\alpha v_t(z,t) - v_{zz}(t,z) = f(v(z,t)), \\ (z,t) \in [0,1] \times (0,+\infty) \end{aligned} \tag{5}$$

satisfying the initial conditions

$$\begin{aligned} v(z,0) = \varphi_0(z), \quad v_t(z,0) = \varphi_1(z), \\ \varphi_0 \in H_0^1(0,1), \varphi_1 \in L^2(0,1) \end{aligned} \tag{6}$$

and boundary conditions

$$v(0,t) = 0, \quad v(1,t) = 0, \tag{7}$$

and $w(z,t)$ is the solutions of

$$\begin{aligned} w_{tt}(z,t) + 2\alpha w_t(z,t) - w_{zz}(t,z) \\ = f(v(z,t) + w(z,t)) - f(v(z,t)) + D(z,t), \end{aligned} \tag{8}$$

satisfying the initial conditions

$$w(z,0) = 0, \quad w_t(z,0) = 0, \quad z \in [0,1] \tag{9}$$

and boundary conditions

$$w(0,t) = d_0(t), \quad w(1,t) = d_1(t). \tag{10}$$

Let us first estimate the solutions to the problem (5)-(7) by means of the following Lyapunov function borrowed from Caughey and Ellison (1975)

$$\begin{aligned} V(v(\cdot,t), v_t(\cdot,t)) = \frac{1}{2}\int_0^1 \Big(v_z^2(z,t) + \\ (v_t + \alpha v(z,t))^2 + \alpha^2 v^2(z,t) - 2\int_0^{v(z,t)} f(s)\,ds\Big)dz \end{aligned} \tag{11}$$

Lemma 1. *The Lyapunov function defined above satisfies the inequality*

$$V(v(\cdot,t), v_t(\cdot,t)) \leq e^{-\frac{2\alpha}{\theta}t} V(\varphi_0(z), \varphi_1(z)), \quad t \geq 0 \tag{12}$$

where $\theta := 1 + \frac{1}{2\pi^2}(2\alpha^2 + L + \sqrt{(2\alpha^2+L)^2 + 4\pi^2\alpha^2})$.

Proof. The derivative of V along solutions of (5)—(7) is

$$\frac{dV}{dt} = \int_0^1 \Big(v_z v_{zt} + (v_t + \alpha v)(v_{tt} + \alpha v_t) + \alpha^2 v v_t - f(v) v_t\Big)dz$$

$$= \int_0^1 \Big(v_z v_{zt} + (v_t + \alpha v)(v_{zz} + f(v) - \alpha v_t) + \alpha^2 v v_t - f(v) v_t\Big)\,dz$$

$$= \int_0^1 v_z v_{zt}\,dz + \int_0^1 v_t v_{zz}\,dz + \alpha\int_0^1 v v_{zz}\,dz + \int_0^1 v_t f(v)\,dz$$

$$+\alpha\int_0^1 v f(v)\,dz - \alpha\int_0^1 v_t^2\,dz - \alpha^2\int_0^1 v v_t\,dz + \alpha^2\int_0^1 v v_t\,dz$$

$$-\int_0^1 v_t f(v)\,dz = v_z(z,t)v_t(z,t)\Big|_{z=0}^{z=1}$$

$$-\int_0^1 v_{zz} v_t\,dz + \int_0^1 v_{zz} v_t\,dz + \alpha(v(z,t)v_z(z,t)\Big|_{z=0}^{z=1}$$

$$-\int_0^1 v_z^2\,dz) - \alpha\int_0^1 v_t^2\,dz + \alpha\int_0^1 v f(v)\,dz$$

$$= -\alpha\int_0^1 (v_t^2(z,t) + v_z^2(z,t))\,dz + \alpha\int_0^1 v f(v)\,dz.$$

For the function $V(v(\cdot,t), v_t(\cdot,t))$ the following estimates hold

$$\begin{aligned} V(v(\cdot,t), v_t(\cdot,t)) \geq \frac{1}{2}\int_0^1 \Big(v_z^2(z,t) + \\ (v_t(z,t) + \alpha v(z,t))^2 + \alpha^2 v^2(z,t)\Big)dz \end{aligned} \tag{13}$$

$$\begin{aligned} V(v(\cdot,t), v_t(\cdot,t)) \leq \frac{1}{2}\int_0^1 \Big(v_z^2(z,t) + \\ (v_t(z,t) + \alpha v(z,t))^2 + (\alpha^2 + L)v^2(z,t)\Big)dz \end{aligned}$$

With help of the inequality $2|a||b| \leq \eta a^2 + \eta^{-1} b^2$, $a, b \in \mathbb{R}$, $\eta > 0$ from the last inequality follows

$$\begin{aligned} V(v(\cdot,t), v_t(\cdot,t)) \leq \frac{1}{2}\int_0^1 \Big(v_z^2(z,t) + (1+\alpha\eta)v_t(z,t)^2 + \\ (2\alpha^2 + L + \alpha/\eta)v^2(z,t)\Big)dz \end{aligned}$$

Then by the Friedrichs's inequality

$$\int_0^1 v^2(z,t)\,dz \leq \frac{1}{\pi^2}\int_0^1 v_z^2(z,t)\,dz$$

we obtain

$$\begin{aligned} V(v(\cdot,t), v_t(\cdot,t)) \leq \frac{1}{2}\int_0^1 \Big((1 + \frac{1}{\pi^2}(2\alpha^2 + L + \alpha/\eta))v_z^2(z,t) \\ +(1+\alpha\eta)v_t^2(z,t)\Big)dz \end{aligned}$$

So far $\eta > 0$ can be chosen arbitrary. Let η be such that

$$\alpha\eta = \frac{1}{\pi^2}(2\alpha^2 + L + \alpha/\eta),$$

that is $\eta = \frac{2\alpha^2 + L + \sqrt{(2\alpha^2+L)^2 + 4\pi^2\alpha^2}}{2\pi^2\alpha}$. Then we get

$$V(v(\cdot,t), v_t(\cdot,t)) \leq \frac{\theta}{2}\int_0^1 \Big(v_z^2(z,t) + v_t^2(z,t)\Big)dz \tag{14}$$

Hence,

$$\frac{dV}{dt} \le -\alpha \int_0^1 (v_t^2(z,t) + v_z^2(z,t))\,dz \le -\frac{2\alpha}{\theta} V(v(\cdot,t)).$$

which implies

$$V(v(\cdot,t), v_t(\cdot,t)) \le e^{-\frac{2\alpha}{\theta}t} V(\varphi_0(z), \varphi_1(z)), \quad t \ge 0 \qquad (15)$$

This proves the lemma. □

Next, we are going to derive estimates for the solutions to the problem (8)—(10).

Lemma 2. *Let us denote*

$$\gamma = \sum_{n=1}^{[\alpha/\pi]} \frac{1}{\pi^4 n^4} + \sum_{n=[\alpha/\pi]+1}^{\infty} \frac{1}{\alpha^2 \omega_n^2},$$

$$\delta = \sum_{n=1}^{[\alpha/\pi]} \frac{(2\sqrt{2}\pi + 1)^2}{\pi^4 n^3} + \sum_{n=[\alpha/\pi]+1}^{\infty} \frac{(2\sqrt{2}(1+\alpha)+1)^2}{\alpha^2 \omega_n^2},$$

where $\omega_n = \sqrt{\pi^2 n^2 - \alpha^2}$. *Let the Lipschitz constant* L *be small enough:* $L < 1/\sqrt{\gamma}$.

Then

$$\sup_{t\in(0,\infty)} \|w(\cdot,t)\|_{L^2(0,1)} \le \frac{\sqrt{\delta}}{1 - L\sqrt{\gamma}} \|\mathbf{D}\|_{\mathcal{D}}.$$

Proof. Let $\psi_n(x)$, $n \in \mathbb{N}$ be the normalized eigenfunctions of the Sturm-Liouville operator

$$\mathcal{L} = -\frac{d^2}{dz^2}, \quad D(\mathcal{L}) = \{f \in C^2(0,1)\ \ f(0) = f(1) = 0\},$$

that is

$$\psi_n''(z) + \lambda_n \psi_n(z) = 0, \quad \psi_n(0) = \psi_n(1) = 0,$$

then $\lambda_n = \pi^2 n^2$, $\psi_n(z) = \sqrt{2}\sin \pi n z$.

We denote

$$G(z,t) := f(v(z,t) + w(z,t)) - f(v(z,t)) + D(z,t),$$

$$C_n(t) := \int_0^1 w(z,t)\psi_n(z)\,dz,$$

then

$$\ddot{C}_n(t) = \int_0^1 w_{tt}(z,t)\psi_n(z)\,dz = \int_0^1 (-2\alpha w_t(z,t) + w_{zz}(z,t) + G(z,t))\psi_n(z)\,dz$$

$$= -2\alpha \dot{C}_n(t) + \int_0^1 w_{zz}(z,t)\psi_n(z)\,dz + \int_0^1 G(z,t)\psi_n(z)\,dz.$$

Let us apply two times the integration by parts to the last but one integral:

$$\int_0^1 w_{zz}(z,t)\psi_n(z)\,dz = w_z(z,t)\psi_n(z)\Big|_{z=0}^{z=1} - \int_0^1 w_z(z,t)\psi_n'(z)\,dz = -\int_0^1 w_z(z,t)\psi_n'(z)\,dz =$$

$$-\Big(w(z,t)\psi_n'(z)\Big|_{z=0}^{z=1} - \int_0^1 w(z,t)\psi_n''(z)\,dz\Big)$$

$$= d_0(t)\psi_n'(0) - d_1(t)\psi_n'(1) - \lambda_n C_n(t).$$

This implies that $C_n(t)$ is the solution to the initial value problem for the following ordinary differential equation of the second order

$$\ddot{C}_n(t) + 2\alpha \dot{C}_n(t) + \lambda_n C_n(t) = \pi n \xi(t) + g_n(t),$$
$$C_n(0) = 0, \quad \dot{C}_n(0) = 0,$$

where

$$g_n(t) = \int_0^1 G(z,t)\psi_n(z)dz,\ \xi(t) = \sqrt{2}(d_0(t) + (-1)^{n+1} d_1(t)).$$

Hence for $n > \left[\alpha/\pi\right]$ we have

$$C_n(t) = \frac{1}{\omega_n}\int_0^t e^{-\alpha(t-s)} \sin \omega_n(t-s)(\pi n \xi(s) + g_n(s))\,ds,$$

and for $n < \left[\alpha/\pi\right]$ we have

$$C_n(t) = \frac{1}{\beta_n}\int_0^t e^{-\alpha(t-s)} \sinh \beta_n(t-s)(\pi n \xi(s) + g_n(s))\,ds,$$

where $\beta_n = \sqrt{\alpha^2 - \pi^2 n^2}$. For $n = \left[\alpha/\pi\right]$

$$C_n(t) = \int_0^t e^{-\alpha(t-s)}(t-s)(\pi n \xi(s) + g_n(s))\,ds.$$

By means of the Cauchy inequality the following estimate for $g_n(t)$ follows:

$$|g_n(t)| \le \|G(\cdot,t)\|_{L^2(0,1)}$$
$$\le \|f(v(\cdot,t) + w(\cdot,t)) - f(v(\cdot,t))\|_{L^2(0,1)} + \|D(\cdot,t)\|_{L^2(0,1)}$$
$$\le L\|w(\cdot,t)\|_{L^2(0,1)} + D_\infty.$$

Let $T > 0$, then for all $t \in (0,T]$ we have

$$\sup_{t\in(0,T]} |g_n(t)| \le L \sup_{t\in(0,T]} \|w(\cdot,t)\|_{L^2(0,1)} + D_\infty.$$

Let us estimate $C_n(t)$ for $0 < n \le \left[\alpha/\pi\right]$, $t \in [0,T]$:

$$|C_n(t)| \le \frac{1}{\beta_n}\int_0^t e^{-\alpha(t-s)} \sinh \beta_n(t-s)\,ds \times (2\sqrt{2}\pi n d_\infty + L \sup_{t\in(0,T]} \|w(\cdot,t)\|_{L^2(0,1)} + D_\infty)$$

$$\le \frac{1}{\beta_n}\int_0^t e^{-\alpha s} \sinh \beta_n s\,ds \times (2\sqrt{2}\pi n d_\infty + L \sup_{t\in(0,T]} \|w(\cdot,t)\|_{L^2(0,1)} + D_\infty)$$

$$\le \frac{1}{\beta_n}\int_0^{+\infty} e^{-\alpha s} \sinh \beta_n s\,ds \times (2\sqrt{2}\pi n d_\infty + L \sup_{t\in(0,T]} \|w(\cdot,t)\|_{L^2(0,1)} + D_\infty)$$

$$\le \frac{1}{\pi^2 n^2}(L \sup_{t\in(0,T]} \|w(\cdot,t)\|_{L^2(0,1)} + 2\sqrt{2}\pi n d_\infty + D_\infty),$$

then again by the inequality $(a+b)^2 \le (1+\epsilon)a^2 + (1+\epsilon^{-1})b^2$, $a, b \in \mathbb{R}$, $\epsilon > 0$ we get for $0 < n \le \left[\alpha/\pi\right]$, $t \in (0,T]$ that

$$C_n^2(t) \le \frac{1}{\pi^4 n^4}((1+\epsilon)L^2 \sup_{t\in(0,T]} \|w(\cdot,t)\|^2_{L^2(0,1)} +(1+\epsilon^{-1})(2\sqrt{2}\pi n d_\infty + D_\infty)^2).$$

To estimate $C_n(t)$ for $n > \left[\alpha/\pi\right]$, $t \in (0,T]$ we first apply the integration by parts to the integral

$$\frac{1}{\omega_n}\int_0^t e^{-\alpha(t-s)} \sin\omega_n(t-s)\pi n \xi(s)\,ds = \frac{\pi n}{\omega_n}\int_0^t e^{-\alpha(t-s)} \sin\omega_n(t-s)\xi(s)\,ds = \frac{\pi n}{\omega_n}\int_0^t e^{-\alpha s} \sin\omega_n s \xi(t-s)\,ds$$
$$= -\frac{e^{-\alpha t}(\alpha \sin\omega_n t + \omega_n\cos\omega_n t)}{\omega_n \pi n}\xi(0) + \frac{1}{\pi n}\xi(t) + \frac{1}{\omega_n \pi n}\int_0^t e^{-\alpha s}(\alpha\sin\omega_n s + \omega_n \cos\omega_n s)\dot{\xi}(t-s)\,ds,$$

Taking into account that $\xi(0) = 0$ we get

$$\Big|\frac{1}{\omega_n}\int_0^t e^{-\alpha(t-s)} \sin\omega_n(t-s)\pi n \xi(s)\,ds\Big| \le \frac{2\sqrt{2}d_\infty}{\pi n} + \frac{2\sqrt{2}\widetilde{d}_\infty}{\alpha\omega_n} \le \frac{2\sqrt{2}(\alpha d_\infty + \widetilde{d}_\infty)}{\alpha \omega_n},$$

where

$$d_\infty = \max(\|d_0\|_\infty, \|d_1\|_\infty), \quad \widetilde{d}_\infty = \max(\|\dot{d}_0\|_\infty, \|\dot{d}_1\|_\infty).$$

Hence the following estimate holds

$$|C_n(t)| \le \frac{2\sqrt{2}(d_\infty + \alpha^{-1}\widetilde{d}_\infty)}{\omega_n} + \frac{1}{\alpha\omega_n}(L\|w(\cdot,t)\|_{L^2(0,1)} + D_\infty) = \frac{L}{\alpha\omega_n}\sup_{t\in(0,T]}\|w(\cdot,t)\|_{L^2(0,1)} + \frac{2\sqrt{2}(d_\infty + \alpha^{-1}\widetilde{d}_\infty) + D_\infty/\alpha}{\omega_n},$$

That is for $n > \left[\alpha/\pi\right]$, $t \in (0,T]$ we get

$$C_n^2(t) \le (1+\epsilon)\frac{L^2}{\alpha^2\omega_n^2}\sup_{t\in(0,T]}\|w(\cdot,t)\|^2_{L^2(0,1)} +(1+\epsilon^{-1})\frac{(2\sqrt{2}(d_\infty + \alpha^{-1}\widetilde{d}_\infty) + D_\infty/\alpha)^2}{\omega_n^2}$$

From the Parseval's identity we get for $t \in (0,T]$

$$\|w(\cdot,t)\|^2_{L^2(0,1)} = \int_0^1 w^2(z,t)\,dz = \sum_{n=1}^\infty C_n^2(t) \le (1+\epsilon)\gamma L^2 \sup_{t\in(0,T]}\|w(\cdot,t)\|^2_{L^2(0,1)} + (1+\epsilon^{-1})\delta\|\mathbf{D}\|^2_{\mathcal{D}}.$$

Let $\gamma L^2 < 1$, $0 < \epsilon < \frac{1-\gamma L^2}{\gamma L^2} = \epsilon^*$, then

$$\sup_{t\in(0,T]}\|w(\cdot,t)\|^2_{L^2(0,1)} \le \frac{(1+\epsilon^{-1})\delta\|\mathbf{D}\|^2_{\mathcal{D}}}{1-(1+\epsilon)\gamma L^2}.$$

Hence

$$\sup_{t\in(0,T]}\|w(\cdot,t)\|^2_{L^2(0,1)} \le \inf_{\epsilon\in(0,\epsilon^*)}\frac{(1+\epsilon^{-1})\delta\|\mathbf{D}\|^2_{\mathcal{D}}}{1-(1+\epsilon)\gamma L^2} = \frac{\delta\|\mathbf{D}\|^2_{\mathcal{D}}}{(1-L\sqrt{\gamma})^2}.$$

The right hand side in the lst inequality does not depend on T, hence we get

$$\sup_{t\in(0,\infty)}\|w(\cdot,t)\|_{L^2(0,1)} \le \frac{\sqrt{\delta}}{1-L\sqrt{\gamma}}\|\mathbf{D}\|_{\mathcal{D}}.$$

The lemma is proved. □

Theorem 1. *Let the Lipschitz constant for f be such that $L < 1/\sqrt{\gamma}$. Then the problem (1)—(3) satisfies the partial ISS property with respect to the L^2-norm and the following estimate holds true*

$$\|u(\cdot,t)\|_{L^2(0,1)} \le \sqrt{\frac{2}{\pi^2+\alpha^2}}e^{-\frac{\alpha t}{\theta}}\varrho_0 + \frac{\sqrt{\delta}}{1-L\sqrt{\gamma}}\|\mathbf{D}\|_{\mathcal{D}}.$$

where $\varrho_0 = \sqrt{V(\varphi_0,\varphi_1)}$.

The proof is based on the estimates from Lemma1 and Lemma 2 as well as on the Friedrich's inequality, however it is omitted due to space reasons.

4.2 The ISS property

To derive the uniform estimates for the solutions of (1)-(3) we need to require some more regularity from the disturbances $d_i(t)$, $i = 0,1$. Let us assume that $\widehat{d}_\infty = \max(\|\ddot{d}_0\|_\infty, \|\ddot{d}_1\|_\infty) < +\infty$. Let us apply the following substitution

$$U(z,t) = u(z,t) - (zd_1(t) + (1-z)d_0(t)).$$

in (1)-(3), then we get

$$U_{tt} + 2\alpha U_t - U_{zz} = f(U(z,t)) + H(z,t), \quad (z,t) \in [0,1]\times(0,+\infty) \tag{16}$$

with initial conditions

$$U(z,0) = \varphi_0(z), \quad U_t(z,0) = \varphi_1(z), \quad \varphi_0 \in H_0^1(0,1), \varphi_1 \in L^2[0,1], \tag{17}$$

and boundary conditions

$$U(0,t) = 0, \quad U(1,t) = 0. \tag{18}$$

We also assume that the following consistency conditions are satisfied: $d_0(0) = d_1(0) = 0$.

Theorem 2. *The problem (1)–(3) satisfies the ISS property, satisfying the following estimates*

$$\|u(\cdot,t)\|_{C[0,1]} \le e^{-\frac{\alpha}{\theta}t}\rho_0 + \|\mathbf{D}\|_{\mathcal{D}}, \quad t\ge 0,$$
$$\|u_t(\cdot,t)\|_{L^2(0,1)} \le e^{-\frac{\alpha}{\theta}t}\rho_0 + \|\mathbf{D}\|_{\mathcal{D}}, \quad t\ge 0,$$

where $\|\mathbf{D}\|_{\mathcal{D}} = \frac{\theta}{\alpha}(\sqrt{\frac{2}{3}}(\widehat{d}_\infty + 2\alpha\widetilde{d}_\infty + Ld_\infty) + \frac{D_\infty}{\sqrt{2}}) + 2\max\{d_\infty, \widetilde{d}_\infty\}$ *and* $\widehat{d}_\infty = \max_{i=0,1}(\|\ddot{d}_i\|_\infty)$, $\widetilde{d}_\infty = \max_{i=0,1}(\|\dot{d}_i\|_\infty)$, $d_\infty = \max_{i=0,1}(\|d_i\|_\infty)$.

Proof. Estimations for solutions of (16)-(18), will be derived by means of the following Lyapunov function

$$V(U(\cdot,t),U_t(\cdot,t)) = \frac{1}{2}\int_0^1 \Big(U_z^2(z,t) + (U_t(z,t)+\alpha U(z,t))^2 + \alpha^2 U^2(z,t) - 2\int_0^{U(z,t)} f(s)\,ds\Big)dz \tag{19}$$

For this functions we have already derived that

$$V(U(\cdot,t),U_t(\cdot,t)) \ge \frac{1}{2}\int_0^1 \Big(U_z^2(z,t) + (U_t(z,t)+\alpha U(z,t))^2 + \alpha^2 U^2(z,t)\Big)dz \tag{20}$$

$$V(U(\cdot,t),U_t(\cdot,t)) \le \frac{\theta}{2}\int_0^1 \Big(U_z^2(z,t) + U_t^2(z,t)\Big)dz \tag{21}$$

For the derivative of $V(U(\cdot,t),U_t(\cdot,t))$ along solutions of (16)-(18) we have the following expression

$$\frac{dV}{dt} = -\alpha\int_0^1 (U_t^2(z,t) + U_z^2(z,t))\,dz + \alpha\int_0^1 U(z,t)f(U(z,t))\,dz + \int_0^1 (U_t(z,t)+\alpha U(z,t))H(z,t)dz. \tag{22}$$

Let us separately estimate the integral $\int_0^1 (U_t(z,t) + \alpha U(z,t))H(z,t)dz$ by means of the Cauchy inequality

$$\int_0^1 (U_t(z,t)+\alpha U(z,t))H(z,t)dz \le \|U_t(\cdot,t)+\alpha U(\cdot,t)\|_{L^2[0,1]}\|H(\cdot,t)\|_{L^2[0,1]}.$$

From (20) it follows that

$$\|U_t(\cdot,t)+\alpha U(\cdot,t)\|_{L^2(0,1)} \le \sqrt{2V(U(\cdot,t),U_t(\cdot,t))},$$

hence for $H(\cdot,t)$ we get the next estimate

$$\|H(\cdot,t)\|_{L^2[0,1]} \le \frac{2}{\sqrt{3}}\widehat{d}_\infty + \frac{4\alpha}{\sqrt{3}}\widetilde{d}_\infty + \frac{2L}{\sqrt{3}}d_\infty + D_\infty.$$

Let us denote $\Delta = \frac{\sqrt{2}}{\sqrt{3}}(\widehat{d}_\infty + 2\alpha\widetilde{d}_\infty + Ld_\infty) + D_\infty$, then from (22) we obtain the inequality

$$\frac{dV}{dt} \le -\alpha\int_0^1 (U_t^2(z,t)+U_z^2(z,t))\,dz + 2\Delta\sqrt{V(U(\cdot,t),U_t(\cdot,t))}. \tag{23}$$

Taking into account the inequality (14), we obtain the differential inequality

$$\frac{dV}{dt} \le -\frac{2\alpha}{\theta}V(U(\cdot,t),U_t(\cdot,t)) + 2\Delta\sqrt{V(U(\cdot,t),U_t(\cdot,t))}.$$

By theorem for differential inequalities follows that

$$\sqrt{V(U(\cdot,t),U_t(\cdot,t))} \le e^{-\frac{\alpha}{\theta}t}\sqrt{V(\varphi_0(z),\varphi_1(z))} + \frac{\theta}{\alpha}\Big(1-e^{-\alpha t/\theta}\Big)\Delta, \quad t \ge 0. \tag{24}$$

Hence,

$$\|U_z(\cdot,t)\|_{L^2(0,1)} \le e^{-\frac{\alpha}{\theta}t}\rho_0 + \frac{\theta}{\alpha}\Delta, \qquad \|U_t(\cdot,t)\|_{L^2(0,1)} \le e^{-\frac{\alpha}{\theta}t}\rho_0 + \frac{\theta}{\alpha}\Delta \tag{25}$$

and from the boundary conditions (18) by means of the Cauchy inequality we obtain that

$$|U(z,t)| = \Big|\int_0^z U_z(s,t)\,ds\Big| \le \|U(\cdot,t)\|_{L^2(0,1)} \le e^{-\frac{\alpha}{\theta}t}\rho_0 + \frac{\theta}{\alpha}\Delta.$$

Hence

$$\sup_{z\in[0,1]} |u(z,t)| \le e^{-\frac{\alpha}{\theta}t}\rho_0 + \frac{\theta}{\alpha}\Delta + 2d_\infty.$$

and

$$\|u_t(\cdot,t)\|_{L^2(0,1)} \le e^{-\frac{\alpha}{\theta}t}\rho_0 + \frac{\theta}{\alpha}\Delta + \frac{2}{\sqrt{3}}\widetilde{d}_\infty.$$

The theorem is proved. □

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