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Tube-based Anticipative Robust MPC for Systems with Multiplicative Uncertainty ⋆

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Abstract: Many control problems contain time-varying or parameter-varying dynamics. With model predictive control (MPC), it is possible to include known plant variations into the controller for improving control performance. Unfortunately, perfect knowledge of the plant is rarely available and the accurateness of models may change over time and operating points. Robust control approaches consider worst-case realizations with a static model which ensure constraint satisfaction and stability but may yield conservative performance. The control algorithm presented in this paper uses anticipative information about future uncertainties and varying models to improve control performance while ensuring stability and feasibility. Possible system trajectories are bounded by polytopic tubes and recursive feasibility is achieved by the use of a terminal set. The controller properties are evaluated in a numerical example and compared to a similar control algorithm.

Keywords: Predictive control, linear parameter-varying systems, time-varying systems

1. INTRODUCTION

Model predictive control (MPC) is a popular control technique for systems with state and input constraints. The control task is formulated as an online optimization problem which allows an explicit handling of system constraints. At each sample step, a sequence of control inputs over a finite prediction horizon gets calculated but only the first element will be used. For linear time-invariant systems, there exist efficient algorithms which ensure asymptotic stability and feasibility of the problem for all times, e.g. in (Rawlings and Mayne, 2015).

The main reason for using feedback control is the existence of uncertainty regarding additive disturbances and modeling errors. These uncertainties are explicitly considered in *robust* MPC, where recursive feasibility and stability can be assured despite bounded uncertainties. When uncertainty in the system dynamics is present, the number of possible state trajectories grows exponentially over the prediction horizon. There exist several approaches in robust MPC to reduce this growth in computational complexity.

Early algorithms describe the plant uncertainty in its impulse response using FIR models (Zheng and Morari, 1993) where the filter coefficients lie in a bounded range. Other approaches use linear matrix inequalities (LMIs) to find robustly stabilizing control laws. In (Kothare et al., 1996), all possible system realizations have to lie in a convex polytopic set and a state-dependent state-feedback law is computed online. In order to reduce the computational burden, a state-feedback law and a terminal set are computed offline in (Kouvaritakis et al., 2000). Online, free control moves over the prediction horizon are determined by solving a simple optimization problem. In (Rawlings and Mayne, 2015, Ch. 3.5), the multiplicative uncertainty is formulated as an additive disturbance which yields low computational complexity. Due to a worst-case considerations of the disturbance, this approach is rather conservative. A robust MPC formulation which considers all extreme state trajectories is described in (Pannocchia, 2004). Although the number of decision variable is low, the number of constraints grows rapidly with the length of the prediction horizon. An approach using multi-parametric programming is presented in (Bemporad et al., 2003) which has very low online complexity. Robust multi-stage approaches, e.g. in (Subramanian et al., 2018), yield low levels of conservatism but the number of constraints grow exponentially with the prediction horizon.

Algorithms using high-complexity polytopes as tubes, e.g. in (Evans et al., 2012), may yield less conservative results than using ellipsoidal tubes based on LMIs. In addition, the online problem can be formulated as an ordinary quadratic problem and the number of constraints grows linearly with the length of the prediction horizon. A tube-based MPC algorithm based on vertex control laws is proposed in (Brunner et al., 2013) and (Hanema et al., 2016), where a control input is computed for every vertex of the model uncertainty and state tube at each prediction step. Set-inclusion methods for constructing tubes are used in (Evans et al., 2012) and (Fleming et al., 2015) where half-spaces of tubes are scaled online based on deviations from a nominal model and the current state.

In some control applications, information about the current and/or future system realizations are available, for example weather forecasts for renewable energy systems, maps and forward-looking sensors for vehicles, and temperature predictions for chemical reactions. This information can consist of nominal predicted models and error

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bounds on the models. Incorporating this information in the MPC problem leads to an anticipative behavior which may improve control performance compared to a static nominal model and uncertainty. There exists only few literature about robust MPC with a varying nominal model which is summarized next.

When the current realization of the uncertainty can be perfectly measured, the system is said to be linear-parameter-varying (LPV). In (Suzukia and Sugie, 2006), anticipative behavior of an LPV system is achieved by bounding the rate of variation of the system dynamics. A very flexible parametrization of the uncertainty is described in (Hanema et al., 2016), where one can choose from perfect knowledge of the future to arbitrarily fast variations of the plant. In addition, the algorithm can be tailored to handle LPV and robust control problems. As a downside, no nominal prediction model can be incorporated.

Another anticipative MPC approach is presented in (Gonzalez et al., 2011), where perfect knowledge of the time-varying system dynamics is assumed but additive disturbances are allowed. Moreover, the tubes are based on online reachable set calculations which are computationally expensive.

This work extends the robust MPC algorithm presented in (Fleming et al., 2015) to use information about future system evolutions. The tube cross-section presented in this paper depends on the anticipated evolution of the plant instead of the worst-case uncertainty. In contrast to (Hanema et al., 2016), a nominal system model and a quadratic cost function are used which can be beneficial in many control applications. In addition, a reduction of constraints can be expected due to a different method of constructing tubes. Knowledge of future plant realizations can vary from perfect, full knowledge to a robust setting with arbitrarily fast variations of the plant. As the controller is robustly designed, it can recover to a robust control mode after is has operated with information about future plant realizations.

This paper has the following structure: The notations used in this paper are described in Section 2. Section 3 provides the description of the uncertain system, the time-varying uncertainty and the separation into nominal and error dynamics. In Section 4, time-varying polytopic tubes are derived which bound the error evolution over the prediction horizon. A terminal set for the tube parameters and nominal states is considered in Section 5. The MPC algorithm using time-varying tubes is presented in Section 6 and illustrated by a numerical example in Section 7.

2. NOTATION

A set which is convex, compact and contains the origin in its interior is called a C-set. The identity matrix I_n is of dimension $n \times n$, and a vector containing only ones or zeros is denoted as $\underline{1}$ or $\underline{0}$, respectively.

The scaled C-set γS is defined as $\gamma S = \{x \,|Vx \leq \gamma \underline{1}\}$ with $S = \{x \,|Vx \leq \underline{1}\}$, $\gamma \in [0, \infty)$ and $V \in \mathbb{R}^{n_\mathrm{V} \times n_\mathrm{x}}$. For a λ-contractive C-set S and system $x_{k+1} = f(x_k)$ holds that if $x \in S$ then $x_{k+1} \in \lambda S$ with $\lambda \in [0, 1)$. The set is called invariant for $\lambda = 1$.

The transpose of a matrix Q is written as Q'. In addition, the notation $\|x\|_Q = x'Qx$ is used and $\|x\|_\infty$ denotes the maximum norm. The spectral radius of a square matrix X is defined by $\rho(X) = \max(\mathrm{abs}(\mathrm{eig}(X)))$ and the joint spectral radius as $\hat{\rho}(\mathcal{M})$ for a family of matrices $\mathcal{M}$.

In addition, $\mathrm{Co}\{(\cdot)\}$ denotes the convex hull formed by the corresponding vertices.

3. SYSTEM DESCRIPTION

Consider a discrete-time linear time-varying (LTV) system

$$x_{k+1} = A_k x_k + B u_k \tag{1}$$

with the system plant $A_k \in \mathrm{Co}\{A_k^{(j)} | j \in \mathbb{N}_{[1,n_\mathrm{P}]}\}$. The time-varying vertex plant realizations $A_k^{(j)}$ at each time step k are given by

$$A_k^{(j)} = \sum_{i=1}^{n_\mathrm{P}} (\theta_k^{(j)})_i A^{(i)}, \sum_{i=1}^{n_\mathrm{P}} (\theta_k^{(j)})_i = 1\,, (\theta_k^{(j)})_i \in \mathbb{R}_{[0,1]}, \tag{2}$$

where $(\cdot)_i$ denotes the ith vector row and $A^{(i)}$ is a vertex of the convex set of all possible plant realizations known offline. The time-varying scheduling parameters $\theta_k^{(j)}$ form the scheduling set $\Theta_k = \mathrm{Co}\{\theta_k^{(j)} | j \in \mathbb{N}_{[1,n_\mathrm{P}]}\}$ which describes all potential plant realizations at time step k. The set $\Theta = \mathrm{Co}\{\theta^{(i)} | i \in \mathbb{N}_{[1,n_\mathrm{P}]}\} \supseteq \Theta_k$ contains all possible scheduling parameters. In addition, $x \in \mathbb{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$ denotes the state vector and $u \in \mathbb{U} \subseteq \mathbb{R}^{n_\mathrm{u}}$ the input vector. The state and input constraints are given by the compact sets $\mathbb{X}$ and $\mathbb{U}$ which are expressed as

$$F x_k + G u_k \leq \underline{1}\,. \tag{3}$$

A linear feedback law

$$u_k = \begin{cases} K x_k + c_k, & k = 0, \dots, N-1\,, \\ K x_k, & k = N, \dots, \infty \end{cases} \tag{4}$$

is used to pre-stabilize the system and c_k denotes free control moves over the prediction horizon determined by the MPC algorithm. The linear feedback K has to be chosen such that it stabilizes asymptotically all systems defined by the vertices $A^{(i)}$. Minimizing the worst-case cost in the unconstrained case is a reasonable objective for selecting K, e.g. with the method described in (Kouvaritakis and Cannon, 2015, Ch. 3.2). Pre-stabilizing helps to reduce the effect of uncertainty over time as the system's eigenvalues usually get smaller and bounds on errors tighter. In addition, it also enables the computation of invariant sets in case of unstable plants.

The state x_k can be separated into $x_k = z_k + e_k$ with z_k as the nominal state and e_k as the error. Using (1) and (4), one can write

$$\begin{aligned} x_{k+1} &\in \mathrm{Co}\{A_k^{(j)} x_k + BK x_k + B c_k | j \in \mathbb{N}_{[1,n_\mathrm{P}]}\} \\ &= \mathrm{Co}\{(A_k^{(0)} + BK) z_k + (A_k^{(j)} + BK) e_k + \\ &\qquad \Delta_k^{(j)} z_k + B c_k | j \in \mathbb{N}_{[1,n_\mathrm{P}]}\} \end{aligned} \tag{5}$$

for $k \in \mathbb{N}_{[1,N-1]}$, where the vertex plant $A_k^{(j)}$ is divided into a nominal model $A_k^{(0)} \in \mathrm{Co}\{A_k^{(j)} | j \in \mathbb{N}_{[1,n_\mathrm{P}]}\}$ and the uncertainty $\Delta_k^{(j)} = A_k^{(j)} - A_k^{(0)}$. Separating into nominal and error part yields

$$z_{k+1} = (A_k^{(0)} + BK) z_k + B c_k\,, \tag{6a}$$

$$e_{k+1} \in \mathrm{Co}\{(A_k^{(j)} + BK) e_k + \Delta_k^{(j)} z_k | j \in \mathbb{N}_{[1,n_\mathrm{P}]}\}\,. \tag{6b}$$

with $A_k^{(j)} = A_k^{(0)} + \Delta_k^{(j)}$ and $k \in \mathbb{N}_{[1,N-1]}$. Defining $\Phi_k^{(j)} = A_k^{(j)} + BK$ gives

$$z_{k+1} = \begin{cases} \Phi_k^{(0)} z_k + B c_k\,, & k = 0, \dots, N-1\,, \\ \Phi_k^{(0)} z_k, & k = N, \dots, \infty \end{cases} \tag{7}$$

for the nominal state trajectory and

$$e_{k+1} \in \mathrm{Co}\{\Phi_k^{(j)} e_k + \Delta_k^{(j)} z_k | j \in \mathbb{N}_{[1,n_\mathrm{P}]}, k \in \mathbb{N}_{[0,\infty)}\} \quad (8)$$

for the error term.

4. POLYTOPIC TUBES

Trying to directly evaluate (8) leads to a number of constraints which grows exponentially over the prediction horizon. In order to overcome this issue, the tube-based approach described in (Fleming et al., 2015) can be used. Using a set-inclusion method based on Farkas' Lemma, one can ensure that the error is bounded by consecutive polytopic tube cross-sections.

Proposition 1. (Dorea and Hennet, 1999) Given two polytopes $S_1 = \{x \,|\, F_1 x \leq g_1\}$ and $S_2 = \{x \,|\, F_2 x \leq g_2\}$, then $S_1 \subseteq S_2$ if and only if there exists a non-negative matrix H such that $HF_1 = F_2$ and $Hg_1 \leq g_2$.

In order to increase the region of attraction, a slow growth of the tube cross-sections is desired. This can be achieved by computing an autonomous, robust contractive set for the uncertain system (1).

Definition 2. A set $\mathcal{S} \subseteq \mathbb{X}$ is called robust λ-contractive for system (1) with $u = Kx \in \mathbb{U}$ if $\forall x \in \mathcal{S} : (A^{(i)} + BK)x \in \lambda\mathcal{S}$, $\forall i \in \mathbb{N}_{[1,n_\mathrm{P}]}$.

Let $\mathcal{S}$ be parametrized as $\mathcal{S} = \{e_k | Ve_k \leq \underline{1}\}$ with $V \in \mathbb{R}^{n_\mathrm{V} \times n_\mathrm{x}}$ under the uncertain dynamics $e_{k+1} = \Phi^{(i)} e_k$, $i \in \mathbb{N}_{[1,n_\mathrm{P}]}$. The tube cross-section at each prediction step can be then parametrized as $\mathcal{T}_k = \{e_k | Ve_k \leq \alpha_k\}$ with $\alpha_k \leq \underline{1}$ as a parameter to be determined online. The matrix V determines the general structure of the cross-section while α_k scales each half-space.
Let the cross-section at time $k+1$ be defined as $\mathcal{T}_{k+1} = \{e_{k+1} | Ve_{k+1} \leq \alpha_{k+1}\}$. Using the error model (8), one can express $\mathcal{T}_{k+1}$ in terms of e_k as

$$\begin{aligned}\mathcal{T}_{k+1} &= \{e_k | V(\Phi_k^{(j)} e_k + \Delta_k^{(j)} z_k) \leq \alpha_{k+1}, \forall j \in \mathbb{N}_{[1,n_\mathrm{P}]}\} \\ &= \{e_k | V\Phi_k^{(j)} e_k \leq \alpha_{k+1} - V\Delta_k^{(j)} z_k, \forall j \in \mathbb{N}_{[1,n_\mathrm{P}]}\}.\end{aligned} \quad (9)$$

Hence, the error state in $\mathcal{T}_k$ will lie in $\mathcal{T}_{k+1}$ if $\mathcal{T}_k \subseteq \mathcal{T}_{k+1}(e_k)$ holds. Proposition 1 can be used now to establish this relationship between two consecutive cross-sections by requiring

$$H_k^{(j)} V = V\Phi_k^{(j)} \,, \quad (10a)$$

$$H_k^{(j)} \alpha_k \leq \alpha_{k+1} - V\Delta_k^{(j)} z_k \,, \quad (10b)$$

where $H_k^{(j)}$ is a non-negative matrix. The last inequality is non-linear in $H_k^{(j)}$ and α_k if they are concurrently optimized. In (Fleming et al., 2015), matrix $H_k^{(j)}$ is time-invariant and can be designed offline which makes the inequality linear but may add conservatism. As the scheduling set may change from prediction step to prediction step, online computations of $H_k^{(j)}$ may require too much computation time. Therefore, it is proposed here to compute $H^{(i)}$ offline using the extreme plant realizations $\Phi^{(i)}$ and requiring

$$H^{(i)} V = V\Phi^{(i)} \,. \quad (11)$$

Multiplying equation (11) with the scalar scheduling variable $(\theta_k^{(j)})_i$ for $i = 1, \ldots, n_\mathrm{P}$ and summing up yields

$$\sum_{i=1}^{n_\mathrm{P}} (\theta_k^{(j)})_i H^{(i)} V = H_k^{(j)} V = V\Phi_k^{(j)} \,. \quad (12)$$

When $H^{(i)}$ is designed offline, inequality (10) is not necessary any more but becomes only sufficient (Kouvaritakis and Cannon, 2015, Ch. 5.5). The introduced conservatism can be reduced by minimizing the sum of each row in $H^{(i)}$ with the linear program

$$((H^{(i)})_n)' = \underset{h \in \mathbb{R}^{n_\mathrm{V}}}{\arg\min}\, h \quad \text{s.t. } h'V = (V)_n \Phi^{(i)}, h \geq 0 \,, \quad (13)$$

where $(\cdot)_n$ denotes the nth matrix row.
The system constraints (3) can be also separated into a nominal part and an error part

$$(F + GK)z_k + (F + GK)e_k + Gc_k \leq \underline{1} \,, \quad (14)$$

where control law (4) has been used. As e_k always lies in $\mathcal{T}_k$, one can apply Proposition 1 to the constraint (14)

$$H_\mathrm{c} V = F + GK \,, \quad (15a)$$

$$H_\mathrm{c} \alpha_k \leq \underline{1} - (F + GK)z_k - Gc_k \quad (15b)$$

with $c_k = 0$ for $k \geq N$. The non-negative matrix $H_\mathrm{c} \in \mathbb{R}^{n_\mathrm{C} \times n_\mathrm{V}}$ can be then derived with

$$(H_\mathrm{c})'_n = \underset{h \in \mathbb{R}^{n_\mathrm{V}}}{\arg\min}\, h \quad \text{s.t. } h'V = (F + GK)_n, h \geq 0 \quad (16)$$

which reduces the conservatism by making the rows in H_c small.

5. COMPUTATION OF THE TERMINAL SET

As the MPC problem considers the state evolution only over a finite amount of time, it is important to ensure feasibility and stability beyond the prediction horizon. This can be achieved by requiring that the last state of the prediction horizon has to lie in a terminal set. The terminal set has to be invariant for the system state while never violating state and input constraints. Here, the system is already pre-stabilized by a linar control law which is also used inside the terminal set. As the nominal system model can be arbitrarily chosen, the uncertain dynamics of the terminal set must contain all possible nominal models. When the system has entered the terminal set $\mathcal{X}_\mathrm{f}$, the constraint

$$H_\mathrm{c} \alpha_k + (F + GK)z_k = F_\mathrm{c} \begin{bmatrix} \alpha_k \\ z_k \end{bmatrix} \leq \underline{1} \quad (17)$$

must hold for all times. In addition, inequality (10b) is fulfilled inside the terminal set for every pair of nominal plant and plant uncertainty when the dynamics

$$A_\mathrm{f}^{(n,m)} = \begin{bmatrix} H^{(m)} & V(\Phi^{(m)} - \Phi^{(n)}) \\ 0 & \Phi^{(n)} \end{bmatrix} \quad (18)$$

are used for the computation of the terminal set with $n, m \in \mathbb{N}_{[1,n\mathrm{P}]}$. This can be seen by multiplying matrix (18) with the scalar scheduling variables $(\theta^{(i)})_m$ and $(\theta^{(j)})_n$ and summing up. Every combination of nominal plant $\Phi^{(0)} = \sum_{n=1}^{n_\mathrm{P}} (\theta^{(0)})_n \Phi^{(n)}$ and uncertainty realizations $\Phi^{(j)} = \sum_{m=1}^{n_\mathrm{P}} (\theta^{(j)})_m \Phi^{(m)}, H^{(j)} = \sum_{m=1}^{n_\mathrm{P}} (\theta^{(j)})_m H^{(m)}$ are covered by the terminal set.

Lemma 3. The uncertain dynamics $[\alpha_{k+1}, z_{k+1}]' = A_\mathrm{f}[\alpha_k, z_k]'$, $A_\mathrm{f} \in Co\{A_\mathrm{f}^{(n,m)}\}$ are asymptotically stable.

Proof. See proof for Lemma 2 in (Peschke and Görges, 2019).

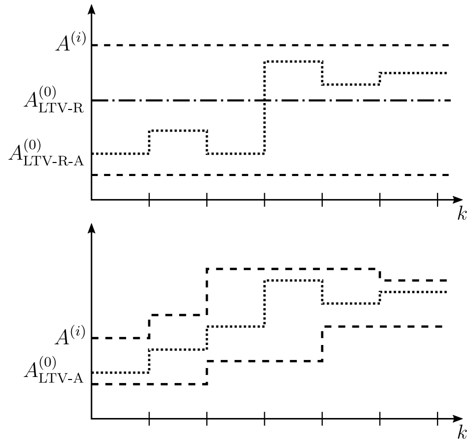

Fig. 1. Examplary Nominal Model Trajectories and Boundaries

An invariant set can be computed within a finite number of steps, e.g. with the algorithm from (Miani and Savorgnan, 2005), when the system defined by (18) is robust asymptotically stable, (Blanchini and Miani, 2015, Ch. 5.4), which is the case for (18) according to Lemma 3.

6. MPC ALGORITHM

In order to ensure stability and recursive feasibility, the nominal plant $A_k^{(0)}$ and the scheduling set Θ_k may not be chosen arbitrarily at each sampling step.

Assumption 4. At each sample time k, let a sequence of nominal plants $A_{i|k}^{(0)}, i \in \mathbb{N}_{[0,N-1]}$ be given. At the next time instant $k+1$, it holds that $A_{i|k+1}^{(0)} = A_{i+1|k}^{(0)} \, \forall i \in \mathbb{N}_{[0,N-2]}$ and $A_{N-1|k+1}^{(0)} \in \text{Co}\{A^{(j)} | j \in \mathbb{N}_{[1,n_{\text{P}}]}\}$.

Assumption 4 requires that the nominal plant is constant in terms of absolute times over the prediction horizon. At the end of the prediction horizon, a new nominal plant is added which can be chosen arbitrarily from the convex set of all allowed plant realizations.

A similar assumption also holds for the scheduling set Θ_k:

Assumption 5. At each sample step k, let a sequence of scheduling sets $\Theta_{i|k}, i \in \mathbb{N}_{[0,N-1]}$ be given. At the next time instant $k+1$, it holds that $\Theta_{i|k+1} \subseteq \Theta_{i+1|k} \, \forall i \in \mathbb{N}_{[0,N-2]}$ and $\Theta_{N-1|k+1} \in \text{Co}\{\theta^{(i)} | i \in \mathbb{N}_{[1,n_{\text{P}}]}\}$.

Assumption 5 requires that the uncertainty at a predicted time does not increase at the next sampling step. At the end of the prediction horizon, an arbitrary scheduling set can be chosen as long as it does not exceed the predefined bounds.

Both assumptions together enable a flexible parametrization of future nominal plants and error bounds, for example bounded growth of uncertainty over the prediction horizon, a varying nominal model with fixed error bounds or perfect knowledge over future plant variations.

In (Hanema et al., 2016), an interesting classification for different types of scheduling sets is described which shall be modified and extended here:

(1) LTV-R (robust): $\boldsymbol{\Theta} = \{\Theta, \Theta, \ldots, \Theta\}$,
(2) LTV-A (anticipative): $\boldsymbol{\Theta} = \{\Theta_{0|k}, \Theta_{1|k}, \ldots, \Theta_{N-1|k}\}$,
(3) LTV-O (oracle): $\boldsymbol{\Theta} = \{\theta_{0|k}, \theta_{1|k}, , \ldots, \theta_{N-1|k}, \}$.

The class 'LTV-R' describes the classical robust MPC setup, which has no knowledge about future plant realizations. In Figure 1, the class 'LTV-R' is indicated by constant matrices $A^{(i)}$ for extreme plant realizations and a constant nominal model $A^{(0)}_{\text{LTV-R}}$. Future knowledge about the nominal plant is incorporated in class 'LTV-R-A' where the nominal model $A^{(0)}_{\text{LTV-R-A}}$ is time-varying. This paper addresses the case of 'LTV-A' in particular, where the uncertainty may be anticipated and possibly tightened over the prediction horizon when additional knowledge exists. As an extreme setup, 'LTV-O' requires perfect knowledge of the plant over the prediction horizon. With the algorithm presented in this paper, all setups described above can be handled as long as Assumptions 4 and 5 are valid.

In order to improve the system performance by using a nominal model, the infinite-time cost function

$$J = \sum_{k=0}^{\infty} \|z_k\|_Q + \|u_k\|_R \tag{19}$$

can be used. Inserting control law (4) yields

$$\begin{aligned} J \leq \hat{J} = \sum_{k=0}^{N-1} &\|z_k\|_{Q+K'RK} + 2z_k K' R c_k + \\ &\|c_k\|_R + \|z_N\|_P \,, \end{aligned} \tag{20}$$

where P solves the Lyapunov equation

$$\Phi^{(i)\prime} P \Phi^{(i)} - P \preceq -(Q + K'RK) \quad \forall\, i \in \mathbb{N}_{[1,n_{\text{P}}]} \tag{21}$$

with positive definite matrices Q and R.

The MPC algorithm can be now stated as:

Offline:

(1) Choose K and P so that system (1) is robustly stabilized for all $A^{(i)}$ and equation (21) is fulfilled.
(2) Compute the λ-contractive set S, matrices $H^{(i)}$ and H_{c}, and terminal set $\mathcal{X}_{\text{f}}$.

Online

(1) Compute tube matrices $H_{i|k}^{(j)}$ using (12) and scheduling sets $\Theta_{i|k}$.
(2) Solve

$$\begin{aligned}
\min_{\underline{z},\underline{c}} \quad & \hat{J}(x_{0|k}, z_{1|k-1}) && \text{(22a)} \\
\text{s.t.} \quad & z_{i+1|k} = \Phi_{i|k}^{(0)} z_{i|k} + B c_{i|k} \,, && \text{(22b)} \\
& V(x_{0|k} - z_{0|k}) \leq \alpha_{0|k} \,, && \text{(22c)} \\
& - V(x_{0|k} - z_{1|k-1}) \leq -\alpha_{0|k} \,, && \text{(22d)} \\
& H_{i|k}^{(j)} \alpha_{i|k} + V \Delta_{i|k}^{(j)} z_{i|k} \leq \alpha_{i+1|k} \,, && \text{(22e)} \\
& H_c \alpha_{i|k} + (F + GK) z_{i|k} + G c_{i|k} \leq \underline{1} \,, && \text{(22f)} \\
& (\alpha_{N|k}, z_{N|k}) \in \mathcal{X}_{\text{f}} \,, && \text{(22g)} \\
& i \in \mathbb{N}_{[0,N-1]} \,, j \in \mathbb{N}_{[1,n_{\text{P}}]} \,.
\end{aligned}$$

with initial condition $z_{1|-1} = x_{0|0}$. The underlined variables $\underline{z}_k$ and $\underline{c}_k$ denote sequences of the respective variable over the prediction horizon.

Inequality (22d) is important to ensure stability and to exploit the nominal prediction model. If inequality (22d) did not exist, the optimal solution would set $z_{0|k}$ close to the origin, where the system is controlled by the predesigned robust control gain K to the origin. Moreover, the input vector $\underline{c}_k$ would be merely used to fulfill the input constraints and not for anticipative behavior. Therefore, the nominal state $z_{0|k}$ must lie close to the current true system state x_k so that the MPC algorithm uses $\underline{c}_k$ actively for optimizing the nominal cost. Setting $\alpha_{0|k} = \underline{0}$ may lead to the most anticipative behavior but stability cannot be ensured as the cost may increase at consecutive time steps. Hence, inequality (22d) requires $\alpha_{0|k} \leq V(x_{0|k} - z_{1|k-1})$ which ensures that the cost can decrease as predicted in the previous step. In addition, the constraint imposes tight limits between measured and nominal state as needed for good anticipative behavior. For example, in case of a perfect match between nominal model and true plant, $\alpha_{0|k} = \underline{0}$ can be chosen by the algorithm.

Theorem 6. System (1) is recursively feasible and the nominal state is asymptotically stable under algorithm (22) for all x_0 inside the N-step reachable set $\mathcal{F}_N$ for $\mathcal{X}_\mathrm{f}$. In addition, the true system state x converges to the origin.

Proof. The optimal solution is marked by * in the following.
Nominal stability: Following standard proofs in MPC, e.g. in (Rawlings and Mayne, 2015), one can express the difference in cost between two consecutive time steps

$$\begin{aligned}\Delta J &= \hat{J}(z^*_{k+1}, c^*_{k+1}) - \hat{J}(z^*_k, c^*_k)\\ &\leq \hat{J}([z^*_{1:N|k}, z_{N|k+1}], [c^*_{1:N-1|k}, 0]) - \hat{J}(z^*_k, c^*_k)\\ &\leq \|z_{N|k}\|_{\Phi'_{N|k+1} P \Phi_{N|k+1}} + \|z_{N|k}\|_{Q+K'RK} - \|z_{N|k}\|_P\\ &\leq 0\,,\end{aligned} \tag{23}$$

where P is designed as in equation (21), Q and R are positive definite, $z^*_{1:N|k}$ denotes the $N-1$ last elements from the solution at time k and $\Phi^{(0)}_{i|k+1} = \Phi^{(0)}_{i+1|k}$ due to Assumption 4. Asymptotic stability of the nominal state follows from the negative definiteness of the change in cost.
Feasiblity: For all x_k in $\mathcal{F}_N$ there exist feasible sequences $\underline{z}_k, \underline{c}_k$ and $\underline{\alpha}_k$. The input sequence $\underline{c}_{k+1}$ can be constructed from the last $N-1$ elements of $\underline{c}_k$ and appending 0 as defined in (4). Due to Assumption 4, the nominal plant stays constant at absolute times and hence, the nominal state sequence $\underline{z}_{k+1}$ can be formed of the last $N-1$ elements of $\underline{z}_k$ and the last element is $z_{N|k+1} = \Phi^{(0)}_{N-1|k+1} z_{N|k}$ which is feasible by definition of the terminal set.
Due to Assumption 4 and 5, a feasible solution for the first $N-1$ element of $\underline{\alpha}_{k+1}$ can be constructed from the $N-1$ last elements of $\underline{\alpha}_k$ as inequality (22e) holds for all realizations of H and Δ inside the time-varying scheduling sets. The last element of $\underline{\alpha}_{k+1}$ is feasible due to the definition of the terminal set as every combination of nominal plant $\Phi^{(0)}_k$, uncertainty $V\Delta^{(j)}_k$ and tube matrices $H^{(j)}_k$ can be formed by a convex combination of $A^{(i,j)}_\mathrm{f}$. Inequalities (22c) and (22d) are also feasible for the shifted sequence due to the design of $\alpha_{1|k}$ in (22e) with $\alpha_{0|k+1} = \alpha_{1|k}$ and $z_{0|k+1} = z_{1|k}$.
Convergence of system state: Due to the asymptotic stability of the nominal state z_k, it follows that $z_k \to 0$. This implies $e_k \to 0$ because of the dynamics (8) and hence, $x_k \to 0$.

The quadratic program (22) has Nn_x equality constraints, $N(n_\mathrm{V}n_\mathrm{p} + n_\mathrm{C}) + 2n_\mathrm{V} + n_\mathrm{F}$ inequality constraints and $N(n_\mathrm{x} + n_\mathrm{u} + n_\mathrm{V}) + n_\mathrm{x} + n_\mathrm{V}$ decision variables. In addition, the number of inequalities n_V depends on the desired contraction factor λ and n_F denotes the number of constraints of the terminal set. For contraction factors close to the spectral radius of the set of all $\Phi^{(j)}$, the number of inequality constraints increases rapidly. On the other hand, the tube cross-section grows slower over the prediction horizon which leads to a increased region of attraction.

7. NUMERICAL EXAMPLE

In this section, a numerical example is given to demonstrate the advantages of the algorithm presented in Section 6. An uncertain, unstable, discrete-time system is given by the vertices

$$\begin{aligned}A^{(1)} &= \begin{bmatrix} 1 & 1.5 \\ -0.4 & 1 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.85 \end{bmatrix},\\ A^{(3)} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\end{aligned} \tag{24}$$

and is pre-stabilized by $K = [-0.6237\ -1.8654]$. The states are constrained by $|x|_\infty \leq 7$ and for the input $|u| \leq 1$ holds. The tube matrices $H^{(j)}$ are chosen so that the corresponding set is 0.9-contractive and the terminal set $\mathcal{X}_\mathrm{f}$ is invariant for z_k and α_k. Weights $Q = 10I$ and $R = 1$ are used with I as the identity matrix.
The algorithm is evaluated in four different scenarios with a prediction horizon of five. In each scenario, the plant switches from one vertex realization to the next one at each time instant. Scenario one has no specific knowledge about plant variations and uses the average of all three $A^{(i)}$ as a nominal model. This corresponds to the 'LTV-R'-type described in Section 6. In scenario two, perfect knowledge about the nominal plant is assumed but the uncertainty stays the same as in scenario one. Hence, it is a 'LTV-R' system with anticipative knowledge about the nominal plant ('LTV-R-A'). The uncertainty is tightened in scenario three: Each uncertainty description contains the nominal plant as a vertex and the systems which are given by averaging the nominal plant with its adjacent $A^{(i)}$ ('LTV-A'). Scenario four has perfect knowledge about the plant and no uncertainty is present ('LTV-O').
Moreover, a robust version of the algorithm presented in (Hanema et al., 2016) is used for comparison. As described in the paper, the constraints are changed so that the solution does not depend on the current or future plant realizations. In addition, the terminal set is robustly invariant instead of control invariant and hence, the LPV assumption can be dropped. A min-max cost function is used in (Hanema et al., 2016) in contrast to a quadratic cost in this paper. The performance of both algorithms is evaluated by the total quadratic cost

$$J_\mathrm{QP} = \sum_{k=0}^{\infty} \|x_k\|_Q + \|u_k\|_R \tag{25}$$

and the total achieved maximum cost

$$J_\mathrm{max} = \sum_{k=0}^{\infty} \|Qx_k\|_\infty + \|Ru_k\|_\infty\,. \tag{26}$$

For each scenario and control algorithm, the total costs (25) and (26) are averaged over a dense grid of initial conditions. Only initial conditions which are feasible for both algorithms and all scenarios are considered in order to account for different sizes of domains of attractions

Table 1. Results from Algorithm in Section 6.

Scenario	Avg. J_{QP}	Avg. J_{MM}	Area of DOA
LTV-R	22.96	29.79	3.63
LTV-R-A	19.27	21.56	3.63
LTV-A	18.61	20.48	8.11
LTV-O	18.61	20.48	47.97

Table 2. Results from adapted algorithm in (Hanema et al., 2016).

Scenario	Avg. J_{QP}	Avg. J_{MM}	Area of DOA
LTV-R	23.48	30.76	3.60
LTV-R-A	23.48	30.76	3.60
LTV-A	20.23	21.95	9.25
LTV-O	19.85	17.57	47.98

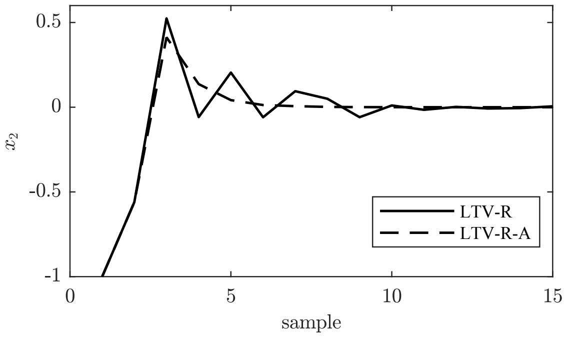

Fig. 2. Examplary State Trajectories

(DOA). In additions, the area of the DOA is also given for each configuration.

Using anticipative knowledge ('LTV-R-A') reduces the quadratic cost by 16 % compared to 'LTV-R' as noted in Table 1. The advantage of anticipative knowledge can be seen in Fig. 2 where a faster and smoother converge to the origin is achieved. When the uncertainty is tightened ('LTV-A'), the cost can be further reduced and the domain of attraction increases. The smaller uncertainty enables the control to reduce the cost as the constraints can be fulfilled more easily. Having perfect knowledge ('LTV-O') does not decrease the cost further in this example but heavily increases the domain of attraction. The costs for the 'LTV-R' and 'LTV-R-A' scenarios stay the same for the adapted algorithm from Hanema et al. (2016) as listed in Table 7. No nominal model can be used with this algorithm and the costs can be only decreased by reducing the uncertainty as in the 'LTV-A' scenario. Compared to the algorithm in this paper, the achieved costs are in general higher as no use of a good nominal model is made. The domain of attraction is greater in the 'LTV-A' scenario and is roughly the same otherwise.

The complexity for algorithm Hanema et al. (2016) depends linearly on the prediction horizon N but also on the number of vertices q_{F} of the terminal set and n_{P}. For the above example, the algorithm uses 49 decision variables, three equality constraints and 656 inequality constraints. The algorithm presented in this paper needs 41 decision variables, 10 equality constraints and 134 inequality constraints. Hence, a comparable or better performance in case of a good nominal model can be achieved by using significantly less inequality constraints.

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