The \textit{J}-Orthogonal Square-Root Fifth-Degree Cubature Kalman Filtering Method for Nonlinear Stochastic Systems

Gennady Yu. Kulikov * Maria V. Kulikova *

* CEMAT, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1, 1049-001 LISBOA, Portugal (emails: gennady.kulikov@tecnico.ulisboa.pt, maria.kulikova@ist.utl.pt).

Abstract: This paper addresses the issue of square-rooting in the \textit{Fifth-Degree Cubature Kalman Filtering (5D-CKF)} method grounded in the Itô-Taylor approximation of order 1.5 and designed by Santos-Díaz, Haykin and Hurd in 2018. That filter is rather accurate and efficient in treating nonlinear continuous-discrete stochastic systems of practical value and shown to outperform many other algorithms in a radar tracking scenario. However, the cited authors mention “the lack of a square-root implementation” of the filter under consideration as a principle shortcoming reducing its applied potential. Here, we address the reported lack and resolve it with a hyperbolic QR factorization used for devising the filter’s \textit{J}-orthogonal square-root version, which possesses an exceptional robustness to round-off and other disturbances. Our square-root implementation of the 5D-CKF technique is justified theoretically and examined and compared numerically to its non-square-root predecessor in a flight control scenario with ill-conditioned measurements.

Keywords: Continuous-discrete nonlinear stochastic model, fifth-degree cubature Kalman filter, square-root implementation, radar tracking, maneuvering target, ill-conditioned measurements.

1. INTRODUCTION

Many modern control tasks in nonlinear stochastic systems arisen in applied science and engineering demand accurate and efficient state estimation tools for continuous process models with discrete measurements, which are of the form

\[ dX(t) = F(X(t))dt + GdW(t), \quad t > 0, \quad (1) \]

\[ Z_k = h(X_k) + V_k, \quad k \geq 1. \quad (2) \]

The process model (1) is supposed to be an Itô-type \textit{Stochastic Differential Equation (SDE)}, in which the unknown stochastic process \( X(t) \) represents the state of the plant of size \( n \) at time \( t \), the known nonlinear vector-function \( F : \mathbb{R}^n \to \mathbb{R}^n \) describes its dynamic behavior, the diffusion matrix \( G \) is assumed to be time-invariant and of size \( n \times n \) in the driving noise used, and the random disturbance \( \{W(t), t > 0\} \) is a multivariate Wiener process with independent zero-mean Gaussian increments \( dW(t) \) having a covariance of the form \( Qdt \) of size \( n \times n \) where the matrix \( Q \) is positive definite and fixed in time. The initial state can also be a Gaussian variable \( X(0) \sim \mathcal{N}(X_0, \Pi_0) \) with mean \( X_0 \) and covariance \( \Pi_0 > 0 \) in SDE (1). Next, the discrete-time measurement model (2) with \( k \) being a discrete time index (i.e. \( X_k \) means \( X(t_k) \)) establishes a nonlinear in general link \( h : \mathbb{R}^n \to \mathbb{R}^m \) between the distribution of the state \( X_k \) in the dynamic process at hand and its measurement \( Z_k \) of size \( m \) corrupted by a zero-mean Gaussian variable \( \{V_k, k \geq 1\} \) with its covariance \( R_k > 0 \) at every sampling instant \( t_k \). The measurements \( Z_k \) arrive uniformly and with the sampling rate \( \delta = t_k - t_{k-1} \) in our setting. This time interval \( \delta \) is also known as the sampling period in filtering theory. Furthermore, all realizations of the noises \( dW(t), V_k \) and the initial state \( X(0) \) are taken from mutually independent Gaussian distributions. The continuous-discrete state estimation scenarios are often encountered in practical modeling and motivated in Santos-Díaz et al. (2018); Jazwinski (1970); Särkkä (2007), etc.

A conventional state estimation setting in stochastic systems of the above form (1) and (2) is to obtain an optimal estimate of the dynamic model obeying SDE (1) grounded in measurements \( \{Z_1, \ldots, Z_k\} \) realized up to each sampling instant \( t_k \). Here, we stick to the Kalman formulation and look for the optimal estimation of the random process \( X(t) \) in the mean least square sense, which is presented by the conditional mean \( \hat{X}_{t_k} \). Based on the Gaussianity assumption of the a priori and a posteriori random distributions, the solution to this state estimation task demands multidimensional Gaussian-weighted integrals of the form

\[ \int_{\mathbb{R}^n} g(X)\mathcal{N}(X; \bar{X}, P_X)dX = \int_{\mathbb{R}^n} g(\bar{X} + S_X)\mathcal{N}(X; 0_n, I_n)dX \]

(3)

to be accurately computed. In integral (3), the functions \( g(X) \equiv X \) and \( g(X) = (X - \bar{X})(X - \bar{X})^\top \) are employed in calculations of the predicted and filtering mean vectors and covariance matrices, respectively, \( 0_n := [0, \ldots, 0]^\top \in \mathbb{R}^n \), \( I_n \) stands for the identity matrix of size \( n \), \( \mathcal{N}(X; \bar{X}, P_X) \) denotes the Gaussian probability density function with its expectation \( \bar{X} \) and covariance matrix \( P_X \) and the matrix...
$S_X$ of size $n \times n$ refers to the Square Root (SR) of the covariance, which is determined by the following property:

$$P_X = S_X S_X^T,$$  \hspace{1cm} (4)

This SR $S_X$ is often accepted to be the lower triangular Cholesky factor of covariance in practical estimation tasks.

Arasaratnam and Haykin (2009) gave rise to a novel and effective state estimation technology rooted in cubature rules applied for numerical approximations of integrals (3) and termed it the Cubature Kalman Filter (CKF). Later on, Arasaratnam et al. (2010) explained that the first CKF based on the third-degree spherical-radial cubature rule is equivalent to a particular version of the Unscented Kalman Filter (UKF) presented by Julier et al. (1995, 2000); Julier and Uhlmann (2004); and Santos-Diaz et al. (2018). This justifies its sound state estimation potential in nonlinear stochastic radar tracking scenarios. However, these can suffer severely from “the lack of a square-root implementation” reported by Santos-Diaz et al. (2018).

In what follows, we address the mentioned lack and devise one SR implementation of the fifth-degree CKF presented in the latter cited paper. Our solution is rooted in the Itô-Taylor SDE approximation of order 1.5 (IT-1.5), which is employed by Santos-Diaz et al. (2018) as well. Furthermore, this SR implementation uses the concept of a J-orthogonal transformation implemented by means of a hyperbolic QR factorization because of potential negativity of some weights in the fifth-degree cubature rule applied. We remark that J-orthogonal QR decompositions are commonly utilized in the realm of $H_2$ filtering and other tasks with indefinite inner products. Here, we stick to the J-orthogonal QR factorization of Bojanczyk et al. (2003), which combines the Householder reflections and hyperbolic rotations and considered to be numerically robust and efficient. The factorization implemented gives rise to our novel J-orthogonal Square-Root Fifth-Degree Cubature Kalman Filtering (JSR-5D-CKF) technique. Its exceptional robustness to round-off and other disturbances resulting in the superiority of the JSR-5D-CKF method towards its predecessor presented by Santos-Diaz et al. (2018) is exposed within the target tracking scenario of Arasaratnam et al. (2010), in which an aircraft executes a coordinated turn. That numerical examination setup is employed with ill-conditioned measurements in this paper.

2. J-ORTHOGONAL SQUARE-ROOT 5D-CKF

For yielding our new filter, we have to square-root the time and measurement update steps in the IT-1.5-based 5D-CKF method presented by Santos-Diaz et al. (2018). We start off at the modification of its time update, below.

2.1 The Time Update Step in the JSR-5D-CKF

Following Santos-Diaz et al. (2018), our state estimator enjoys the IT-1.5-based discretization of the strong order 1.5.

With use of an equidistant mesh consisting of $L-1$ equally spaced subdivision nodes (with a user-supplied prefixed quantity $L$) introduced in each sampling interval $[t_{k-1}, t_k]$ of size $\delta$, this IT-1.5 approximation casts SDE (1) into the corresponding discrete-time stochastic system of the form

$$X_{k+1} = f_d(X_{k-1}) + Q^{1/2}W_1 + LF(X_{k-1})W_2$$  \hspace{1cm} (5)

where $Q^{1/2}$ stands for the lower triangular factor (SR) in the Cholesky decomposition of the process noise covariance $Q$ and the discretized drift coefficient obeys the formula

$$F_d(X_{k-1}) := X_{k-1} + \tau F(X_{k-1}) + \tau^2 L_0 F(X_{k-1})/2.$$  \hspace{1cm} (6)

Here and below, the random variable $X_{k-1}$ denotes the IT-1.5-based approximation to the solution $X(t_{k-1})$ of SDE (1) at a particular time instant $t_{k-1} := t_k - \delta l$, $l = 0, 1, \ldots, L$, and $F(\cdot)$ is the drift function of the process model. The scalar $\tau := \delta/L$ denotes the step size in the equidistant subdivision (mesh) introduced in each sampling interval $[t_{k-1}, t_k]$ underlying the L-step discretization of the form (5) and (6). Also, the differential operators $L_0$ and $L_j$ utilized in formulas (5) and (6) are defined as follows:

$$L_0 := \sum_{i=1}^n F_i \frac{\partial}{\partial X_i} + \frac{1}{2} \sum_{i,j,p,r=1}^n \tilde{G}_{ij;p;r} \frac{\partial^2}{\partial X_i \partial X_r},$$

$$L_j := \sum_{i=1}^n \tilde{G}_{ij} \frac{\partial}{\partial X_i}, \hspace{1cm} j = 1, 2, \ldots, n,$$

where each scalar $\tilde{G}_{ij}$ refers to the $(i,j)$-entry in $\tilde{G} := Q^{1/2}$. The notation $L F(X_{k-1})$ in the discretized stochastic system (5) means the square matrix whose $(i,j)$-entry is a value of the above operator $L_0 F(X_{k-1})$ and the last summand is computed by means of the operator $L_0 F(X_{k-1})$ in the discrete-time drift coefficient (6). Furthermore, the pair of correlated $n$-dimensional Gaussian variables $W_1$ and $W_2$ is generated from the pair of uncorrelated $n$-dimensional standard Gaussian variables $U_1$ and $U_2$ by the rule: $W_1 := \sqrt{\tau} U_1$, $W_2 := \tau^{3/2}(U_1 + U_2/\sqrt{3})/2$.

Our further intention is to establish mean and covariance time-propagation schemes for the discrete-time stochastic process (5) and (6). In other words, given the mean $X_{k-1}|k-1|$ and covariance matrix $S_{k-1}|k-1|^T$ of the random variable $X_{k-1}$ (i.e. $P_{k-1}|k-1] = [S_{k-1}|k-1]|S_{k-1}|k-1|^T$), we have to advance a step in the discretized process model and compute the mean $X_{k+1}|k-1|$ and covariance matrix $S_{k+1}|k-1|$ of the random variable $X_{k+1}$ derived by equations (5) and (6) whose time-updated covariance satisfies condition (4), i.e. $P_{k+1}|k-1] = [S_{k+1}|k-1]|S_{k+1}|k-1|^T$.

The non-SR state mean and covariance time-propagation schemes have been already developed in the form of the rather complicated formulas (46) and (47) presented in Santos-Diaz et al. (2018). Then, for facilitating our square-rooting technique and making the calculations effective in MATLAB, we amend first the cited mean and covariance evolutions to a more compact matrix-vector multiplication fashion. With this goal in mind, we set the fifth-degree spherical-radial cubature rule’s coefficients in the form of the following vector and square matrix of size $2n^2 + 1$:...
\[ \mathbf{W} := \begin{pmatrix} \mathbf{I}_{2n^2+1} - \mathbf{I}_{2n^2+1} \otimes w \end{pmatrix} \mathbf{w} + \begin{pmatrix} \mathbf{I}_{2n^2+1} - \mathbf{I}_{2n^2+1} \otimes w \end{pmatrix} \mathbf{w}^\top, \]  
(7)

where the notation \( \mathbf{I}_{2n^2+1} \) and \( \mathbf{I}_{2n^2+1} \) stands for the unitary column-vectors of size \( 2n, 2n \) and \( 2n^2 + 1 \), respectively. \( \mathbf{I}_{2n^2+1} \) refers to the identity matrix of size \( 2n^2 + 1 \), \( \mathbf{w} \) denotes the diagonal matrix whose diagonal entries are given by the entries of the vector defined in formula (7) and \( \otimes \) is the conventional Kronecker tensor product fulfilled by the MATLAB’s command kron.

In addition, we assemble the column of cubature nodes
\[ \mathcal{X}_{1, k-1 | k-1} := \left[ \mathcal{X}_{1, k-1 | k-1}, \mathcal{X}_{2, k-1 | k-1}, \ldots, \mathcal{X}_{2n^2+1, k-1 | k-1} \right], \]
where columns are determined in line with the formula
\[ \mathcal{X}_{i, k-1 | k-1} := \mathcal{X}_{i, k-1 | k-1} + \mathcal{X}_{l, k-1 | k-1}, \quad i, l = 1, 2, \ldots, 2n^2 + 1. \]

(9)
The vector \( \Gamma \), in rule (10) is the \( i \)th column in the matrix
\[ \Gamma := \sqrt{n + 2} \mathbf{0}_n, E^+ - E^+, -E^- I_n - I_n, \]
whose submatrices \( E^+ \) and \( E^- \) (both of size \( n \times n \)) consist of the column-vectors of size \( n \times n \) defined in the following rule:
\[ E^+ := \left\{ (e_i + e_j) \sqrt{2} : i < j, i, j = 1, 2, \ldots, n \right\}, \]
\[ E^- := \left\{ (e_i - e_j) \sqrt{2} : i < j, i, j = 1, 2, \ldots, n \right\}, \]
where \( e_i \) and \( e_j \) stand for the \( i \)th and \( j \)th columns in the identity matrix \( I_n \), respectively, and \( \mathbf{0}_n \) is the zero column-vector of size \( n \). We recall that the state mean \( \mathcal{X}_{1, k-1 | k-1} \) and covariance \( \mathcal{S}_{1, k-1 | k-1} \) given by formula (10) are assumed to be known at time \( t_{k-1} := t_{k-1} + \tau, \) \( \ell = 0, 1, \ldots, L - 1. \)

Next, with use of the discretized drift coefficient (6), we modify matrix (9) of the cubature nodes to the form
\[ \mathcal{Y}_{1, k-1 | k-1} := \left[ \mathcal{Y}_{1, k-1 | k-1}, \mathcal{Y}_{2, k-1 | k-1}, \ldots, \mathcal{Y}_{2n^2+1, k-1 | k-1} \right], \]
\( \mathcal{Y}_{l, k-1 | k-1} := F_{\alpha}(\mathcal{X}_{l, k-1 | k-1}), \quad i = 1, 2, \ldots, 2n^2 + 1. \)

(14)

Eventually, the theory of Särkkä (2007) allows the mean and covariance time evolutions in Santos-Díaz et al. (2018) to be casted into the simple and convenient form as follows:
\[ \mathcal{X}_{1, k-1 | k-1} = \mathcal{Y}_{1, k-1 | k-1} + \mathbf{w}, \]
\[ \mathcal{P}_{k-1 | k-1} = \begin{pmatrix} \mathcal{Y}_{1, k-1 | k-1} \end{pmatrix} \mathbf{W} \begin{pmatrix} \mathcal{Y}_{1, k-1 | k-1} \end{pmatrix}^\top + \tau \begin{pmatrix} \mathcal{G}_{l, k-1 | k-1} \end{pmatrix} \times \begin{pmatrix} \mathcal{G}_{l, k-1 | k-1} \end{pmatrix}^\top + \frac{\tau^2}{3} \begin{pmatrix} \mathcal{P}_{l, k-1 | k-1} \end{pmatrix}^2 \begin{pmatrix} \mathcal{P}_{l, k-1 | k-1} \end{pmatrix}^\top. \]

(16)

In formula (17), we have utilized the notation \( \mathcal{G}_{l, k-1 | k-1} := \mathcal{G} + \mathcal{P}_{l, k-1 | k-1} \) and \( \mathcal{P}_{l, k-1 | k-1} := \tau \mathcal{L} F(\mathcal{X}_{l, k-1 | k-1}) \) where the square matrix \( \mathcal{L} F(\mathcal{X}_{l, k-1 | k-1}) \) of size \( n \) is explained after formula (6). The mean evolution (16) takes its final fashion, but the covariance one (17) is to be square-rooted.

First of all, we need an SR of the coefficient matrix (8). Taking into account the negativity of the last 2\( n \) entries in the coefficient vector (7) when \( n > 4 \), we replace these with their magnitude and arrive at the modified coefficient matrix SR defined by the following two formulas:
\[ |w|^1/2 := \sqrt{\frac{\sqrt{n^2 - 2} - \sqrt{2n(n-1)} - \sqrt{2(n+2)}}{2(n+2)}}, \]
\[ |w|^1/2 := \left( \sqrt{n^2 - 2} - \sqrt{2n(n-1)} - \sqrt{2(n+2)} \right) \mathbf{w}. \]

(18)

(19)

We stress that the entries of row-vector (18) influence only the second factor on the right-hand side of the coefficient matrix SR definition (19), whereas the first one is found by means of column-vector (7). In addition, we set the signature matrix for this second diagonal factor as follows:
\[ \mathcal{J} := \begin{pmatrix} 1, \mathbf{I}_{2n(n-1)} \end{pmatrix} \mathbf{w} \quad \text{where the function } \mathbf{w} \text{ determines the size of its last 2\( n \) diagonal entries by the rule: } \mathbf{w} = \mathbf{I}_{n \times 1} \text{ if } n \leq 4 \text{ and } \mathbf{w} = \mathbf{I}_{n \times 1} = \mathbf{0} \text{ if } n > 4. \]

It is worthwhile to remark that formulas (8) and (18)–(20) entail the obvious equality
\[ \mathcal{W} = |\mathcal{W}|^{1/2} \mathbf{J} |\mathcal{W}|^{1/2}, \]
where \( |\mathcal{W}|^{1/2} \) stands for the transpose of the SR \( |\mathcal{W}|^{1/2} \).

Next, we assemble the predicted covariance pre-array
\[ S_X := \begin{pmatrix} \sqrt{\tau} \mathcal{G}^t_{k-1 | k-1} \sqrt{\sqrt{\tau} \mathcal{G}^t_{k-1 | k-1} \mathcal{G}^t_{k-1 | k-1} \mathcal{W}^{1/2} \mathbf{J} |\mathcal{W}|^{1/2}}. \]

(22)

Formulas (21) and (22) cast equation (17) into the form
\[ P_{k-1 | k-1} = S_X J S_X^\top + \mathbf{J} := \mathcal{I}_{2n, \mathcal{J}}. \]

(23)

Further, the notion of J-orthogonality is a background in our approach to square-rooting the covariance time evolution scheme (17). Higham (2003) defines a J-orthogonal matrix as follows: A square matrix \( \mathcal{J} \) of size \( n \times n \) is said to be J-orthogonal with a signature matrix \( \mathcal{J} := \mathcal{J} \mathbf{J} = \mathbf{J} \mathbf{J} \) where all positive entries are placed in the beginning of its main diagonal and the remaining negative ones complete it. That is why we have changed the order of the summands on the right hand-side of formula (17) and, hence, the order of the blocks in pre-array (22) and the signature \( \mathcal{J} \) in (23).

This hyperbolic QR decomposition method applied to the transposed matrix of pre-array (22) with the signature \( \mathcal{J} \) from equation (23) returns the lower triangular post-array
\[ \mathbf{R}^\top = \begin{pmatrix} \mathcal{S}_{k-1 | k-1} \end{pmatrix} \mathbf{0}_n \times (2n^2 + n + 1), \]

(25)

which is of size \( n \times (2n^2 + 2n + 1) \), with the notation \( \mathbf{0}_n \times (2n^2 + n + 1) \) standing for the zero-block of size \( n \times (2n^2 + n + 1) \). Eventually, we read-off the square block \( \mathcal{S}_{k-1 | k-1} \) of size \( n \), which constitutes the requested time-updated covariance matrix SR because equations (22)–(25) prove its SR condition (4) by the following formula chain:
\[ P_{k-1 | k-1} = S_X J S_X^\top = \mathbf{R}^\top \mathbf{J} \mathbf{R} = \mathbf{R}^\top \mathbf{J} \mathbf{R} = \begin{pmatrix} \mathcal{S}_{k-1 | k-1} \end{pmatrix} \mathcal{W}^{1/2} \mathbf{J} |\mathcal{W}|^{1/2}. \]

(26)
The signature matrix $J$ becomes the identity one after multiplication of its negative part with $0_{n \times (2n^2+n+1)}$ in post-array (25) and, hence, vanishes in proof (26). This completes the time update in our JSR-5D-CKF. For convenience of practical use, we summarize the time update of this filter in the following condensed algorithmic form:

Given $\hat{X}_{k-1|k-1}$ and $S_{k-1|k-1}$ at time $t_{k-1}$, compute the predicted state mean $\hat{X}_{k|k-1}$ and covariance SR $S_{k|k-1}$ at time $t_k$. Set the local initial values $\hat{X}_{k|k-1}^0 := \hat{X}_{k-1|k-1}$ and $S_{k-1|k-1}^0 := S_{k-1|k-1}$ and fulfill the $L$-step time-update procedure with $\tau := (t_k - t_{k-1})/L$ as follows:

1) Assemble matrix (9) by means of formulas (10)–(13);
2) Set up the measurement-function-modified matrix (29);
3) Set up the unified coupled covariance pre-array (34);
4) Compute the time update of our JSR-5D-CKF and the vectors $\bar{Z}_k$ and covariance matrix $S_{k|k}$ after the time update elaborated in Sec. 2.1. For convenience of the post-array (25). (27)
5) Fulfil the time update elaborated in Sec. 2.1. For convenience of the calculation of the innovations, cross- and filtering covariances as follows:

$$ R_k = P_{k|k-1}$$

$$ P_{k|k} = P_{k|k-1} + \Psi_k P_{zz,k|k-1} \Psi_k^T$$

where $R_k$ stands for the covariance of the measurement noise in model (2) and the Kalman gain obeys the formula

$$ \Psi_k := P_{zz,k|k-1}^{-1}$$

It is commonly accepted to square-root all the covariance matrices calculated by formulas (30)–(32) in the form of a unified coupled pre-array. Here, it is set up by the formula

$$ B := \begin{bmatrix} R_k^{1/2} & Z_{k|k-1}^{-1/2} & 0_{m \times n} & X_{k|k-1}^{-1/2} \\ 0_m \times n & 0_{n \times (2n^2+n+1)} & 0_{n \times (2n^2+n+1)} & 0_{n \times (2n^2+n+1)} \end{bmatrix}$$

(34)

where $R_k^{1/2}$ refers to the lower triangular Cholesky factor (SR) of the measurement noise covariance, i.e. $R_k = R_k^{1/2} R_k^{1/2}$. Similar to the time update presented in Sec. 2.1, the above-mentioned hyperbolic QR decomposition code is applied to the transposed matrix of pre-array (34) with the signature matrix $J := \text{diag} \{I_m, J\}$, in which $I_m$ is the identity matrix of size $m$ and $J$ obeys formula (20). The latter factorization returns the lower triangular post-array

$$ R_T = \begin{bmatrix} P_{zz,k|k-1}^{1/2} & 0_m \times n & 0_{m \times (2n^2+n+1)} \\ P_{zz,k|k-1}^{1/2} & Z_{k|k-1}^{-1/2} & S_{k|k}^{-1/2} \end{bmatrix}$$

(35)

which is of size $(m+n) \times (2n^2+n+1)$, with the matrix $F_{zz,k|k-1}$ denoting the modified cross-covariance. This matrix amends the Kalman gain computation formula (33) to the more convenient and robust form

$$ W_k = P_{zz,k|k-1}^{-1/2} F_{zz,k|k-1}^{-1/2}$$

(36)

Next, we construct the measurement mean by the inner product

$$ \hat{Z}_k := \hat{X}_{k|k-1}$$

and complete this measurement update with calculating

$$ \hat{X}_{k|k} = \hat{X}_{k|k-1} + W_k (Z_k - \hat{X}_{k|k-1})$$

(39)

We recall that the predicted state mean $\hat{X}_{k|k-1}$ comes from the time update elaborated in Sec. 2.1. For convenience of practical use, we present this measurement update of JSR-5D-CKF in the following condensed algorithmic form:

Given the predicted state mean $\hat{X}_{k|k-1}$ and covariance matrix $S_{k|k-1}$ := $S_{k|k-1}$, compute the filtering covariance SR $S_{k|k}$ after the time update elaborated in Sec. 2.1. For convenience of the calculation of the innovations, cross- and filtering covariances as follows:

1) Assemble matrix (27) by means of formulas (10)–(13) and (28);
2) Set up the measurement-function-modified matrix (29);
3) Set up the unified coupled covariance pre-array (34);
4) Fulfil the $J$-orthogonal QR factorization of the transposed pre-array (34) with the signature $J := \text{diag}(I_m, J)$; 
5) Compute the Kalman gain $W_k$ by formula (36); 
6) Compute the filtering mean $\hat{X}_{k|k}$ by formulas (38), (39); 
7) Read-off the filtering covariance SR $S_{k|k}$ in array (37).

Further, we examine the novel JSR-5D-CKF method presented in Sec. 2.1 and 2.2 and compare it to its non-SR predecessor 5D-CKF designed by Santos-Díaz et al. (2018) in severe conditions of tackling a radar tracking problem of Arasaratnam et al. (2010), where an aircraft executes a coordinated turn. It is implemented with ill-conditioned measurements in our nonlinear stochastic scenario, below.

3. AIR TRAFFIC CONTROL SCENARIO WITH ILL-CONDITIONED MEASUREMENTS

The flight control scenario under consideration is a famous one in nonlinear filtering theory, which has been published with all particulars by Arasaratnam et al. (2010); Kulikov and Kulikova (2016, 2017a), etc. So, the interested reader is referred to the cited papers for more details. We simulate the turning aircraft dynamics for 150 s and set its angular velocity $\omega = 3^\circ/s$. The performance of our novel algorithm JSR-5D-CKF and its non-SR predecessor 5D-CKF devised by Santos-Díaz et al. (2018) is assessed in the sense of the Accumulated Root Mean Square Errors in position ($\text{ARMSE}_p$) and in velocity ($\text{ARMSE}_v$) defined as follows:

$$ \text{ARMSE}_p := \left[ \frac{1}{100K} \sum_{m=1}^{100} \sum_{k=1}^{K} (x_{mc}(t_k) - \hat{x}_{mc})(t_k) - \hat{x}_{mc}(t_k) \right]^2 + \left( \frac{1}{100K} \sum_{m=1}^{100} \sum_{k=1}^{K} (y_{mc}(t_k) - \hat{y}_{mc}(t_k) - \hat{y}_{mc}(t_k)) \right)^2 \right]^{1/2}, $$

$$ \text{ARMSE}_v := \left[ \frac{1}{100K} \sum_{m=1}^{100} \sum_{k=1}^{K} (\dot{x}_{mc}(t_k) - \hat{\dot{x}}_{mc}(t_k))^2 + \left( \frac{1}{100K} \sum_{m=1}^{100} \sum_{k=1}^{K} (\dot{y}_{mc}(t_k) - \hat{\dot{y}}_{mc}(t_k)) \right)^2 \right]^{1/2}, $$

where $x_{mc}(t_k)$, $y_{mc}(t_k)$, $\dot{x}_{mc}(t_k)$, $\dot{y}_{mc}(t_k)$, $z_{mc}(t_k)$ and $\dot{z}_{mc}(t_k)$ stand for the aircraft’s position and velocity simulated by the Euler-Maruyama method with the small step size $\tau := 0.0005$ at time $t_k$ in the $m$th Monte Carlo run (out of 100 independent simulations), $z_{mc}$, $\dot{z}_{mc}$, $\ddot{z}_{mc}$ denote the aircraft’s position and velocity estimated by each filtering algorithm, $k$ means the particular sampling time $t_k$ and $K$ refers to the total number of samples in the simulation interval $[0, 150]$. The sampling rate is limited to $\delta = 1$ s in this flight control task.

In contrast to Arasaratnam et al. (2010), for provoking numerical instabilities in the filters under examination, we utilize the artificial measurement equation of the form

$$ Z_k = \begin{bmatrix} \begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} & 0 \end{bmatrix} X_k + V_k $$

(40)

where the aircraft’s state $X_k := [x_k \ y_k \ \dot{x}_k \ \dot{y}_k \ z_k \ \dot{z}_k \ \omega_k]^T$ is estimated at time $t_k$ and $\sigma$ denotes a small positive real number determining ill-conditioning of model (40). Here, we address the cases of $\sigma = 1.0e-01, 1.0e-02, \ldots, 1.0e-11$.

All, each measurement $Z_k$ is supposed to be corrupted by a normally distributed noise $V_k \sim N(0, R_\sigma)$ with the covariance $R_\sigma = \sigma^2I_2$ depending on the ill-conditioning parameter $\sigma$. Measurements (40) with the measurement noise covariance matrix $R_\sigma$ are typical means in numerical stability studies of various KF including the continuous-discrete and discrete-discrete methods presented by Dyer and McReynolds (1969); Grewal and Andrews (2001); Kulikov and Kulikova (2017c, 2018b, 2019). These correspond to the third reason of ill-conditioning elaborated by Grewal and Andrews (2001) because the matrix inversions in the Kalman gain computations (33) and (36) become increasingly ill-conditioned in line with the vanishing scalar $\sigma$.

We abbreviate our novel SR filter to JSR-5D-CKF and its non-SR predecessor published by Santos-Díaz et al. (2018) to 5D-CKF, respectively. These methods are coded and run in MATLAB. The state estimators under consideration enjoy $L = 64$ subdivision steps in each sampling period. The last 2n entries in the fifth-degree spherical-radial cubature rule’s coefficient vector (7) are negative because $n > 4$ in the target tracking scenario in use. This serves for the effective examination of JSR-5D-CKF and its valued comparison to 5D-CKF in the presence of the increasingly ill-conditioned measurement model (40), i.e. when $\sigma \to 0$.

Fig 1 exhibits that the SR filter and its non-SR predecessor work identically and expose the same $\text{ARMSE}_p$ and $\text{ARMSE}_v$ when the ill-conditioning parameter $\sigma \geq 1.0e-03$, i.e. when our air traffic control scenario is rather well-
conditioned. Then, we see that the non-SR 5D-CKF fails at $\sigma = 1.0e-04$ because its covariance matrix computed loses the positivity and, hence, the Cholesky factorization may not be fulfilled. In contrast, the JSR-5D-CKF succeeds in producing the decent state estimates for all the values of the ill-conditioning parameter $\sigma$ accepted in our case study. This confirms the sound numerical robustness of the filtering algorithm presented in Sec. 2 and establishes a solid background for its successful applications in practice.

4. CONCLUSION

This paper has addressed the issue of “the lack of a square-root implementation” and devised a square-root version of the Itô-Taylor-based Fifth-Degree Cubature Kalman Filter presented by Santos-Díaz et al. (2018). Taking into account the negativity of some weights in the fifth-degree spherical-radial cubature rule, which are possible in continuous-discrete stochastic scenarios of large size, we have applied the hyperbolic QR decomposition for designing our novel $J$-orthogonal square-root state estimator, which has been examined in severe conditions of tackling a radar tracking problem, where an aircraft executes a coordinated turn, in the presence of ill-conditioned measurements. The sound state estimation potential of this filter has been proven theoretically and evidenced numerically within the mentioned challenging stochastic flight control scenario.

5. ACKNOWLEDGMENTS

The authors thank Prof Nicholas J. Higham sincerely for his hyperbolic QR decomposition code employed successfully in their novel $J$-orthogonal Square-Root Fifth-Degree Cubature Kalman Filtering (JSR-5D-CKF) state estimation algorithm presented and examined in this paper.

REFERENCES