

Sustainability Analysis of Interconnected Food Production Systems via Theory of Barriers

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Abstract: Controlled environment agriculture (CEA) is used for efficient food production. Efficiency can be increased further by interconnecting different CEA systems (e.g. plants and insect larvae or fish and larvae), using products and by-products of one system in the other. These interconnected systems define an overall system that can be described by models of interacting species. It is necessary to identify system parameters (e.g. initial species concentration, harvest rate, feed quality, etc.) such that the resources are not exhausted. For such systems with interacting species, modelled by the Lotka-Volterra equations, a set-based approach based on the recent results of the theory of barriers to exactly determine the so-called *admissible set* (also known as viability kernel) and the *maximal robust positively invariant set* is presented. Using an example of a larvae-fish based production system, steps to obtain special trajectories which are the boundaries of the admissible set are shown. This admissible set is used to prevent the under and over population of the species in the CEA. Furthermore, conditions of the system parameters are stated, such that the existence of these trajectories can be guaranteed.

Keywords: Nonlinear systems, Continuous time systems, System analysis, Constraint satisfaction problems, Sets, Controlled environment agriculture

1. INTRODUCTION

In recent years the general interest in sustainable ecosystems has increased, because of environmental changes and publicly well known challenges concerning, for example conservation of biodiversity, overexploitation of marine ecosystems, and exhaustion of fossil resources. It has also been recognised that human interventions yield ecological stresses with irreversible consequences. For instance, the loss of species or destruction of habitat. Therefore, sustainable management of ecological systems becomes more and more important as described in De Lara and Doyen (2008).

Food production industry is one of the biggest contributor to this ecosystem imbalance (Barnosky et al., 2011; Funabashi, 2018). Modern farming techniques such as controlled environment agriculture (CEA), also used in space research, are evaluated to improve the productivity and address the sustainability aspects through the use of different organisms (e.g. plant, fishes, insects) interacting through resource exchange (Conrad et al., 2017). Such studies are new in CEA and can utilize methods and tools developed for conventional farming and ecosystem management.

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The Lotka-volterra equations for describing the species interaction and the viability theory for finding the viability kernel to prevent species exhaustion, are widely used in the marine- and terrestrial ecosystem studies. For example, in Bayen and Rapaport (2019) the viability kernel was found to prevent the extinction of prey, in Eisenack et al. (2006) the viability theory was applied to design management framework for fisheries, and for exhaustible natural resource a viable control approach is presented via viability theory in (Martinet and Doyen, 2007).

The aforementioned species interaction dynamics are also used to address different aspects of agriculture and food production. Development of protocols for food production optimization (Fort et al., 2017), effects of organic management practices in pest control (Schmidt et al., 2014), optimization of harvesting policy (Zhang et al., 2000), design of ecosystems for nitrification in aquaponics system (Graham et al., 2007), and modeling the growth of *Hermetia illucens* (predator) (Gligorescu et al., 2019) are some of the examples applying these interaction dynamics.

In this paper we analyse the *admissible set* (also known as the viability kernel) and the *maximal robust positively invariant* (MRPI) set of the widely studied Lotka-Volterra model, with state and input constraints, on a food production system with species interaction. We use the theory of

barriers in constrained nonlinear systems as developed in De Dona and Lévine (2013) and Esterhuizen et al. (2019) to analyse these sets, describing how to construct them, and stating algebraic conditions of the system parameters under which they exist. We use these sets to obtain information pertaining to management strategies for sustainable operation of food production systems. We illustrate our results for a fish and insect-larvae example derived from a combination of aquaculture and insect farming under controlled environments.

The outline of this paper is as follows. In Section 2 a brief overview of the model and its relationship with controlled environment agriculture is established. In Section 3 we briefly cover the theory of barriers. Section 4 contains the main result of this paper with the analysis of the admissible and positively invariant set for the Lotka-Volterra equations. Section 5 presents a numerical examples and in Section 6 we discuss our analysis. We conclude the paper in Section 7 with a summary and ideas for future research.

2. IDEA OF SUSTAINABLE INTERCONNECTED CONTROLLED ENVIRONMENT AGRICULTURE

In an interconnected CEA system producing different biomass (plant, fish, insect larvae) interaction between the connected systems is established through the exchange of products and by-products (e.g. plant waste as feed for larvae, larvae as feed for fish, fish waste as nutrients for plant). In case of a single CEA system the interconnection can be represented by the species interacting with a food source (e.g. larvae feeding on larvae feed, plant consuming nutrients etc.)

Dynamics of such interactions between the species or between species and resources can be modelled using Lotka-Volterra equations. These equations are defined as follows

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 - \beta x_1 x_2 + u_1 x_1, \\ \dot{x}_2 &= \delta x_1 x_2 - \gamma x_2 + u_2 x_2, \end{aligned} \quad (1)$$

where $x_1 \in \mathbb{R}_{\geq 0}$ denotes the number of prey (e.g. plant, larvae) and $x_2 \in \mathbb{R}_{\geq 0}$ denotes the number of predators. (e.g. larvae, fish) The parameters α, β, γ and δ are positive constants. As per Getz (2012), the best applicable interpretation of the parameters of the Lotka-Volterra equations for biomass production are: α , the intrinsic growth rate of the resource (prey); β , the extraction rate per unit resource per unit consumer (predator); $\delta = \eta\beta$, where η is the biomass conversion parameter; and γ , the intrinsic rate of decline of the consumer in the absence of resource. Furthermore, $u_i(t) \in [u_i^{\min}, u_i^{\max}] \subset \mathbb{R}$ for all t , with $u_i^{\min} \leq u_i^{\max}$ $i = 1, 2$, are constrained inputs that model the intervention into the predator-prey system. In our framework of a CEA system the input is either the introduction of prey or the removal (harvest) of predators. Additionally we have to introduce the limits for the number of prey and predators. The lower limit is naturally given by zero, whereas for the goal of preventing extinction of a species, this number has to be positive. We also introduce an upper limit, since, we act in a controlled environment with a given limit of the habitat for the species, thus, we also want to prevent overpopulation (overproduction). Therefore we need to consider the following constraints $x_1(t) \in [\underline{x}_1, \bar{x}_1] \forall t, 0 \leq \underline{x}_1 < \bar{x}_1, x_2(t) \in [\underline{x}_2, \bar{x}_2] \forall t,$

$0 \leq \underline{x}_2 < \bar{x}_2$. These constraints describe the limits for the under- and overpopulation.

As mentioned in the introduction, the viability theory plays a decisive role for sustainable strategies related to different ecosystems. In particular, the identification of the viability kernel (or admissible set) for predator-prey systems yields information to deduce harvesting strategies which prevent the extinction of a species. In detail, this set describes the population number of predators and prey such that there exists at least one intervention strategy that results in the population remaining within the defined limits. On the boundary of the admissible set there exist *exactly one* intervention strategy that prevents under- or over-population. The MRPI is interpretable as the population of predators and prey for which every intervention is sustainable. In other word, there does not exist the possibility of crossing the limits for overpopulation or underpopulation. In contrast to the MRPI, every population number *outside* of the admissible set will lead to a violation of the constraints and therefore to an inevitable overpopulation or underpopulation.

3. SUMMARY OF THE THEORY OF BARRIERS

This section presents a succinct review of the theory of barriers as presented in De Dona and Lévine (2013) and Esterhuizen et al. (2019).

The considered nonlinear system subjected to state and input constraints is defined as follows:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0, \quad u \in \mathcal{U}, \quad (2)$$

$$g_i(x(t)) \leq 0, \forall t \in [t_0, \infty[, \quad i = 1, 2, \dots, p, \quad (3)$$

where $x(t) \in \mathbb{R}^n$ denotes the state; x_0 is the initial condition at the initial time, t_0 ; $u(t) \in \mathbb{R}^m$ denotes the input, and \mathcal{U} is the set of all Lebesgue measurable functions that map $[t_0, \infty[\subset \mathbb{R}$ into $U \subset \mathbb{R}^m$, with U compact and convex. The functions $g_i, i = 1, \dots, p$, are constraint functions imposed on the state. We impose the same assumptions on the problem data as those specified in De Dona and Lévine (2013) and Esterhuizen et al. (2019). Briefly, we assume that the functions f and g_i for $i = 1, \dots, p$ are C^2 with respect to their arguments on appropriate open sets; that all solutions of the system remain bounded on finite intervals; and that the set $\{f(x, u) : u \in U\}$ is convex. To lighten our notation, we introduce the following definitions:

$$G \triangleq \{x : g_i(x) \leq 0, \quad \forall i \in \{1, 2, \dots, p\}\}, \quad (4)$$

$$G_- \triangleq \{x : g_i(x) < 0, \quad \forall i \in \{1, 2, \dots, p\}\},$$

$$G_0 \triangleq \{x : \exists i \in \{1, 2, \dots, p\} \text{ s.t. } g_i(x) = 0\}.$$

The set of indices of active constraints at x is denoted by $\mathbb{I}(x) = \{i : g_i(x) = 0\}$. Furthermore, we denote by $L_f g(x, u)$ the Lie derivative of a continuously differentiable function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ with respect to $f(x, u)$ at the point x . Thus, $L_f g(x, u) = Dg(x)f(x, u)$. The boundary of a set S is denoted by ∂S . By $\mathbb{R}_{\geq 0}$ we refer to the set of nonnegative real numbers. In the following definitions, let $x^{(u, x_0, t_0)}(t)$ denote the solution to the differential equation at time t , initiating at x_0 at time t_0 , with input $u \in \mathcal{U}$.

Definition 1. The *admissible set*¹ of the system (2)-(3), denoted by \mathcal{A} , is the set of initial states for which there

¹ also called the viability kernel in Aubin (2009)

exists a $u \in \mathcal{U}$ such that the corresponding solution satisfies the constraints (3) for all future time.

$$\mathcal{A} \triangleq \left\{ x_0 \in \mathbb{R}^n : \exists u \in \mathcal{U}, x^{(u, x_0, t_0)}(t) \in G \ \forall t \in [t_0, \infty[\right\}.$$

Definition 2. A set $\Omega \subset \mathbb{R}^n$ is said to be a *robust positively invariant set* (RPI) of the system (2) provided that $x^{(u, x_0, t_0)}(t) \in \Omega$ for all $t \in [t_0, \infty[$, for all $x_0 \in \Omega$ and for all $u \in \mathcal{U}$.

Definition 3. The *maximal robust positively invariant set* (MRPI) of the system (2)-(3) contained in G , is the union of all RPIs that are subsets of G . Equivalently²

$$\mathcal{M} \triangleq \left\{ x_0 \in \mathbb{R}^n : x^{(u, x_0, t_0)}(t) \in G, \ \forall u \in \mathcal{U}, \ \forall t \in [t_0, \infty[\right\}.$$

We introduce: $[\partial\mathcal{A}]_- \triangleq \partial\mathcal{A} \cap G_-$ called the *barrier* and $[\partial\mathcal{M}]_- \triangleq \partial\mathcal{M} \cap G_-$, called the *invariance barrier*. These parts of the sets' boundaries are special, in that for any initial condition located on the barrier (resp. invariance barrier) there exists an input such that the resulting curve remains on the barrier (resp. invariance barrier) and satisfies a minimum-like (resp. maximum-like) principle. This is summarised in the following theorem.

Theorem 1. Under the assumptions in De Dona and Lévine (2013) and Esterhuizen et al. (2019), every integral curve $x^{\bar{u}}$ on $[\partial\mathcal{A}]_-$ (resp. $[\partial\mathcal{M}]_-$) and the corresponding input function \bar{u} satisfy the following necessary conditions. There exists a nonzero absolutely continuous maximal solution $\lambda^{\bar{u}}$ to the adjoint equation:

$$\dot{\lambda}^{\bar{u}}(t) = - \left(\frac{\partial f}{\partial x}(x^{\bar{u}}(t), \bar{u}(t)) \right)^T \lambda^{\bar{u}}(t),$$

such that

$$\min_{u \in U} \{ \lambda^{\bar{u}}(t)^T f(x^{\bar{u}}(t), u) \} = \lambda^{\bar{u}}(t)^T f(x^{\bar{u}}(t), \bar{u}(t)) = 0 \quad (5)$$

$$\left(\text{resp. } \max_{u \in U} \{ \lambda^{\bar{u}}(t)^T f(x^{\bar{u}}(t), u) \} = \lambda^{\bar{u}}(t)^T f(x^{\bar{u}}(t), \bar{u}(t)) = 0 \right), \quad (6)$$

for almost all t . Moreover, if $x^{\bar{u}}$ intersects G_0 in finite time, we have: $\lambda^{\bar{u}}(\bar{t}) = (Dg_{i^*}(z))^T$, where

$$\min_{u \in U} \max_{i \in \mathbb{I}(z)} L_f g_i(z, u) = L_f g_{i^*}(z, \bar{u}(\bar{t})) = 0 \quad (7)$$

$$\left(\text{resp. } \max_{u \in U} \max_{i \in \mathbb{I}(z)} L_f g_i(z, u) = L_f g_{i^*}(z, \bar{u}(\bar{t})) = 0 \right), \quad (8)$$

\bar{t} denotes the time at which G_0 is reached, and $z \triangleq x^{(\bar{u}, \bar{x}, t_0)}(\bar{t}) \in G_0$.

In addition to using the necessary conditions to analyse the predator-prey model (for which we will exactly describe the special control \bar{u} , and present conditions under which certain parts of the set \mathcal{A} exists), we will also use them to construct the sets as follows: first, identify points of *ultimate tangentiality* on G_0 , via (7) (resp. (8)). Second, determine the input realisation associated with $[\partial\mathcal{A}]_-$ (resp. $[\partial\mathcal{M}]_-$) using the *Hamiltonian minimisation* (resp. *maximisation*) condition (5) (resp. (6)). Third, integrate the system dynamics and the adjoint equations backwards in time from the points of ultimate tangentiality to obtain the barrier curves.

² shown in Esterhuizen et al. (2019), Proposition 2

Since the conditions in Theorem 1 are necessary, it may happen that some integral curves obtained through this method (or parts of them) do not define parts of the boundary of \mathcal{A} or \mathcal{M} and may have to be ignored. Thus, in general we refer to trajectories obtained from these conditions as *candidate barrier* and *candidate invariance barrier* trajectories.

4. SET-BASED ANALYSIS OF THE LOTKA-VOLTERRA EQUATIONS

In this section we present the analysis of the aforementioned sets. We also derive conditions under which candidate barrier trajectories exist.

Consider the Lotka-Volterra equations (1), the system's axes form two trivial robustly invariant manifolds, with the direction of flow along them determined by the signs of $\alpha + u_1^{\max}$ and $-\gamma + u_2^{\max}$. We use the necessary conditions of Theorem 1 to show that the constrained Lotka-Volterra model has a non-trivial MRPI if and only if $u_i^{\min} = u_i^{\max}$, $i = 1, 2$. We use the well-known fact that with a constant input the integral curves of the Lotka-Volterra model trace out periodic orbits, for example, see Lemma 1 in Bayen and Rapaport (2019).

Proposition 2. The Lotka-Volterra model as in (1) with $x(t) \in G \subset [0, \infty[\times [0, \infty[$ for all t , with G as defined in (4), and $u(t) \in U \triangleq [u_1^{\min}, u_1^{\max}] \times [u_2^{\min}, u_2^{\max}]$, has an MRPI that includes points not on the axes, if and only if U is a singleton.

Proof. Suppose the system has an MRPI as described, labelled \mathcal{M} , and let us concentrate on the invariance barrier trajectories on $[\partial\mathcal{M}]_-$. From Theorem 1, any curve on $[\partial\mathcal{M}]_-$ satisfies the Hamiltonian maximisation condition (6) for almost all t . Suppose now that there exists a $\tilde{u} \in U$ such that $\lambda^{\tilde{u}}(\hat{t})^T f(x^{\tilde{u}}(\hat{t}), \tilde{u}) < 0$ where \hat{t} is an arbitrary Lebesgue point of \bar{u} . Then, we could specify the constant input $u(t) \equiv \arg \min \lambda^{\tilde{u}}(\hat{t})^T f(x^{\tilde{u}}(\hat{t}), u)$, which would result in a periodic orbit that intersects \mathcal{M}^c , contradicting the fact that \mathcal{M} is robustly invariant. We conclude that for any trajectory on $[\partial\mathcal{M}]_-$, we have $\lambda^{\bar{u}}(t)^T f(x^{\bar{u}}(t), u) = 0$ for all $u \in U$ for almost all t . In particular,

$$\max_{u \in U} \lambda^{\bar{u}}(t)^T f(x^{\bar{u}}(t), u) = \min_{u \in U} \lambda^{\bar{u}}(t)^T f(x^{\bar{u}}(t), u) = 0, \quad (9)$$

for almost all t .

Let $\hat{u}(t) = (\hat{u}_1(t), \hat{u}_2(t))^T \triangleq \arg \max_u \lambda^{\bar{u}}(t)^T f(x^{\bar{u}}(t), u)$ and $\check{u}(t) = (\check{u}_1(t), \check{u}_2(t))^T \triangleq \arg \min_u \lambda^{\bar{u}}(t)^T f(x^{\bar{u}}(t), u)$. From the form of the Hamiltonian we see that the functions \hat{u}_i and \check{u}_i are always saturated, that is $\hat{u}_i(t) \in \{u_i^{\min}, u_i^{\max}\}$ and $\check{u}_i(t) \in \{u_i^{\min}, u_i^{\max}\}$, $i = 1, 2$. From (9) we can conclude that:

$$\lambda_1^{\bar{u}}(t) x_1^{\bar{u}}(t) (u_1^{\min} - u_1^{\max}) + \lambda_2^{\bar{u}}(t) x_2^{\bar{u}}(t) (u_2^{\min} - u_2^{\max}) = 0,$$

for almost all t . We now argue that $\lambda_1^{\bar{u}}(t)$ and $\lambda_2^{\bar{u}}(t)$ are nonzero for almost all time. Indeed, suppose $\lambda_1^{\bar{u}}(t) = 0$ over an interval of time. Then, $\lambda_1^{\bar{u}}(t) = 0$ over this same interval, which gives $-\delta x_2(t) \lambda_2^{\bar{u}}(t) = 0$, and thus, because the state is assumed to be positive, $\lambda_2^{\bar{u}}(t) = 0$ over this same interval. But this is impossible because $\lambda^{\bar{u}}(t) \neq 0$, from Theorem 1. A similar argument holds for $\lambda_2^{\bar{u}}$. Therefore, (9) holds if and only if $u_1^{\max} = u_1^{\min}$ and $u_2^{\max} = u_2^{\min}$ for almost all t .

Turning our attention to the set $[\partial\mathcal{M}]_0$, by a similar argument we require that

$$\min_{u \in U} \max_i L_f g_i(z, u) = \max_{u \in U} \max_i L_f g_i(z, u) = 0 \quad (10)$$

for points on $[\partial\mathcal{M}]_0$. Thus, either i) $Dg_{i^*} = 0$ and $u_2^{\max} = u_2^{\min}$; or ii) $Dg_{i^*2} = 0$ and $u_1^{\max} = u_1^{\min}$; or iii) $u_1^{\max} = u_1^{\min}$ and $u_2^{\max} = u_2^{\min}$, where $Dg_{i^*} = (Dg_{i^*1}, Dg_{i^*2})$, must hold on $[\partial\mathcal{M}]_0$. Focusing on case i), we see that this condition only holds if $x_1 = \frac{\gamma - u_2}{\delta}$, (with $u_2 = u_2^{\max} = u_2^{\min}$) and thus the condition only holds at an isolated point on the active constraint g_{i^*} . In order for this point to be robustly invariant, we would require $\dot{x}_1 = 0$, which can only be true if $u_1^{\max} = u_1^{\min}$. A similar argument for case ii) leads to $u_2^{\max} = u_2^{\min}$ for points on $[\partial\mathcal{M}]_0$. This completes the proof.

Though the constrained Lotka-Volterra model as specified above cannot have a nontrivial MRPI, it can have an admissible set, as we will show. We continue our analysis with the same constraints on u , $u_i(t) \in [u_i^{\min}, u_i^{\max}]$, and we let the state be constrained to a box: $x_i(t) \in [\underline{x}_i, \bar{x}_i]$ for all t , with $0 < \underline{x}_i < \bar{x}_i$.

4.1 Points of ultimate tangentiality for the admissible set

We label the state constraints as follows: $g_1(x) = x_1 - \bar{x}_1$, $g_2(x) = -x_1 + \underline{x}_1$, $g_3(x) = x_2 - \bar{x}_2$, $g_4(x) = -x_2 + \underline{x}_2$, and the i -th ultimate tangentiality point $z_i \triangleq (z_{i1}, z_{i2})^T \in \mathbb{R}^2$. First, we consider the constraints for the prey, i.e. $g_1(x)$ and $g_2(x)$. Invoking condition (7) we get:

$$\begin{aligned} \min_{u \in U} L_f g_1(z, u) &= \min_{u \in U} \{\alpha \bar{x}_1 - \beta \bar{x}_1 z_{12} + u_1 \bar{x}_1\} = 0, \\ \min_{u \in U} L_f g_2(z, u) &= \min_{u \in U} \{-\alpha \underline{x}_1 + \beta \underline{x}_1 z_{22} - u_1 \underline{x}_1\} = 0, \end{aligned}$$

from where we get the two points $z_1 = (\bar{x}_1, \frac{\alpha + u_1^{\min}}{\beta})$ for g_1 and $z_2 = (\underline{x}_1, \frac{\alpha + u_1^{\max}}{\beta})$ for g_2 . Now we focus on the predator constraints, $g_3(x)$ and $g_4(x)$. Again invoking condition (7) we get:

$$\begin{aligned} \min_{u \in U} L_f g_3(z, u) &= \min_{u \in U} \{\delta z_{31} \bar{x}_2 - \gamma \bar{x}_2 + u_2 \bar{x}_2\} = 0, \\ \min_{u \in U} L_f g_4(z, u) &= \min_{u \in U} \{-\delta z_{41} \underline{x}_2 + \gamma \underline{x}_2 - u_2 \underline{x}_2\} = 0, \end{aligned}$$

from where we identify $z_3 = (\frac{\gamma - u_2^{\min}}{\delta}, \bar{x}_2)$ for g_3 and $z_4 = (\frac{\gamma - u_2^{\max}}{\delta}, \underline{x}_2)$ for g_4 .

4.2 Input realisation associated with the barrier

Invoking condition (5) to determine the input associated with the barrier, we get:

$$\begin{aligned} \min_{u \in U} \{\lambda(t)^T f(x(t), u)\} &= \min_{u \in U} \{\lambda_1 (\alpha x_1 - \beta x_1 x_2 + u_1 x_1) \\ &\quad + \lambda_2 (\delta x_1 x_2 - \gamma x_2 + u_2 x_2)\} = 0 \end{aligned}$$

for almost every t , with $\lambda \triangleq (\lambda_1, \lambda_2)^T$. We get:

$$\bar{u}_1(t) = \begin{cases} u_1^{\min} & \text{if } \lambda_1(t) \geq 0 \\ u_1^{\max} & \text{if } \lambda_1(t) < 0, \end{cases} \quad (11)$$

and

$$\bar{u}_2(t) = \begin{cases} u_2^{\min} & \text{if } \lambda_2(t) \geq 0 \\ u_2^{\max} & \text{if } \lambda_2(t) < 0. \end{cases} \quad (12)$$

The adjoint equation is

$$\dot{\lambda} = \begin{pmatrix} -\alpha + \beta x_2 - u_1 & -\delta x_2 \\ \beta x_1 & -\delta x_1 + \gamma - u_2 \end{pmatrix} \lambda, \quad (13)$$

with the final conditions (from $\lambda(\bar{t}) = (Dg_{i^*}(z))^T$), $\lambda(\bar{t}) = (1, 0)^T, (-1, 0)^T, (0, 1)^T, (0, -1)^T$ associated with z_1, z_2, z_3 and z_4 , respectively.

We note that there are four lines in the state space (that intersect the four points of ultimate tangentiality) where the control \bar{u} switches, summarised in the following Proposition.

Proposition 3. Switches in \bar{u} occur as follows:

- If $x^{\bar{u}}(t) \in \{(x_1, x_2) : \delta x_1 - \gamma + u_2^{\max} = 0\}$ and $\lambda_2(t) < 0$, then \bar{u}_1 switches from u_1^{\max} to u_1^{\min} .
- If $x^{\bar{u}}(t) \in \{(x_1, x_2) : \delta x_1 - \gamma + u_2^{\min} = 0\}$ and $\lambda_2(t) > 0$, then \bar{u}_1 switches from u_1^{\min} to u_1^{\max} .
- If $x^{\bar{u}}(t) \in \{(x_1, x_2) : \alpha - \beta x_2 + u_1^{\max} = 0\}$ and $\lambda_1(t) < 0$, then \bar{u}_2 switches from u_2^{\min} to u_2^{\max} .
- If $x^{\bar{u}}(t) \in \{(x_1, x_2) : \alpha - \beta x_2 + u_1^{\min} = 0\}$ and $\lambda_1(t) > 0$, then \bar{u}_2 switches from u_2^{\max} to u_2^{\min} .

Proof. From condition (5), we see that if $\lambda_1(t) = 0$, then $\lambda_2(t)(\delta x_1(t)x_2(t) - \gamma x_2(t) + \bar{u}_2(t)x_2(t)) = 0$, and using the fact that $\lambda_2(t) \neq 0$ and $x_2(t) \neq 0$, we see that if $\lambda_2(t) < 0$ then a switch in $\bar{u}_1(t)$ occurs on the line segment given by $\{(x_1, x_2) : \delta x_1 - \gamma + u_2^{\max}\}$. Because $\lambda_2(t) < 0$, we have $\dot{\lambda}_1(t) > 0$, implying $\lambda_1(t) < 0$ on an interval before t , implying $\bar{u}_1(t) = u_1^{\max}$ before t . This same argument carries over to the remaining three cases.

4.3 Existence of candidate barrier trajectories

As already mentioned in Section 3 the conditions stated in Theorem 1 are necessary. Therefore, some integral curves or parts of them may need to be ignored since they do not define the boundary of the sets. In particular, integral curves evolving outside the constrained state space need to be ignored. The next proposition gives conditions under which a point of ultimate tangentiality is indeed associated with a candidate barrier trajectory evolving backwards into G_- .

Proposition 4. There exists a candidate barrier trajectory associated with \mathcal{A} , partly contained in G_- and ending at the point of ultimate tangentiality

$$\begin{aligned} z_1 &\Leftrightarrow (\delta \bar{x}_1 - \gamma + u_2^{\max}) > 0, \quad z_2 \Leftrightarrow (\delta \underline{x}_1 - \gamma + u_2^{\min}) < 0, \\ z_3 &\Leftrightarrow (\alpha - \beta \bar{x}_2 + u_1^{\min}) < 0, \quad z_4 \Leftrightarrow (\alpha - \beta \underline{x}_2 + u_1^{\max}) > 0. \end{aligned}$$

Proof. Consider g_1 and the corresponding point of ultimate tangentiality z_1 along with the final adjoint $\lambda(\bar{t}) = (1, 0)^T$. From (13) it follows that $\dot{\lambda}_2(\bar{t}) = \beta \bar{x}_1 \lambda_1(\bar{t}) > 0$, which implies that $\lambda_2(t) < 0$ and $\bar{u}_2(t) = u_2^{\max}$ over a time interval before \bar{t} . Because $\lambda_2(t) < 0$ over this interval, the integral curve associated with \bar{u} will evolve backwards into G_- if and only if $\dot{x}_2(\bar{t}) > 0$, which gives us the statement in the first bullet point. Similar arguments carry over to the other three points, which completes the proof.

5. NUMERICAL EXAMPLE

This example is based on the recent trend in aquaculture to feed fish with insect larvae-based food for efficient fish

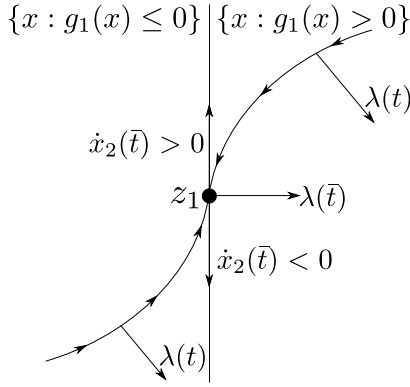


Fig. 1. Clarification of the proof of Proposition 4: knowing that $\lambda_2(t) < 0$ before the barrier intersects z_1 , we must have $\dot{x}_2(\bar{t}) > 0$.

production. The environments of the species are often decoupled. Nevertheless, we assume that the fish are continuously supplied with larvae on which to feed. The prey are represented by the larvae of *Hermetia illucens* insects and the predators by *Oreochromis niloticus* fish. The goal of this example is to show the application of the proposed set-based method for sustainable production of larvae and fish. First, we impose state constraints to prevent underpopulation of both species with $g_2(x) \triangleq -0.5 + x_1$ and $g_4(x) \triangleq -0.5 + x_2$. The input bounds are $u_1 \in [10, 20]$ and $u_2 \in [-10, -20]$. The model parameters α, β, γ and δ are derived from the parameters k_1, k_4 (Diener et al., 2009), k_2, k_3, k_5 (Terpstra, 2015) and η_1 (Rana et al., 2015; Stamer et al., 2014) as listed in Table 1. Since α and γ are vastly smaller than β and δ as well

Table 1. Lotka-Volterra model parameters for fish (*Oreochromis niloticus*) and larvae (*Hermetia illucens*) production system

parameter and meaning	value	unit
k_1 larval growth from egg to adult	30	[d]
k_2 fish growth from egg to adult	225	[d]
k_3 larvae consumption rate of fish	22	[gd ⁻¹]
k_4 weight of adult larva	0.120	[g]
k_5 weight of adult fish	700	[g]
η_1 conversion eff. of larval food	0.55	[-]
α $1/k_1$	0.0014	[h ⁻¹]
β $k_3/(k_4k_5)$	10.8	[kggh ⁻¹]
δ $\eta_1\beta$	5.94	[kggh ⁻¹]
γ $1/k_2$	1.85×10^{-5}	[h ⁻¹]

as the input bounds, they do not play a decisive role for the dynamics. Hence, we set them to zero. We identify: $z_2 \approx (0.5, 1.852)^T$ and $z_4 \approx (1.543, 0.5)^T$. By integrating backwards, we find two candidate barrier trajectories defining the boundary of \mathcal{A} , shown in Figure 2.

Next, we additionally impose state constraints to prevent overpopulation with $g_1(x) \triangleq x_1 - 10$ and $g_3(x) \triangleq x_2 - 10$. We identify: $z_1 \approx (10, 0.926)^T$ and $z_3 \approx (3.086, 10)^T$ and integrate backwards from all four points of tangentiality. We find four candidate barrier trajectories, shown in Figure 3. The difference to the example before is the existence of a trajectory evolving into G_- which does not define a part of the boundary of the admissible set. Hence, we have to ignore the dash-dotted line drawn from z_3 and ending on g_1 because g_2 contains points for

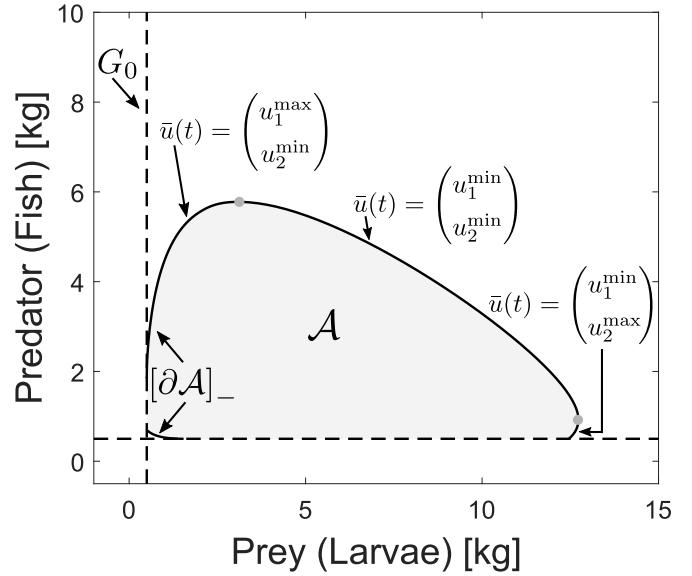


Fig. 2. Admissible set for a system with interacting larvae and fish to prevent underpopulation. The grey points indicate the switching of $\bar{u}(t)$, associated with the barrier trajectory initiating from z_2 .

which $\min_{u \in U} L_f g_2(x, u) > 0$. Thus, the boundary of the admissible set inside the constrained state space is defined by the three integral curves ending on z_1, z_2 , and z_4 shown by the continuous lines.

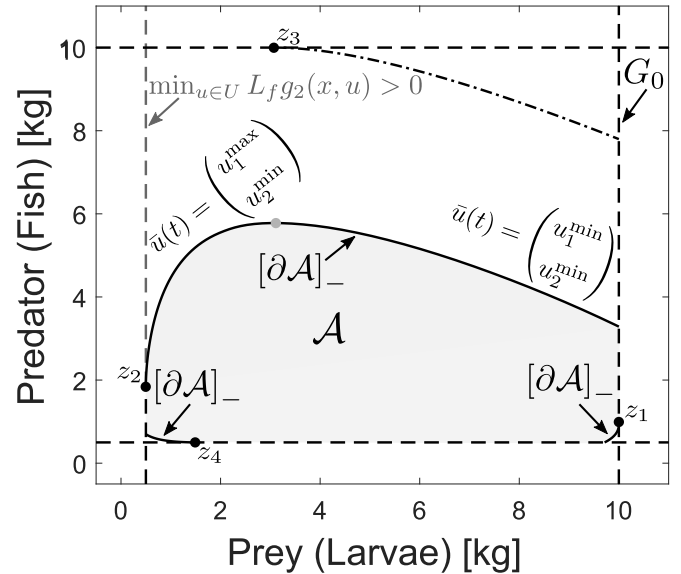


Fig. 3. Admissible set for a system with interacting larvae and fish. The input realisation $\bar{u}(t)$ associated with the barrier trajectory initiating from z_2 switches at the grey point.

6. DISCUSSION

It is interesting to note that other works in the literature that concentrate on biological applications of the viability kernel (or admissible set) describe its boundary, and argue that it is made up of special integral curves of the system. A similar statement is made in the theory of barriers, the main observation being that these curves

satisfy a minimum/maximum-like principle, as described in Theorem 1. In this new paradigm it can be seen that the description of the boundary is much simpler than the arguments in, for example, Bayen and Rapaport (2019).

Recall that the considered predator-prey system never has a (nontrivial) MRPI. Thus, the interpretation is that if one can control the introduction or harvesting of species, then this has to be done carefully, because there always exists an input (intervention strategy) that results in over- or under-population of a species in the future. Thus, it is vital that the population number remains in \mathcal{A} . Furthermore, the unique input on the boundary of \mathcal{A} is always saturated, and it switches according to the statement in Proposition 3. For a given system, similar to the model of the CEA system presented in this work, the proposed method can be used to set up the initial population as well as intervention strategies through the addition and/or removal of the species.

7. CONCLUSION

The main goal of this paper was to analyse CEA systems with interacting species, modelled by the Lotka-Volterra equations, to identify the admissible set (also known as viability kernel) and the MRPI which yields information about sustainable intervention strategies. We used the theory of barriers to determine special trajectories that define the boundaries of the admissible set and discovered new aspects, such as the nonexistence of nontrivial MRPIs in the constrained Lotka-Volterra model. Furthermore, we obtained conditions of the systems' parameters that guarantees the existence of these special trajectories. We illustrated our results based on a CEA system with larvae and fish as interacting species. Future research could focus on extending the results to robust admissible sets, as in Regnier and De Lara (2015), in the context of viability theory. Another interesting idea could be to analyse the sets of an extended Lotka-Volterra model that describes the effects of humans feeding one species to another.

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