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The J-Orthogonal Square-Root Fifth-Degree Cubature Kalman Filtering Method for Nonlinear Stochastic Systems ⋆

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Abstract: This paper addresses the issue of square-rooting in the *Fifth-Degree Cubature Kalman Filtering* (5D-CKF) method grounded in the Itô-Taylor approximation of order 1.5 and designed by Santos-Diaz, Haykin and Hurd in 2018. That filter is rather accurate and efficient in treating nonlinear continuous-discrete stochastic systems of practical value and shown to outperform many other algorithms in a radar tacking scenario. However, the cited authors mention "the lack of a square-root implementation" of the filter under consideration as a principle shortcoming reducing its applied potential. Here, we address the reported lack and resolve it with a hyperbolic QR factorization used for devising the filter's J-orthogonal square-root version, which possesses an exceptional robustness to round-off and other disturbances. Our square-root implementation of the 5D-CKF technique is justified theoretically and examined and compared numerically to its non-square-root predecessor in a flight control scenario with ill-conditioned measurements.

Keywords: Continuous-discrete nonlinear stochastic model, fifth-degree cubature Kalman filter, square-root implementation, radar tracking, maneuvering target, ill-conditioned measurements.

1. INTRODUCTION

Many modern control tasks in nonlinear stochastic systems arisen in applied science and engineering demand accurate and efficient state estimation tools for continuous process models with discrete measurements, which are of the form

$$dX(t) = F\big(X(t)\big)dt + GdW(t), \quad t > 0, \tag{1}$$

$$Z_k = h(X_k) + V_k, \quad k \geq 1. \tag{2}$$

The process model (1) is supposed to be an Itô-type *Stochastic Differential Equation* (SDE), in which the unknown stochastic process $X(t)$ represents the state of the plant of size n at time t, the known nonlinear vector-function $F : \mathbb{R}^n \to \mathbb{R}^n$ describes its dynamic behavior, the diffusion matrix G is assumed to be time-invariant and of size $n \times n$ in the driving noise used, and the random disturbance $\{W(t),\, t > 0\}$ is a multivariate Wiener process with independent zero-mean Gaussian increments $dW(t)$ having a covariance of the form Qdt of size $n \times n$ where the matrix Q is positive definite and fixed in time. The initial state can also be a Gaussian variable $X(0) \sim \mathcal{N}(\bar{X}_0, \Pi_0)$ with mean $\bar{X}_0$ and covariance $\Pi_0 > 0$ in SDE (1). Next, the discrete-time measurement model (2) with k being a discrete time index (i.e. X_k means $X(t_k)$) establishes a nonlinear in general link $h : \mathbb{R}^n \to \mathbb{R}^m$ between the distribution of the state X_k in the dynamic process at hand and its measurement Z_k of size m corrupted by a zero-mean Gaussian variable $\{V_k,\, k \geq 1\}$ with its covariance $R_k > 0$ at every sampling instant t_k. The measurements Z_k arrive uniformly and with the sampling rate $\delta = t_k - t_{k-1}$ in our setting. This time interval δ is also known as the sampling period in filtering theory. Furthermore, all realizations of the noises $dW(t)$, V_k and the initial state $X(0)$ are taken from mutually independent Gaussian distributions. The continuous-discrete state estimation scenarios are often encountered in practical modeling and motivated in Santos-Diaz et al. (2018); Jazwinski (1970); Särkkä (2007), etc.

A conventional state estimation setting in stochastic systems of the above form (1) and (2) is to obtain an optimal estimate of the dynamic model obeying SDE (1) grounded in measurements $\{Z_1, \ldots, Z_k\}$ realized up to each sampling instant t_k. Here, we stick to the Kalman formulation and look for the optimal estimation of the random process $X(t)$ in the mean least square sense, which is presented by the conditional mean $\hat{X}_{k|k}$. Based on the Gaussianity assumption of the a priori and a posteriori random distributions, the solution to this state estimation task demands multidimensional Gaussian-weighted integrals of the form

$$\int_{\mathbb{R}^n} g(X)\mathcal{N}(X; \hat{X}, P_X)dX = \int_{\mathbb{R}^n} g(\hat{X} + S_X X)\mathcal{N}(X; \mathbf{0}_n, I_n)dX \tag{3}$$

to be accurately computed. In integral (3), the functions $g(X) \equiv X$ and $g(X) \equiv (X - \hat{X})(X - \hat{X})^\top$ are employed in calculations of the predicted and filtering mean vectors and covariance matrices, respectively, $\mathbf{0}_n := [0, \ldots, 0]^\top \in \mathbb{R}^n$, I_n stands for the identity matrix of size n, $\mathcal{N}(X; \hat{X}, P_X)$ denotes the Gaussian probability density function with its expectation $\hat{X}$ and covariance matrix P_X and the matrix

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S_X of size $n \times n$ refers to the *Square Root* (SR) of the covariance, which is determined by the following property:

$$P_X = S_X S_X^\top. \tag{4}$$

This SR S_X is often accepted to be the lower triangular Cholesky factor of covariance in practical estimation tasks.

Arasaratnam and Haykin (2009) gave rise to a novel and effective state estimation technology rooted in cubature rules applied for numerical approximations of integrals (3) and termed it the *Cubature Kalman Filter* (CKF). Later on, Arasaratnam et al. (2010) explained that the first CKF based on the third-degree spherical-radial cubature rule is equivalent to a particular version of the *Unscented Kalman Filter* (UKF) presented by Julier et al. (1995, 2000); Julier and Uhlmann (2004); Wan and Van der Merwe (2000); Van der Merwe and Wan (2001); Wan and Van der Merwe (2001). This justifies its sound state estimation potential exposed for target tracking in Gaussian and non-Gaussian scenarios by Kulikov and Kulikova (2016, 2017b,a, 2018a, 2020). Next, Jia et al. (2013); Santos-Diaz et al. (2018); Zhang and Teng (2014) designed and analyzed the fifth-degree variants of that earlier-published CKF technique, which outperform it and many other filtering methods in nonlinear stochastic radar tracking scenarios. However, these can suffer severely from "the lack of a square-root implementation" reported by Santos-Diaz et al. (2018).

In what follows, we address the mentioned lack and devise one SR implementation of the fifth-degree CKF presented in the latter cited paper. Our solution is rooted in the Itô-Taylor SDE approximation of order 1.5 (IT-1.5), which is employed by Santos-Diaz et al. (2018) as well. Furthermore, this SR implementation uses the concept of a J-orthogonal transformation implemented by means of a hyperbolic QR factorization because of potential negativity of some weights in the fifth-degree cubature rule applied. We remark that J-orthogonal QR decompositions are commonly utilized in the realm of H_∞ filtering and other tasks with indefinite inner products. Here, we stick to the J-orthogonal QR factorization of Bojanczyk et al. (2003), which combines the Householder reflections and hyperbolic rotations and considered to be numerically robust and efficient. The factorization implemented gives rise to our novel *J-orthogonal Square-Root Fifth-Degree Cubature Kalman Filtering* (JSR-5D-CKF) technique. Its exceptional robustness to round-off and other disturbances resulting in the superiority of the JSR-5D-CKF method towards its predecessor presented by Santos-Diaz et al. (2018) is exposed within the target tracking scenario of Arasaratnam et al. (2010), in which an aircraft executes a coordinated turn. That numerical examination setup is employed with ill-conditioned measurements in this paper.

2. J-ORTHOGONAL SQUARE-ROOT 5D-CKF

For yielding our new filter, we have to square-root the time and measurement update steps in the IT-1.5-based 5D-CKF method presented by Santos-Diaz et al. (2018). We start off at the modification of its time update, below.

2.1 The Time Update Step in the JSR-5D-CKF

Following Santos-Diaz et al. (2018), our state estimator enjoys the IT-1.5-based discretization of the strong order 1.5. With use of an equidistant mesh consisting of $L-1$ equally spaced subdivision nodes (with a user-supplied prefixed quantity L) introduced in each sampling interval $[t_{k-1}, t_k]$ of size δ, this IT-1.5 approximation casts SDE (1) into the corresponding discrete-time stochastic system of the form

$$X_{k-1}^{l+1} = F_d\big(X_{k-1}^l\big) + GQ^{1/2}W_1 + \mathbb{L}F\big(X_{k-1}^l\big)W_2 \tag{5}$$

where $Q^{1/2}$ stands for the lower triangular factor (SR) in the Cholesky decomposition of the process noise covariance Q and the discretized drift coefficient obeys the formula

$$F_d\big(X_{k-1}^l\big) := X_{k-1}^l + \tau F\big(X_{k-1}^l\big) + \tau^2 \mathbb{L}_0 F\big(X_{k-1}^l\big)/2. \tag{6}$$

Here and below, the random variable X_{k-1}^l denotes the IT-1.5-based approximation to the solution $X(t_{k-1}^l)$ of SDE (1) at a particular time instant $t_{k-1}^l := t_{k-1} + l\tau$, $l = 0, 1, \ldots, L$, and $F(\cdot)$ is the drift function of the process model. The scalar $\tau := \delta/L$ denotes the step size in the equidistant subdivision (mesh) introduced in each sampling interval $[t_{k-1}, t_k]$ underlying the L-step discretization of the form (5) and (6). Also, the differential operators $\mathbb{L}_0$ and $\mathbb{L}_j$ utilized in formulas (5) and (6) are defined as follows:

$$\mathbb{L}_0 := \sum_{i=1}^n F_i \frac{\partial}{\partial X_i} + \frac{1}{2} \sum_{j,p,r=1}^n \widetilde{G}_{pj} \widetilde{G}_{rj} \frac{\partial^2}{\partial X_p \partial X_r},$$

$$\mathbb{L}_j := \sum_{i=1}^n \widetilde{G}_{ij} \frac{\partial}{\partial X_i}, \quad j = 1, 2, \ldots, n,$$

where each scalar $\widetilde{G}_{ij}$ refers to the (i,j)-entry in $\widetilde{G} := GQ^{1/2}$. The notation $\mathbb{L}F\big(X_{k-1}^l\big)$ in the discretized stochastic system (5) means the square matrix whose (i,j)-entry is a value of the above operator $\mathbb{L}_j F_i\big(X_{k-1}^l\big)$ and the last summand is computed by means of the operator $\mathbb{L}_0 F\big(X_{k-1}^l\big)$ in the discrete-time drift coefficient (6). Furthermore, the pair of correlated n-dimensional Gaussian variables W_1 and W_2 is generated from the pair of uncorrelated n-dimensional standard Gaussian variables U_1 and U_2 by the rule: $W_1 := \sqrt{\tau}U_1$, $W_2 := \tau^{3/2}(U_1 + U_2/\sqrt{3})/2$.

Our further intention is to establish mean and covariance time-propagation schemes for the discrete-time stochastic process (5) and (6). In other words, given the mean $\hat{X}_{k-1|k-1}^l$ and covariance matrix SR $S_{k-1|k-1}^l$ of the random variable X_{k-1}^l (i.e. $P_{k-1|k-1}^l = [S_{k-1|k-1}^l][S_{k-1|k-1}^l]^\top$), we have to advance a step in the discretized process model and compute the mean $\hat{X}_{k-1|k-1}^{l+1}$ and covariance matrix SR $S_{k-1|k-1}^{l+1}$ of the random variable X_{k-1}^{l+1} derived by equations (5) and (6) whose time-updated covariance satisfies condition (4), i.e. $P_{k-1|k-1}^{l+1} = [S_{k-1|k-1}^{l+1}][S_{k-1|k-1}^{l+1}]^\top$.

The non-SR state mean and covariance time-propagation schemes have been already developed in the form of the rather complicated formulas (46) and (47) presented in Santos-Diaz et al. (2018). Then, for facilitating our square-rooting technique and making the calculations effective in MATLAB, we amend first the cited mean and covariance evolutions to a more compact matrix-vector multiplication fashion. With this goal in mind, we set the fifth-degree spherical-radial cubature rule's coefficients in the form of the following vector and square matrix of size $2n^2 + 1$:

$$w := \frac{1}{n+2}\left[2 \;\; \frac{1}{n+2}\mathbf{1}_{2n(n-1)}^\top \;\; \frac{4-n}{2(n+2)}\mathbf{1}_{2n}^\top\right]^\top, \quad (7)$$

$$\mathcal{W} := \big(I_{2n^2+1} - \mathbf{1}_{2n^2+1}^\top \otimes w\big)\,\mathrm{diag}\{w^\top\} \times \big(I_{2n^2+1} - \mathbf{1}_{2n^2+1}^\top \otimes w\big)^\top, \quad (8)$$

where the notation $\mathbf{1}_{2n}$, $\mathbf{1}_{2n(n-1)}$ and $\mathbf{1}_{2n^2+1}$ stands for the unitary column-vectors of sizes $2n$, $2n(n-1)$ and $2n^2+1$, respectively, I_{2n^2+1} refers to the identity matrix of size $2n^2+1$, $\mathrm{diag}\{w^\top\}$ denotes the diagonal matrix whose diagonal entries are given by the entries of the vector defined in formula (7) and $\otimes$ is the conventional Kronecker tensor product fulfilled by the MATLAB's command `kron`. In addition, we assemble the matrix of cubature nodes

$$\mathcal{X}_{k-1|k-1}^l := \left[\,\mathcal{X}_{1,k-1|k-1}^l \;\; \mathcal{X}_{2,k-1|k-1}^l \;\; \ldots \;\; \mathcal{X}_{2n^2+1,k-1|k-1}^l\,\right] \quad (9)$$

whose columns are determined in line with the formula

$$\mathcal{X}_{i,k-1|k-1}^l := \hat{X}_{k-1|k-1}^l + S_{k-1|k-1}^l\Gamma_i, i = 1,2,\ldots,2n^2+1. \quad (10)$$

The vector Γ_i in rule (10) is the ith column in the matrix

$$\Gamma := \sqrt{n+2}\left[\,\mathbf{0}_n \;\; E^+ \;\; -E^+ \;\; E^- \;\; -E^- \;\; I_n \;\; -I_n\,\right] \quad (11)$$

whose submatrices E^+ and E^- (both of size $n\times n(n-1)/2$) consist of the column-vectors yielded as follows:

$$E^+ := \left\{(e_i+e_j)/\sqrt{2} : i<j,\; i,j = 1,2,\ldots,n\right\}, \quad (12)$$

$$E^- := \left\{(e_i-e_j)/\sqrt{2} : i<j,\; i,j = 1,2,\ldots,n\right\} \quad (13)$$

where e_i and e_j stand for the ith and jth columns in the identity matrix I_n, respectively, and $\mathbf{0}_n$ is the zero column-vector of size n. We recall that the state mean $\hat{X}_{k-1|k-1}^l$ and covariance SR $S_{k-1|k-1}^l$ in formula (10) are assumed to be known at time $t_{k-1}^l := t_{k-1} + l\tau$, $l = 0,1,\ldots,L-1$.

Next, with use of the discretized drift coefficient (6), we modify matrix (9) of the cubature nodes to the form

$$\mathcal{Y}_{k-1|k-1}^{l+1} := \left[\,\mathcal{Y}_{1,k-1|k-1}^{l+1} \;\; \mathcal{Y}_{2,k-1|k-1}^{l+1} \;\; \ldots \;\; \mathcal{Y}_{2n^2+1,k-1|k-1}^{l+1}\,\right] \quad (14)$$

whose ith column is found in line with the following rule:

$$\mathcal{Y}_{i,k-1|k-1}^{l+1} := F_d\big(\mathcal{X}_{i,k-1|k-1}^l\big), \quad i = 1,2,\ldots,2n^2+1. \quad (15)$$

Eventually, the theory of Särkkä (2007) allows the mean and covariance time evolutions in Santos-Diaz et al. (2018) to be casted into the simple and convenient form as follows:

$$\hat{X}_{k-1|k-1}^{l+1} = \mathcal{Y}_{k-1|k-1}^{l+1} w, \quad (16)$$

$$P_{k-1|k-1}^{l+1} = \big[\mathcal{Y}_{k-1|k-1}^{l+1}\big]\mathcal{W}\big[\mathcal{Y}_{k-1|k-1}^{l+1}\big]^\top + \tau\big[\mathbb{G}_{k-1|k-1}^l\big] \times \big[\mathbb{G}_{k-1|k-1}^l\big]^\top + \frac{\tau}{3}\big[\mathbb{F}_{k-1|k-1}^l\big]\big[\mathbb{F}_{k-1|k-1}^l\big]^\top. \quad (17)$$

In formula (17), we have utilized the notation $\mathbb{G}_{k-1|k-1}^l := \widetilde{G} + \mathbb{F}_{k-1|k-1}^l$ and $\mathbb{F}_{k-1|k-1}^l := \tau\mathbb{L}F\big(\hat{X}_{k-1|k-1}^l\big)/2$ where the square matrix $\mathbb{L}F\big(\hat{X}_{k-1|k-1}^l\big)$ of size n is explained after formula (6). The mean evolution (16) takes its final fashion, but the covariance one (17) is to be square-rooted.

First of all we need an SR of the coefficient matrix (8). Taking into account the negativity of the last $2n$ entries in the coefficient vector (7) when $n > 4$, we replace these with their magnitude and arrive at the modified coefficient matrix SR defined by the following two formulas:

$$|w|^{1/2} := \left[\frac{\sqrt{2}}{\sqrt{n+2}} \;\; \frac{1}{n+2}\mathbf{1}_{2n(n-1)}^\top \;\; \frac{\sqrt{|4-n|}}{\sqrt{2}(n+2)}\mathbf{1}_{2n}^\top\right], (18)$$

$$|\mathcal{W}|^{1/2} := \big(I_{2n^2+1} - \mathbf{1}_{2n^2+1}^\top \otimes w\big)\,\mathrm{diag}\big\{|w|^{1/2}\big\}. \quad (19)$$

We stress that the entries of row-vector (18) influence only the second factor on the right-hand side of the coefficient matrix SR definition (19), whereas the first one is found by means of column-vector (7). In addition, we set the signature matrix for this second diagonal factor as follows:

$$\mathcal{J} := \mathrm{diag}\left\{1, \mathbf{1}_{2n(n-1)}^\top, \mathrm{sgn}\{4-n\}\mathbf{1}_{2n}^\top\right\} \quad (20)$$

where the function $\mathrm{sgn}\{4-n\}$ determines the sign of its last $2n$ diagonal entries by the rule: $\mathrm{sgn}\{4-n\} = 1$ if $n \le 4$ and $\mathrm{sgn}\{4-n\} = -1$ if $n > 4$. It is worthwhile to remark that formulas (8) and (18)–(20) entail the obvious equality

$$\mathcal{W} = |\mathcal{W}|^{1/2}\mathcal{J}|\mathcal{W}|^{\top/2} \quad (21)$$

where $|\mathcal{W}|^{\top/2}$ stands for the transpose of the SR $|\mathcal{W}|^{1/2}$.

Next, we assemble the predicted covariance pre-array

$$S_X := \left[\,\sqrt{\tau}\,\mathbb{G}_{k-1|k-1}^l \;\; \frac{\sqrt{\tau}}{\sqrt{3}}\mathbb{F}_{k-1|k-1}^l \;\; \mathcal{Y}_{k-1|k-1}^{l+1}|\mathcal{W}|^{1/2}\,\right]. \quad (22)$$

Formulas (21) and (22) cast equation (17) into the form

$$P_{k-1|k-1}^{l+1} = S_X J S_X^\top \quad \text{with} \quad J := \mathrm{diag}\{I_{2n}, \mathcal{J}\}. \quad (23)$$

Further, the notion of J-orthogonality is a background in our approach to square-rooting the covariance time evolution scheme (17). Higham (2003) defines a J-orthogonal matrix as follows: A square matrix Θ of size $n\times n$ is said to be *J-orthogonal* with a signature matrix $J := \mathrm{diag}\{\pm\mathbf{1}_n^\top\}$ of size n, i.e whose diagonal entries equal 1 or -1, when

$$\Theta^\top J\Theta = \Theta J\Theta^\top = J. \quad (24)$$

The J-orthogonality plays a crucial role in the hyperbolic QR factorization employed in our square-rooting method. We apply the above-mentioned algorithm of Bojanczyk et al. (2003), which fulfils a J-orthogonal QR decomposition with a signature of the form $J = \mathrm{diag}\{I_p, -I_s\}$, i.e. where all positive entries are placed in the beginning of its main diagonal and the remaining negative ones complete it. That is why we have changed the order of the summands on the right hand-side of formula (17) and, hence, the order of the blocks in pre-array (22) and the signature J in (23).

This hyperbolic QR decomposition method applied to the transposed matrix of pre-array (22) with the signature J from equation (23) returns the lower triangular post-array

$$R^\top = \left[\,S_{k-1|k-1}^{l+1} \;\; \mathbf{0}_{n\times(2n^2+n+1)}\,\right], \quad (25)$$

which is of size $n\times(2n^2+2n+1)$, with the notation $\mathbf{0}_{n\times(2n^2+n+1)}$ standing for the zero-block of size $n\times(2n^2+n+1)$. Eventually, we read-off the square block $S_{k-1|k-1}^{l+1}$ of size n, which constitutes the requested time-updated covariance matrix SR because equations (22)–(25) prove its SR condition (4) by the following formula chain:

$$P_{k-1|k-1}^{l+1} = S_X J S_X^\top = R^\top Q^\top J Q R = R^\top J R = [S_{k-1|k-1}^{l+1}][S_{k-1|k-1}^{l+1}]^\top. \quad (26)$$

The signature matrix J becomes the identity one after multiplication of its negative part with $\mathbf{0}_{n\times(2n^2+n+1)}$ in post-array (25) and, hence, vanishes in proof (26). This completes the time update in our JSR-5D-CKF. For convenience of practical use, we summarize the *time update* of this filter in the following condensed algorithmic form:

Given $\hat{X}_{k-1|k-1}$ and $S_{k-1|k-1}$ at time t_{k-1}, compute the predicted state mean $\hat{X}_{k|k-1}$ and covariance SR $S_{k|k-1}$ at time t_k. Set the local initial values $\hat{X}^0_{k-1|k-1} := \hat{X}_{k-1|k-1}$ and $S^0_{k-1|k-1} := S_{k-1|k-1}$ and fulfil the L-step time-update procedure with $\tau := (t_k - t_{k-1})/L$ as follows:
For $l = 0, 1, \dots, L-1$ do;
1) Assemble matrix (9) by means of formulas (10)–(13);
2) Set the time-updated matrix (14) with columns (15);
3) Compute the time-updated mean $\hat{X}^{l+1}_{k-1|k-1}$ by rule (16);
4) Set the time-updated predicted covariance array (22);
5) Fulfil the J-orthogonal QR factorization of the transposed array (22) with the signature J from formula (23);
6) Read-off the covariance SR $S^{l+1}_{k-1|k-1}$ in post-array (25).

We remark that vector (7), its absolute value SR (18), matrix (8), its absolute value SR (19) with signature (20) and matrix (11) as well as the process noise covariance Cholesky factorization $Q = Q^{1/2}Q^{\top/2}$ are time-invariant and computed only once before the state estimation run starts off. The predicted state mean $\hat{X}_{k|k-1} := \hat{X}^L_{k-1|k-1}$ and its covariance SR $S_{k|k-1} := S^L_{k-1|k-1}$ are further utilized in the measurement update step of our novel JSR-5D-CKF technique, as explained in the next section.

2.2 The Measurement Update Step in the JSR-5D-CKF

First, we set the matrix of the predicted cubature nodes

$$\mathcal{X}_{k|k-1} := \Big[\, \mathcal{X}_{1,k|k-1} \;\; \mathcal{X}_{2,k|k-1} \;\; \dots \;\; \mathcal{X}_{2n^2+1,k|k-1} \,\Big] \tag{27}$$

whose columns are determined in line with the formula

$$\mathcal{X}_{i,k|k-1} := \hat{X}_{k|k-1} + S_{k|k-1}\Gamma_i, i = 1, 2, \dots, 2n^2+1. \tag{28}$$

In (28), the mean $\hat{X}_{k|k-1}$ and covariance SR $S_{k|k-1}$ come from the time update of our JSR-5D-CKF and the vectors Γ_i have been already derived by (11)–(13) in Sec. 2.1.

Second, matrix (27) is then transformed to the form

$$\mathcal{Z}_{k|k-1} := \Big[\, \mathcal{Z}_{1,k|k-1} \;\; \mathcal{Z}_{2,k|k-1} \;\; \dots \;\; \mathcal{Z}_{2n^2+1,k|k-1} \,\Big] \tag{29}$$

with columns $\mathcal{Z}_{i,k|k-1} := h(\mathcal{X}_{i,k|k-1})$, $i = 1, 2, \dots, 2n^2+1$, i.e. these are yielded by the measurement function $h(\cdot)$ from equation (2). The coefficient matrix (8) and matrices (27) and (29) contribute to computation of the innovations, cross- and filtering covariances as follows:

$$P_{zz,k|k-1} := \mathcal{Z}_{k|k-1}\mathcal{W}\,\mathcal{Z}^\top_{k|k-1} + R_k, \tag{30}$$

$$P_{xz,k|k-1} := \mathcal{X}_{k|k-1}\mathcal{W}\,\mathcal{Z}^\top_{k|k-1}, \tag{31}$$

$$P_{k|k} = P_{k|k-1} - \mathbb{W}_k P_{zz,k|k-1}\mathbb{W}^\top_k \tag{32}$$

where R_k stands for the covariance of the measurement noise in model (2) and the Kalman gain obeys the formula

$$\mathbb{W}_k := P_{xz,k|k-1}P^{-1}_{zz,k|k-1}. \tag{33}$$

It is commonly accepted to square-root all the covariance matrices calculated by formulas (30)–(32) in the form of a unified coupled pre-array. Here, it is set up by the formula

$$B := \begin{bmatrix} R_k^{1/2} & \mathcal{Z}_{k|k-1}|\mathcal{W}|^{1/2} \\ \mathbf{0}_{n\times m} & \mathcal{X}_{k|k-1}|\mathcal{W}|^{1/2} \end{bmatrix} \tag{34}$$

where $R_k^{1/2}$ refers to the lower triangular Cholesky factor (SR) of the measurement noise covariance, i.e. $R_k = R_k^{1/2}R_k^{\top/2}$. Similar to the time update presented in Sec. 2.1, the above-mentioned hyperbolic QR decomposition code is applied to the transposed matrix of pre-array (34) with the signature matrix $J := \mathrm{diag}\{I_m, \mathcal{J}\}$, in which I_m is the identity matrix of size m and $\mathcal{J}$ obeys formula (20). The latter factorization returns the lower triangular post-array

$$R^\top = \begin{bmatrix} P^{1/2}_{zz,k|k-1} & \mathbf{0}_{m\times n} & \mathbf{0}_{m\times(2n^2-n+1)} \\ \bar{P}_{xz,k|k-1} & S_{k|k} & \mathbf{0}_{n\times(2n^2-n+1)} \end{bmatrix}, \tag{35}$$

which is of size $(m+n)\times(2n^2+m+1)$, with the matrix $\bar{P}_{xz,k|k-1}$ denoting the modified cross-covariance. This matrix amends the Kalman gain computation formula (33) to the more convenient and robust form

$$\mathbb{W}_k = \bar{P}_{xz,k|k-1}P^{-1/2}_{zz,k|k-1} \tag{36}$$

where the factor $P^{-1/2}_{zz,k|k-1}$ stands for the inverse of the SR $P^{1/2}_{zz,k|k-1}$. Note that the Kalman gain calculations (33) and (36) are mathematically equivalent, but possess the different numerical robustness to round-off. Formula (33) demands the full innovations covariance matrix $P_{zz,k|k-1}$ to be inverted, that is rather time-consuming and sensitive to round-off error accumulations. Whereas, formula (36) benefits from using the innovations covariance SR $P^{1/2}_{zz,k|k-1}$, which is a lower triangular matrix and, hence, entails the much cheaper and more stable inversion.

We further read-off the lower triangular $(m+n)\times(m+n)$-block S in the resulting post-array (35), which has the form

$$S := \begin{bmatrix} P^{1/2}_{zz,k|k-1} & \mathbf{0}_{m\times n} \\ \bar{P}_{xz,k|k-1} & S_{k|k} \end{bmatrix}. \tag{37}$$

It contains the innovations covariance SR $P^{1/2}_{zz,k|k-1}$, the modified cross-covariance $\bar{P}_{xz,k|k-1}$ and the filtering covariance SR $S_{k|k}$. These are read-off from matrix (37).

Next, we find the measurement mean by the inner product

$$\hat{Z}_{k|k-1} := \mathcal{Z}_{k|k-1}w \tag{38}$$

and complete this measurement update with calculating

$$\hat{X}_{k|k} = \hat{X}_{k|k-1} + \mathbb{W}_k(Z_k - \hat{Z}_{k|k-1}). \tag{39}$$

We recall that the predicted state mean $\hat{X}_{k|k-1}$ comes from the time update elaborated in Sec. 2.1. For convenience of practical use, we present this *measurement update* of JSR-5D-CKF in the following condensed algorithmic form:

Given the predicted state mean $\hat{X}_{k|k-1} := \hat{X}^L_{k-1|k-1}$ and covariance matrix SR $S_{k|k-1} := S^L_{k-1|k-1}$, compute the filtering state mean $\hat{X}_{k|k}$ and covariance matrix SR $S_{k|k}$ based on the measurement Z_k fulfilled at time t_k as follows:
1) Assemble matrix (27) by means of (11)–(13) and (28);
2) Set up the measurement-function-modified matrix (29);
3) Set up the unified coupled covariance pre-array (34);

4) Fulfil the J-orthogonal QR factorization of the transposed pre-array (34) with the signature $J := \mathrm{diag}\{I_m, \mathcal{J}\}$;
5) Compute the Kalman gain $\mathbb{W}_k$ by formula (36);
6) Compute the filtering mean $\hat{X}_{k|k}$ by formulas (38), (39);
7) Read-off the filtering covariance SR $S_{k|k}$ in array (37).

Further, we examine the novel JSR-5D-CKF method presented in Sec. 2.1 and 2.2 and compare it to its non-SR predecessor 5D-CKF designed by Santos-Diaz et al. (2018) in severe conditions of tackling a radar tracking problem of Arasaratnam et al. (2010), where an aircraft executes a coordinated turn. It is implemented with ill-conditioned measurements in our nonlinear stochastic scenario, below.

3. AIR TRAFFIC CONTROL SCENARIO WITH ILL-CONDITIONED MEASUREMENTS

The flight control scenario under consideration is a famous one in nonlinear filtering theory, which has been published with all particulars by Arasaratnam et al. (2010); Kulikov and Kulikova (2016, 2017a), etc. So, the interested reader is referred to the cited papers for more details. We simulate the turning aircraft dynamics for 150 s and set its angular velocity $\omega = 3^\circ$/s. The performance of our novel algorithm JSR-5D-CKF and its non-SR predecessor 5D-CKF devised by Santos-Diaz et al. (2018) is assessed in the sense of the *Accumulated Root Mean Square Errors* in position (ARMSE$_p$) and in velocity (ARMSE$_v$) defined as follows:

$$\mathrm{ARMSE}_p := \Big[\frac{1}{100K}\sum_{mc=1}^{100}\sum_{k=1}^{K}\big(x_{mc}^{\mathrm{true}}(t_k) - \hat{x}_{k|k}^{mc}\big)^2 + \big(y_{mc}^{\mathrm{true}}(t_k) - \hat{y}_{k|k}^{mc}\big)^2 + \big(z_{mc}^{\mathrm{true}}(t_k) - \hat{z}_{k|k}^{mc}\big)^2\Big]^{1/2},$$

$$\mathrm{ARMSE}_v := \Big[\frac{1}{100K}\sum_{mc=1}^{100}\sum_{k=1}^{K}\big(\dot{x}_{mc}^{\mathrm{true}}(t_k) - \hat{\dot{x}}_{k|k}^{mc}\big)^2 + \big(\dot{y}_{mc}^{\mathrm{true}}(t_k) - \hat{\dot{y}}_{k|k}^{mc}\big)^2 + \big(\dot{z}_{mc}^{\mathrm{true}}(t_k) - \hat{\dot{z}}_{k|k}^{mc}\big)^2\Big]^{1/2}$$

where $x_{mc}^{\mathrm{true}}(t_k)$, $y_{mc}^{\mathrm{true}}(t_k)$, $z_{mc}^{\mathrm{true}}(t_k)$ and $\dot{x}_{mc}^{\mathrm{true}}(t_k)$, $\dot{y}_{mc}^{\mathrm{true}}(t_k)$, $\dot{z}_{mc}^{\mathrm{true}}(t_k)$ stand for the aircraft's position and velocity simulated by the the Euler-Maruyama method with the small step size $\tau := 0.0005$ at time t_k in the mc-th Monte Carlo run (out of 100 independent simulations), $\hat{x}_{k|k}^{mc}$, $\hat{y}_{k|k}^{mc}$, $\hat{z}_{k|k}^{mc}$ and $\hat{\dot{x}}_{k|k}^{mc}$, $\hat{\dot{y}}_{k|k}^{mc}$, $\hat{\dot{z}}_{k|k}^{mc}$ denote the aircraft's position and velocity estimated by each filtering algorithm, k means the particular sampling time t_k and K refers to the total number of samples in the simulation interval $[0, 150\,\mathrm{s}]$. The sampling rate is limited to $\delta = 1\,\mathrm{s}$ in this flight control task.

In contrast to Arasaratnam et al. (2010), for provoking numerical instabilities in the filters under examination, we utilize the artificial measurement equation of the form

$$Z_k = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1+\sigma \end{bmatrix} X_k + V_k \qquad (40)$$

where the aircraft's state $X_k := [\, x_k \; \dot{x}_k \; y_k \; \dot{y}_k \; z_k \; \dot{z}_k \; \omega_k \,]^\top$ is estimated at time t_k and σ denotes a small positive real number determining ill-conditioning of model (40). Here, we address the cases of $\sigma = 1.0\text{e-}01, 1.0\text{e-}02, \ldots, 1.0\text{e-}11$. Also, each measurement Z_k is supposed to be corrupted by a normally distributed noise $V_k \sim \mathcal{N}(0, R_\sigma)$ with the covariance $R_\sigma = \sigma^2 I_2$ depending on the ill-conditioning parameter σ. Measurements (40) with the measurement

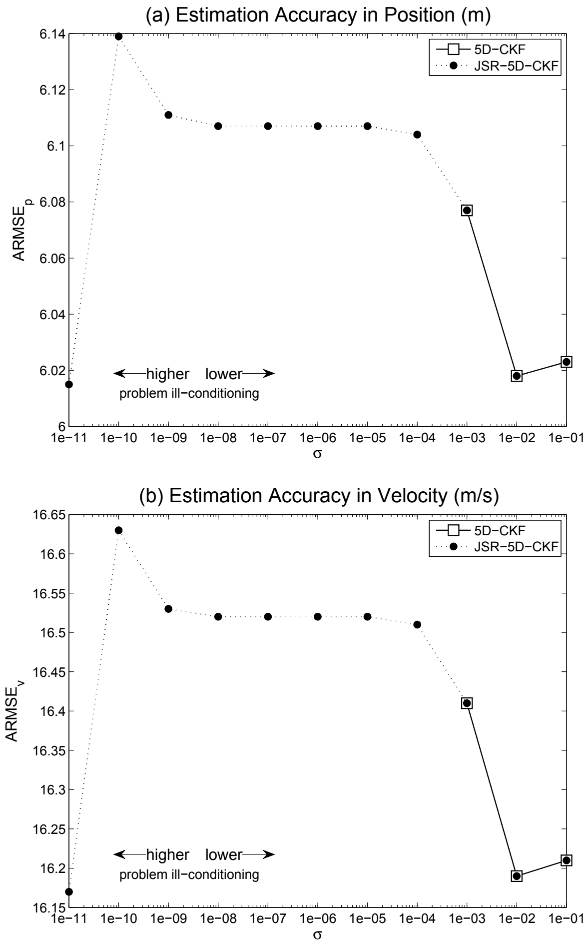

Fig. 1. The ARMSE$_p$ and ARMSE$_v$ observed in the target tracking scenario with the increasingly ill-conditioned measurement model (40) for the ill-conditioning parameter values $\sigma = 1.0\text{e-}01, 1.0\text{e-}02, \ldots, 1.0\text{e-}11$.

noise covariance matrix R_σ are typical means in numerical stability studies of various KF including the continuous-discrete and discrete-discrete methods presented by Dyer and McReynolds (1969); Grewal and Andrews (2001); Kulikov and Kulikova (2017c, 2018b, 2019). These correspond to the third reason of ill-conditioning elaborated by Grewal and Andrews (2001) because the matrix inversions in the Kalman gain computations (33) and (36) become increasingly ill-conditioned in line with the vanishing scalar σ.

We abbreviate our novel SR filter to `JSR-5D-CKF` and its non-SR predecessor published by Santos-Diaz et al. (2018) to `5D-CKF`, respectively. These methods are coded and run in MATLAB. The state estimators under consideration enjoy $L = 64$ subdivision steps in each sampling period. The last $2n$ entries in the fifth-degree spherical-radial cubature rule's coefficient vector (7) are negative because $n > 4$ in the target tracking scenario in use. This serves for the effective examination of `JSR-5D-CKF` and its valued comparison to `5D-CKF` in the presence of the increasingly ill-conditioned measurement model (40), i.e. when $\sigma \to 0$.

Fig 1 exhibits that the SR filter and its non-SR predecessor work identically and expose the same ARMSE$_p$ and ARMSE$_v$ when the ill-conditioning parameter $\sigma \geq 1.0\text{e-}03$, i.e. when our air traffic control scenario is rather wellconditioned. Then, we see that the non-SR `5D-CKF` fails at $\sigma = 1.0$e-04 because its covariance matrix computed loses the positivity and, hence, the Cholesky factorization may not be fulfilled. In contrast, the `JSR-5D-CKF` succeeds in producing the decent state estimates for all the values of the ill-conditioning parameter σ accepted in our case study. This confirms the sound numerical robustness of the filtering algorithm presented in Sec. 2 and establishes a solid background for its successful applications in practice.

4. CONCLUSION

This paper has addressed the issue of "the lack of a square-root implementation" and devised a square-root version of the Itô-Taylor-based Fifth-Degree Cubature Kalman Filter presented by Santos-Diaz et al. (2018). Taking into account the negativity of some weights in the fifth-degree spherical-radial cubature rule, which are possible in continuous-discrete stochastic scenarios of large size, we have applied the hyperbolic QR decomposition for designing our novel J-orthogonal square-root state estimator, which has been examined in severe conditions of tackling a radar tracking problem, where an aircraft executes a coordinated turn, in the presence of ill-conditioned measurements. The sound state estimation potential of this filter has been proven theoretically and evidenced numerically within the mentioned challenging stochastic flight control scenario.

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