Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Data-Driven Tracking MPC for Changing Setpoints

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Abstract: We propose a data-driven tracking model predictive control (MPC) scheme to control unknown discrete-time linear time-invariant systems. The scheme uses a purely data-driven system parametrization to predict future trajectories based on behavioral systems theory. The control objective is tracking of a given input-output setpoint. We prove that this setpoint is exponentially stable for the closed loop of the proposed MPC, if it is reachable by the system dynamics and constraints. For an unreachable setpoint, our scheme guarantees closed-loop exponential stability of the optimal reachable equilibrium. Moreover, in case the system dynamics are known, the presented results extend the existing results for model-based setpoint tracking to the case where the stage cost is only positive semidefinite in the state. The effectiveness of the proposed approach is illustrated by means of a practical example.

Keywords: Predictive control, data-based control, tracking.

1. INTRODUCTION

In the behavioral approach to control, it was proven by Willems et al. (2005) that a single, persistently exciting trajectory of a linear time-invariant (LTI) system suffices to reproduce all system trajectories, without explicitly identifying the system. Recently, various contributions have exploited this result to develop data-driven analysis and control methods in the classical state-space control framework. For instance, the work of Willems et al. (2005) is used for data-driven dissipativity verification in Romer et al. (2019) and it is extended to certain classes of nonlinear systems in Berberich and Allgöwer (2020). Further, state- and output-feedback controllers from noise-free data are designed in De Persis and Tesi (2019), whereas Berberich et al. (2020) consider robust controller design from noisy data. Finally, van Waarde et al. (2019) investigate data-driven control problems without requiring persistently exciting data. These works illustrate a potential advantage of direct data-driven methods compared to identification-based controller design, leading to simple implementations with desirable end-to-end-guarantees. Similarly, data-driven model predictive control (MPC) schemes based on Willems et al. (2005) have been suggested in Yang and Li (2015) as well as Coulson et al. (2019a,b). A first theoretical analysis of such an MPC scheme is provided in Berberich et al. (2019), which utilizes terminal equality constraints to prove closed-loop exponential stability, also for the case of output measurement noise. The present paper extends the approach of Berberich et al. (2019) to tracking of setpoints, which may be unreachable and change online, by introducing an artificial equilibrium to the MPC scheme which is optimized online. Compared to the MPC scheme of Berberich et al. (2019), this has the additional advantages of improved robustness, a larger region of attraction, and the possibility to consider setpoints, which are not feasible equilibria for the *unknown* system. The basic idea of optimizing an artificial setpoint online was introduced in the model-based setting in Limón et al. (2008), and extended, e.g., to nonlinear systems in Limón et al. (2018) or to dynamic reference trajectories in Köhler et al. (2019). In this paper, we prove closed-loop recursive feasibility, constraint satisfaction, as well as exponential stability of the desired equilibrium, if it is reachable by the dynamics and constraints, and of the optimal reachable equilibrium otherwise. Our work provides a solution to the data-driven tracking problem, which has only recently been addressed under significantly more restrictive conditions on the data, cf. Salvador et al. (2019). Further, our main technical contribution is also an extension of the model-based tracking MPC approach of Limón et al. (2008), where so far only positive definite stage costs have been considered.

The paper is structured as follows. After introducing some preliminaries in Section 2, we state the data-driven tracking MPC scheme in Section 3, and we prove the desired properties of the closed loop. The applicability of the presented method is illustrated via an example in Section 4, and the paper is concluded in Section 5.

⋆ This work was funded by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germanys Excellence Strategy - EXC 2075 - 390740016. The authors thank the International Max Planck Research School for Intelligent Systems (IMPRS-IS) for supporting Julian Berberich, and the International Research Training Group Soft Tissue Robotics (GRK 2198/1).

2. PRELIMINARIES

2.1 Notation

$\mathbb{I}_{[a,b]}$ denotes the set of integers in the interval $[a,b]$. For a vector x and a symmetric matrix $P = P^\top \succeq 0$, we write $\|x\|_P = \sqrt{x^\top P x}$. Further, we denote the minimal and maximal eigenvalue of P by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively. For multiple matrices P_i, $i = 1,\dots,n$, we write $\lambda_{\min}(P_1,\dots,P_n) = \min\{\lambda_{\min}(P_1),\dots,\lambda_{\min}(P_n)\}$, and similarly for the maximal eigenvalue. Moreover, $\|x\|_2$, $\|x\|_1$, and $\|x\|_\infty$ denote the standard Euclidean, ℓ_1-, and ℓ_∞-norm of $x \in \mathbb{R}^n$, respectively. If the argument is matrix-valued, then we mean the induced norm. For a sequence $\{x_k\}_{k=0}^{N-1}$, we define the Hankel matrix

$$H_L(x) := \begin{bmatrix} x_0 & x_1 & \dots & x_{N-L} \\ x_1 & x_2 & \dots & x_{N-L+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{L-1} & x_L & \dots & x_{N-1} \end{bmatrix}.$$

A stacked window of the sequence is written as

$$x_{[a,b]} = \begin{bmatrix} x_a \\ \vdots \\ x_b \end{bmatrix}.$$

Moreover, we write x either for the sequence itself or for the stacked vector $x_{[0,N-1]}$.

2.2 Data-driven system representation

We consider the following standard definition of persistence of excitation.

Definition 1. We say that a signal $\{u_k\}_{k=0}^{N-1}$ with $u_k \in \mathbb{R}^m$ is persistently exciting of order L if $\text{rank}(H_L(u)) = mL$.

Throughout the paper, we consider discrete-time LTI systems of order n with m inputs and p outputs. It will be assumed that the system order n is known, but all of our results remain true if n is replaced by an upper bound.

Definition 2. We say that an input-output sequence $\{u_k, y_k\}_{k=0}^{N-1}$ is a trajectory of an LTI system G, if there exists an initial condition $\bar{x} \in \mathbb{R}^n$ as well as a state sequence $\{x_k\}_{k=0}^{N}$ such that

$$x_{k+1} = Ax_k + Bu_k,\ \ x_0 = \bar{x}$$
$$y_k = Cx_k + Du_k,$$

for $k = 0,\dots,N-1$, where (A,B,C,D) is a minimal realization of G.

Throughout this paper, we assume that the input-output behavior of the unknown system can be explained by a (controllable and observable) minimal realization (cf. Definition 2). The following is the main result of Willems et al. (2005) and will be of central importance throughout the paper. While it is originally formulated in the behavioral approach, we employ the formulation in the state-space control framework from Berberich and Allgöwer (2020).

Theorem 3. Suppose $\{u_k^d, y_k^d\}_{k=0}^{N-1}$ is a trajectory of an LTI system G, where u^d is persistently exciting of order $L+n$. Then, $\{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1}$ is a trajectory of G if and only if there exists $\alpha \in \mathbb{R}^{N-L+1}$ such that

$$\begin{bmatrix} H_L(u^d) \\ H_L(y^d) \end{bmatrix} \alpha = \begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix}. \tag{1}$$

Theorem 3 states that the data-dependent Hankel matrices in (1) span the trajectory space of the system G, given that the input data is persistently exciting. Equivalently, one input-output trajectory can be used to reconstruct any other system trajectory, by forming linear combinations of its time-shifts. In Section 3, we use Theorem 3 to set up a data-driven MPC scheme and we prove desirable closed-loop properties. In particular, we show closed-loop exponential stability of the optimal reachable input-output equilibrium, as defined in the following section.

2.3 Input-output equilibria

Since state measurements are not available in the present setting, we define an equilibrium in terms of an input-output pair as follows.

Definition 4. We say that an input-output pair $(u^s, y^s) \in \mathbb{R}^{m+p}$ is an equilibrium of an LTI system G, if the sequence $\{\bar{u}_k, \bar{y}_k\}_{k=0}^{n-1}$ with $(\bar{u}_k, \bar{y}_k) = (u^s, y^s)$ for all $k \in \mathbb{I}_{[0,n-1]}$ is a trajectory of G.

Thus, an input-output pair (u^s, y^s) is an equilibrium if, while applying n times the input u^s to the system, the output stays constant at y^s. This implies that the internal state in any minimal realization, denoted by x^s, stays constant. We write u_n^s and y_n^s for vectors containing n times the input and output component of an equilibrium, respectively. In this paper, we use MPC to steer the system towards a target input-output setpoint, while the input and output satisfy pointwise-in-time constraints, i.e., $u_t \in \mathbb{U}, y_t \in \mathbb{Y}$, for all $t \in \mathbb{N}$. The sets $\mathbb{U}$ and $\mathbb{Y}$ are assumed to be convex. We do not require that the target setpoint $(u^T, y^T) \in \mathbb{R}^{m+p}$ is an equilibrium of the system or satisfies the constraints, but our scheme will achieve convergence to the *optimal reachable equilibrium*. For matrices $S \succeq 0, T \succ 0$, the optimal reachable equilibrium (u^{sr}, y^{sr}) is defined as the feasible equilibrium (u^s, y^s), which minimizes $\|u^s - u^T\|_S^2 + \|y^s - y^T\|_T^2$. Due to a local controllability argument in the proof of our main result, only equilibria which are strictly inside the constraints can be considered, i.e., equilibria which lie in some convex set $\mathbb{U}^s \times \mathbb{Y}^s \subseteq \text{int}(\mathbb{U} \times \mathbb{Y})$. Let $\{u^d, y^d\}_{k=0}^{N-1}$ be a measured trajectory of G and suppose that u^d is persistently exciting of order $2n$. Then, (u^{sr}, y^{sr}) is the minimizer of

$$J_{eq}^*(u^T, y^T) = \min_{u^s, y^s, \alpha} \ \|u^s - u^T\|_S^2 + \|y^s - y^T\|_T^2 \tag{2}$$
$$s.t. \quad \begin{bmatrix} H_n(u^d) \\ H_n(y^d) \end{bmatrix} \alpha = \begin{bmatrix} u_n^s \\ y_n^s \end{bmatrix},$$
$$u^s \in \mathbb{U}^s,\ y^s \in \mathbb{Y}^s.$$

Although this is not required for the implementation of the proposed scheme, Problem (2) can be used to compute (u^{sr}, y^{sr}) from measured data. Clearly, Problem (2) is convex. Similar to Limón et al. (2008); Köhler et al. (2019), we require that it is even strongly convex, as captured in the following assumption.

Assumption 5. The optimization problem (2) is strongly convex w.r.t. (u^s, y^s).

Assumption 5 implies that the optimal reachable equilibrium (u^{sr}, y^{sr}) is unique, and it is, e.g., satisfied if $S \succ 0$.

More generally, in case that $S \succeq 0$, it is also satisfied if $D = 0$ and the matrix B has full column rank in some (and hence any) minimal realization, since these two conditions imply that there exists a unique equilibrium input for any equilibrium output. Throughout this paper, it will be assumed that B has full column rank. This assumption excludes over-actuated systems and guarantees the existence of a unique equilibrium input-output pair for any steady state. To be more precise, it holds for any equilibrium (u^s, y^s) with corresponding steady state x^s that

$$(I - A)x^s = Bu^s, \quad y^s = Cx^s + Du^s. \tag{3}$$

Since B has full column rank, (3) implies

$$\|u^s\|_2^2 + \|y^s\|_2^2 \leq c_{x,1}\|x^s\|_2^2, \tag{4}$$

for $c_{x,1} = \|C\|_2^2 + (1+\|D\|_2^2)\left\|B^\dagger(I - A)\right\|_2^2$, where $B^\dagger$ is the Moore-Penrose inverse of B. Conversely, since we consider minimal realizations, it follows directly from the system dynamics that there exists a constant $c_{x,2} > 0$ such that

$$\|x^s\|_2^2 \leq c_{x,2}(\|u^s\|_2^2 + \|y^s\|_2^2). \tag{5}$$

3. DATA-DRIVEN TRACKING MPC

In this section, we propose a data-driven MPC scheme for setpoint tracking, which is essentially a combination of the schemes from Berberich et al. (2019) and Limón et al. (2008). To be more precise, the scheme relies on Theorem 3 to predict future trajectories and contains a standard tracking cost as well as terminal equality constraints w.r.t. an input-output setpoint, similar to Berberich et al. (2019). Moreover, in line with Limón et al. (2008), this setpoint is optimized online and its deviation from the desired target setpoint is penalized in the cost. After stating the scheme in Section 3.1, we prove recursive feasibility, closed-loop constraint satisfaction, and exponential stability of the optimal reachable equilibrium in Section 3.2.

3.1 Tracking MPC scheme

To set up a data-driven MPC scheme based on Theorem 3, we require three ingredients: a) an initially measured input-output data trajectory $\{u_k^d, y_k^d\}_{k=0}^{N-1}$ used for prediction via Theorem 3, b) past n input-output measurements $u_{[t-n,t-1]}$, $y_{[t-n,t-1]}$ to specify initial conditions, and c) a target input-output setpoint (u^T, y^T). Given these components, we define the open-loop optimal control problem

$$J_L^*(u_{[t-n,t-1]}, y_{[t-n,t-1]}, u^T, y^T) =$$

$$\min_{\substack{\alpha(t),u^s(t),y^s(t)\\ \bar{u}(t),\bar{y}(t)}} \sum_{k=0}^{L-1} \|\bar{u}_k(t) - u^s(t)\|_R^2 + \|\bar{y}_k(t) - y^s(t)\|_Q^2$$

$$+ \|u^s(t) - u^T\|_S^2 + \|y^s(t) - y^T\|_T^2$$

$$s.t. \quad \begin{bmatrix}\bar{u}_{[-n,L-1]}(t)\\ \bar{y}_{[-n,L-1]}(t)\end{bmatrix} = \begin{bmatrix}H_{L+n}(u^d)\\ H_{L+n}(y^d)\end{bmatrix}\alpha(t), \tag{6a}$$

$$\begin{bmatrix}\bar{u}_{[-n,-1]}(t)\\ \bar{y}_{[-n,-1]}(t)\end{bmatrix} = \begin{bmatrix}u_{[t-n,t-1]}\\ y_{[t-n,t-1]}\end{bmatrix}, \tag{6b}$$

$$\begin{bmatrix}\bar{u}_{[L-n,L-1]}(t)\\ \bar{y}_{[L-n,L-1]}(t)\end{bmatrix} = \begin{bmatrix}u_n^s(t)\\ y_n^s(t)\end{bmatrix}, \tag{6c}$$

$$\bar{u}_k(t) \in \mathbb{U},\ \bar{y}_k(t) \in \mathbb{Y},\ k \in \mathbb{I}_{[0,L-1]}, \tag{6d}$$

$$(u^s(t), y^s(t)) \in \mathbb{U}^s \times \mathbb{Y}^s. \tag{6e}$$

The constraint (6a) replaces the model and parametrizes all possible trajectories of the unknown LTI system, assuming persistence of excitation of the input u^d. Moreover, the initial and terminal constraints over n steps in (6b) and (6c) imply that the states $\bar{x}_0(t)$ and $\bar{x}_{L-1}(t)$, corresponding to the predicted input-output trajectory $\{\bar{u}_k(t), \bar{y}_k(t)\}_{k=0}^{L-1}$, are equal to the internal state x_t and the setpoint x^s, respectively, in any minimal realization. Problem (6) is similar to the nominal MPC problem proposed in Berberich et al. (2019), with the main difference that the desired input-output setpoint is replaced by an artificial equilibrium $(u^s(t), y^s(t))$, which is optimized online. Moreover, its distance w.r.t. (u^T, y^T) is penalized in the cost. Note that the terminal constraint (6c) and the dynamics (6a) imply that $(u^s(t), y^s(t))$ is indeed an equilibrium of the system.

Problem (6) requires only a single measured input-output trajectory and can thus be implemented directly, without any model knowledge. As an advantage over the existing data-driven MPC scheme of Berberich et al. (2019), the target setpoint (u^T, y^T) can be arbitrary and is not required to be reachable or even an equilibrium for the unknown system dynamics. If $\mathbb{U}$ and $\mathbb{Y}$ are polytopic (quadratic), then (6) is a convex (quadratically constrained) quadratic program, which can be solved efficiently. As is standard in MPC, Problem (6) is solved in a receding horizon fashion, compare Algorithm 6.

Algorithm 6. Data-Driven Tracking MPC Scheme

(1) At time t, take the past n measurements $u_{[t-n,t-1]}, y_{[t-n,t-1]}$ and solve (6).
(2) Apply the input $u_t = \bar{u}_0^*(t)$.
(3) Set $t = t + 1$ and go back to (1).

We assume for the stage cost that $Q, R \succ 0$. The open-loop cost and the optimal open-loop cost of (6) are denoted by $J_L\left(x_t, u^T, y^T, u^s(t), y^s(t), \alpha(t)\right)$ and $J_L^*\left(x_t, u^T, y^T\right)$, where x_t is the state at time t, corresponding to $(u_{[t-n,t-1]}, y_{[t-n,t-1]})$ in some minimal realization.

3.2 Closed-loop guarantees

In this section, we prove that the MPC scheme defined via (6) exponentially stabilizes the optimal reachable equilibrium state x^{sr} and thus, also the input and output converge exponentially to u^{sr} and y^{sr}, respectively. For this, (u^T, y^T) is not required to satisfy the constraints or to be an equilibrium in the sense of Definition 4, in which case $(u^{sr}, y^{sr}) \neq (u^T, y^T)$. As in the model-based case, recursive feasibility of the scheme will be guaranteed, even if the target setpoint (u^T, y^T) changes online. As an additional technical contribution, the result of this section extends the (model-based) setpoint tracking MPC analysis of Limón et al. (2008) to the case that the stage cost is only positive semidefinite in the state. This is relevant in a model-based setting, e.g., if input-output models are used for prediction.

As will become clear in the proof of our main result, the fact that the stage cost may not be positive definite in the state complicates the analysis of the MPC scheme. To overcome this issue, we analyze the closed loop of the proposed (1-step) MPC scheme over n consecutive time steps and show a desired Lyapunov function decay over n steps. Moreover, as in Berberich et al. (2019), we exploit detectability of the stage cost via an input-output-to-state stability (IOSS) Lyapunov function. For some state in an (observable) minimal realization, there exists an IOSS Lyapunov function $W(x) = \|x\|_P^2$ with $P \succ 0$ which satisfies

$$W(Ax+Bu) - W(x) \leq -\frac{1}{2}\|x\|_2^2 + c_1\|u\|_2^2 + c_2\|y\|_2^2,$$

for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y = Cx + Du$, with some $c_1, c_2 > 0$, cf. Cai and Teel (2008). For some $\gamma > 0$, we define a Lyapunov function candidate based on the IOSS Lyapunov function W, the optimal cost J_L^*, and the cost of the optimal reachable equilibrium $J_{eq}^*(u^T, y^T)$, cf. (2), as

$$V(x_t, u^T, y^T) = J_L^*(x_t, u^T, y^T) + \gamma W(x_t - x^{sr}) - J_{eq}^*(u^T, y^T),$$

where x^{sr} is the state corresponding to (u^{sr}, y^{sr}). Using this Lyapunov function candidate, the following result proves recursive feasibility, constraint satisfaction, and exponential stability of the closed loop.

Theorem 7. Suppose that $\mathbb{U}$ and $\mathbb{Y}$ are compact, $L \geq 2n$, Assumption 5 holds, and u^d is persistently exciting of order $L + 2n$. If the MPC problem (6) is feasible at initial time $t = 0$, then

(a) it is feasible at any $t \in \mathbb{N}$,
(b) the closed loop satisfies the constraints, i.e., $u_t \in \mathbb{U}$ and $y_t \in \mathbb{Y}$ for all $t \in \mathbb{N}$,
(c) the optimal reachable equilibrium x^{sr} is exponentially stable for the resulting closed loop.

Proof. (a). Recursive Feasibility
For the artificial equilibrium, we choose as a candidate at time $t+1$ the previously optimal one, i.e., $u^{s\prime}(t+1) = u^{s*}(t), y^{s\prime}(t+1) = y^{s*}(t)$. Moreover, for the input-output predictions, we consider the standard candidate solution, consisting of the previously optimal solution shifted by one step and appended by $(u^{s\prime}(t+1), y^{s\prime}(t+1))$, i.e.,

$$\bar{u}_k'(t+1) = \bar{u}_{k+1}^*(t),\ \bar{y}_k'(t+1) = \bar{y}_{k+1}^*(t),\ k \in \mathbb{I}_{[-n,L-2]},$$

and $\bar{u}_{L-1}'(t+1) = u^{s\prime}(t+1), \bar{y}_{L-1}'(t+1) = y^{s\prime}(t+1)$. Finally, according to Theorem 3, there exists an $\alpha'(t+1)$ satisfying (6a).

(b). Constraint Satisfaction
This follows directly from recursive feasibility, together with Theorem 3 and the constraints of (6).

(c). Exponential Stability
We first show that the Lyapunov function candidate V is quadratically lower and upper bounded. Thereafter, we prove that V is non-increasing and decreases exponentially over n time steps, which implies exponential stability for the closed loop.

(c.1). Lower Bound on V
Using that $J_{eq}^*(u^T, y^T) \leq \|u^s - u^T\|_S^2 + \|y^s - y^T\|_T^2$ for any equilibrium (u^s, y^s) satisfying the constraints of (2), V is quadratically lower bounded as

$$\gamma\lambda_{\min}(P)\|x_t - x^{sr}\|_2^2 \leq V(x_t, u^T, y^T).$$

(c.2). Local upper Bound on V
Let x_t satisfy $\|x_t - x^{sr}\|_2 \leq \delta$ for a sufficiently small $\delta > 0$. Since $(u^{sr}, y^{sr}) \in \mathbb{U}^s \times \mathbb{Y}^s \subseteq \text{int}(\mathbb{U} \times \mathbb{Y})$, $L \geq 2n$, and by controllability, there exists a feasible input-output trajectory steering the state to x^{sr} in $n \leq L - n$ steps, while satisfying

$$\left\| \begin{bmatrix} \bar{u}_{[0,L-1]}(t) - u_L^{sr} \\ \bar{y}_{[0,L-1]}(t) - y_L^{sr} \end{bmatrix} \right\|_2^2 \leq \Gamma_{uy}\|x_t - x^{sr}\|_2^2 \tag{7}$$

for some $\Gamma_{uy} > 0$. For the artificial equilibrium, we consider the candidate solution $u^s(t) = u^{sr}, y^s(t) = y^{sr}$. Finally, by Theorem 3, there exists an $\alpha(t)$ satisfying (6a), which implies that the defined trajectory satisfies all constraints of (6). Hence, a local quadratic upper bound on V can be obtained as

$$V(x_t, u^T, y^T) \leq (\Gamma_{uy}\lambda_{\max}(Q,R) + \gamma\lambda_{\max}(P))\|x_t - x^{sr}\|_2^2.$$

(c.3). Exponential Decay of V
Define n candidate solutions for $i \in \mathbb{I}_{[1,n]}$, similar to Part (a) of the proof, as

$$\bar{u}_k'(t+i) = \bar{u}_{k+1}^*(t+i-1),\ \bar{y}_k'(t+i) = \bar{y}_{k+1}^*(t+i-1),$$

for $k \in \mathbb{I}_{[-n,L-2]}$, and $\bar{u}_{L-1}'(t+i) = u^{s\prime}(t+i), \bar{y}_{L-1}'(t+i) = y^{s\prime}(t+i)$. The candidate artificial equilibria are defined as

$$u^{s\prime}(t+i) = u^{s*}(t+i-1),\ y^{s\prime}(t+i) = y^{s*}(t+i-1),$$

and $\alpha'(t+i)$ as a corresponding solution to (1). Using this candidate solution, it is readily shown that, for any $i \in \mathbb{I}_{[1,n]}$, the optimal cost is non-increasing with

$$\begin{aligned} &J_L^*(x_{t+i}, u^T, y^T) \\ \leq &J_L(x_{t+i}, u^T, y^T, u^{s\prime}(t+i), y^{s\prime}(t+i), \alpha'(t+i)) \\ = &J_L^*(x_{t+i-1}, u^T, y^T) - \|u_{t+i-1} - u^{s*}(t+i-1)\|_R^2 \\ &- \|y_{t+i-1} - y^{s*}(t+i-1)\|_Q^2. \end{aligned} \tag{8}$$

We derive now a decay-bound of J_L^* over n steps by studying different cases.

Case 1: Assume

$$\begin{aligned} &\sum_{i=0}^{n-1}\|u_{t+i} - u^{s*}(t+i)\|_R^2 + \|y_{t+i} - y^{s*}(t+i)\|_Q^2 \\ &\geq \gamma_1\sum_{i=0}^{n-1}\|u^{s*}(t+i) - u^{sr}\|_S^2 + \|y^{s*}(t+i) - y^{sr}\|_T^2, \end{aligned} \tag{9}$$

for a constant $\gamma_1 > 0$, which will be fixed later in the proof. It follows from (8) that the optimal cost over n steps decreases as

$$\begin{aligned} &J_L^*(x_{t+n}, u^T, y^T) - J_L^*(x_t, u^T, y^T) \\ &= \sum_{i=0}^{n-1} J_L^*(x_{t+i+1}, u^T, y^T) - J_L^*(x_{t+i}, u^T, y^T) \\ &\leq -\sum_{i=0}^{n-1}\left(\|u_{t+i} - u^{s*}(t+i)\|_R^2 + \|y_{t+i} - y^{s*}(t+i)\|_Q^2\right), \end{aligned} \tag{10}$$

where a telescoping sum argument is used for the first equality. Using $a^2 + b^2 \geq \frac{1}{2}(a+b)^2$, (10) implies

$$\begin{aligned} &J_L^*(x_{t+n}, u^T, y^T) - J_L^*(x_t, u^T, y^T) \\ &\overset{(9)}{\leq} -\frac{\gamma_1}{2}\sum_{i=0}^{n-1}\left(\|u^{s*}(t+i) - u^{sr}\|_S^2 + \|y^{s*}(t+i) - y^{sr}\|_T^2\right) \\ &- \frac{1}{2}\sum_{i=0}^{n-1}\left(\|u_{t+i} - u^{s*}(t+i)\|_R^2 + \|y_{t+i} - y^{s*}(t+i)\|_Q^2\right) \\ &\leq -c_3\frac{\min\{1,\gamma_1\}}{4}\sum_{i=0}^{n-1}\left(\|u_{t+i} - u^{sr}\|_2^2 + \|y_{t+i} - y^{sr}\|_2^2\right), \end{aligned} \tag{11}$$

where $c_3 = \lambda_{\min}(Q,R,S,T)$.
Case 2: Assume

$$\sum_{i=0}^{n-1}\|u_{t+i}-u^{s*}(t+i)\|_R^2+\|y_{t+i}-y^{s*}(t+i)\|_Q^2 \le \gamma_1\sum_{i=0}^{n-1}\|u^{s*}(t+i)-u^{sr}\|_S^2+\|y^{s*}(t+i)-y^{sr}\|_T^2. \quad (12)$$

Case 2a: Assume further

$$\sum_{i=0}^{n-1}\|x_{t+i}-x^{s*}(t+i)\|_2^2 \le \gamma_2\sum_{i=0}^{n-1}\|x^{s*}(t+i)-x^{sr}\|_2^2, \quad (13)$$

for a constant $\gamma_2>0$, which will be fixed later in the proof. We consider now a different candidate solution at time $t+1$ with artificial equilibrium $\hat{u}^s(t+1)=\lambda u^{s*}(t)+(1-\lambda)u^{sr}$, $\hat{y}^s(t+1)=\lambda y^{s*}(t)+(1-\lambda)y^{sr}$ for some $\lambda\in(0,1)$, which will be fixed later in the proof. Clearly, this is a feasible equilibrium and it holds for the corresponding equilibrium state that $\hat{x}^s(t+1)=\lambda x^{s*}(t)+(1-\lambda)x^{sr}$. Further,

$$\hat{x}^s(t+1)-x^{s*}(t)=(1-\lambda)(x^{sr}-x^{s*}(t)), \quad (14)$$

and similarly for the input and output. Due to compactness of $\mathbb{U},\mathbb{Y}$, the right-hand side of (13) is bounded from above by $\gamma_2 x_{\max}$ for some $x_{\max}>0$. Thus, if γ_2 is sufficiently small, then $\sum_{i=0}^{n-1}\|x_{t+i}-x^{s*}(t+i)\|_2^2$ is arbitrarily small as well. Hence, if in addition $(1-\lambda)$ is sufficiently small, then, by controllability and since $(\hat{u}^s(t+1),\hat{y}^s(t+1))\in\text{int}(\mathbb{U}\times\mathbb{Y})$, there exists a feasible input-output trajectory $\hat{u}(t+1),\hat{y}(t+1)$ steering the system to $(\hat{u}^s(t+1),\hat{y}^s(t+1))$ in n steps. Moreover, there exists a corresponding $\hat{\alpha}(t+1)$ satisfying all constraints of (6). Further, it holds that

$$\begin{aligned}
&\|\hat{u}^s(t+1)-u^T\|_S^2-\|u^{s*}(t)-u^T\|_S^2\\
=&(\hat{u}^s(t+1)-u^{s*}(t))^\top S\left(\hat{u}^s(t+1)+u^{s*}(t)-2u^T\right)\\
=&(1-\lambda)\left(u^{sr}-u^{s*}(t)\right)^\top S\\
&\cdot\left((1+\lambda)u^{s*}(t)+(1-\lambda)u^{sr}-2u^T\right)\\
=&-(1-\lambda^2)\|u^{s*}(t)-u^{sr}\|_S^2\\
&-2(1-\lambda)(u^{s*}(t)-u^{sr})^\top S(u^{sr}-u^T)\\
\le&-(1-\lambda^2)\|u^{s*}(t)-u^{sr}\|_S^2,
\end{aligned}$$

where the last inequality follows from noting that $2S(u^{sr}-u^T)$ is the gradient of $\|u-u^T\|_S^2$ evaluated at u^{sr}, and the directional derivative of this function towards any other feasible direction increases, due to convexity of (2) by Assumption 5 (compare Köhler et al. (2019) for details). Similarly,

$$\begin{aligned}
\|\hat{y}^s(t+1)-y^T\|_T^2-\|y^{s*}(t)-y^T\|_T^2&\\
\le-(1-\lambda^2)\|y^{s*}(t)-y^{sr}\|_T^2&.
\end{aligned}$$

By controllability, there exists $\Gamma_{uy}>0$ as in (7) such that

$$\begin{aligned}
&\sum_{k=0}^{L-1}\|\hat{u}_k(t+1)-\hat{u}^s(t+1)\|_R^2+\|\hat{y}_k(t+1)-\hat{y}^s(t+1)\|_Q^2\\
\le&\Gamma_{uy}\lambda_{\max}(Q,R)\|x_{t+1}-\hat{x}^s(t+1)\|_2^2\\
\le&2\Gamma_{uy}\lambda_{\max}(Q,R)(\|x_{t+1}-x^{s*}(t)\|_2^2\\
&\qquad+\|x^{s*}(t)-\hat{x}^s(t+1)\|_2^2),
\end{aligned}$$

using again the fact that $(a+b)^2\le 2a^2+2b^2$. The first term can be bounded as

$$\begin{aligned}
\|x_{t+1}-x^{s*}(t)\|_2^2\le&\|A\|_2^2\|x_t-x^{s*}(t)\|_2^2\\
&+\|B\|_2^2\|u_t-u^{s*}(t)\|_2^2\\
\le&\underbrace{(\|A\|_2^2+\|B\|_2^2\Gamma_{uy})}_{c_4:=}\|x_t-x^{s*}(t)\|_2^2,
\end{aligned}$$

where the last inequality follows again from a controllability argument. Combining the bounds leads to

$$\begin{aligned}
&J_L^*(x_{t+1},u^T,y^T)-J_L^*(x_t,u^T,y^T)\\
&\le 2c_4\Gamma_{uy}\lambda_{\max}(Q,R)\|x_t-x^{s*}(t)\|_2^2\\
&+2\Gamma_{uy}\lambda_{\max}(Q,R)(1-\lambda)^2\|x^{sr}-x^{s*}(t)\|_2^2\\
&-\|u_t-u^{s*}(t)\|_R^2-\|y_t-y^{s*}(t)\|_Q^2\\
&-(1-\lambda^2)(\|u^{s*}(t)-u^{sr}\|_S^2+\|y^{s*}(t)-y^{sr}\|_T^2).
\end{aligned}$$

Using a similar candidate solution at time $t+i$ for $i\in\mathbb{I}_{[2,n]}$, the following can be shown.

$$\begin{aligned}
&J_L^*(x_{t+n},u^T,y^T)-J_L^*(x_t,u^T,y^T)\\
&\le 2c_4\Gamma_{uy}\lambda_{\max}(Q,R)\sum_{i=0}^{n-1}\|x_{t+i}-x^{s*}(t+i)\|_2^2\\
&+2\Gamma_{uy}\lambda_{\max}(Q,R)(1-\lambda)^2\sum_{i=0}^{n-1}\|x^{sr}-x^{s*}(t+i)\|_2^2\\
&-\sum_{i=0}^{n-1}(\|u_{t+i}-u^{s*}(t+i)\|_R^2+\|y_{t+i}-y^{s*}(t+i)\|_Q^2)\\
&-(1-\lambda^2)\sum_{i=0}^{n-1}(\|u^{s*}(t+i)-u^{sr}\|_S^2+\|y^{s*}(t+i)-y^{sr}\|_T^2)\\
&\overset{(5),(13)}{\le}(2c_4\gamma_2+2(1-\lambda)^2)\Gamma_{uy}\lambda_{\max}(Q,R)c_{x,2}\\
&\quad\cdot\sum_{i=0}^{n-1}(\|u^{s*}(t+i)-u^{sr}\|_2^2+\|y^{s*}(t+i)-y^{sr}\|_2^2)\\
&-\sum_{i=0}^{n-1}(\|u_{t+i}-u^{s*}(t+i)\|_R^2+\|y_{t+i}-y^{s*}(t+i)\|_Q^2\\
&-(1-\lambda^2)\sum_{i=0}^{n-1}(\|u^{s*}(t+i)-u^{sr}\|_S^2+\|y^{s*}(t+i)-y^{sr}\|_T^2)\\
&\le -c_5\sum_{i=0}^{n-1}(\|u_{t+i}-u^{sr}\|_2^2+\|y_{t+i}-y^{sr}\|_2^2),
\end{aligned}$$

for some $c_5>0$, where the last inequality holds if γ_2 is sufficiently small and λ is sufficiently close to 1.
Case 2b: Assume

$$\sum_{i=0}^{n-1}\|x_{t+i}-x^{s*}(t+i)\|_2^2\ge\gamma_2\sum_{i=0}^{n-1}\|x^{s*}(t+i)-x^{sr}\|_2^2. \quad (15)$$

This implies the existence of an index $k\in\mathbb{I}_{[0,n-1]}$ such that

$$\|x_{t+k}-x^{s*}(t+k)\|_2^2\ge\frac{\gamma_2}{n}\sum_{i=0}^{n-1}\|x^{s*}(t+i)-x^{sr}\|_2^2. \quad (16)$$

The following auxiliary result will be central for the proof of Case 2b.
Lemma 8. There exist $\gamma_3>0, j\in\mathbb{I}_{[0,n-1]}$ such that

$$\begin{aligned}&\|u_{t+j}-u^{s*}(t+k)\|_2^2+\|y_{t+j}-y^{s*}(t+k)\|_2^2 \\ \geq&\gamma_3\sum_{i=0}^{n-1}\|u^{s*}(t+i)-u^{sr}\|_2^2+\|y^{s*}(t+i)-y^{sr}\|_2^2,\end{aligned}\tag{17}$$

with k as in (16).

Proof. Using the system dynamics, it holds that

$$\begin{aligned}\|x_{t+k}-x^{s*}(t+k)\|_2^2\leq a_1\|x_t-x^{s*}(t+k)\|_2^2 \\ +a_2\|u_{[t,t+k-1]}-u_k^{s*}(t+k)\|_2^2,\end{aligned}\tag{18}$$

for suitable $a_1, a_2 > 0$. Further, for the observability matrix Φ and a suitable matrix Γ, which depends on B, C, D, we obtain

$$\begin{aligned}y_{[t,t+n-1]}-y_n^{s*}(t+k)=\Phi(x_t-x^{s*}(t+k)) \\ +\Gamma(u_{[t,t+n-1]}-u_n^{s*}(t+k)).\end{aligned}$$

By observability, we can solve the latter equation for $x_t - x^{s*}(t+k)$, which leads, together with (18), to

$$\begin{aligned}&\|x_{t+k}-x^{s*}(t+k)\|_2^2 \\ \leq&a_3\sum_{i=0}^{n-1}(\|u_{t+i}-u^{s*}(t+k)\|_2^2+\|y_{t+i}-y^{s*}(t+k)\|_2^2),\end{aligned}\tag{19}$$

for a suitable $a_3 > 0$. Let j be the index, for which

$$\|u_{t+j}-u^{s*}(t+k)\|_2^2+\|y_{t+j}-y^{s*}(t+k)\|_2^2$$

is maximal, which implies

$$\begin{aligned}&\|u_{t+j}-u^{s*}(t+k)\|_2^2+\|y_{t+j}-y^{s*}(t+k)\|_2^2 \\ \geq&\frac{1}{n}\sum_{i=0}^{n-1}\|u_{t+i}-u^{s*}(t+k)\|_2^2+\|y_{t+i}-y^{s*}(t+k)\|_2^2,\end{aligned}\tag{20}$$

which in turn leads to

$$\begin{aligned}&\|x_{t+k}-x^{s*}(t+k)\|_2^2 \\ \leq&a_3n(\|u_{t+j}-u^{s*}(t+k)\|_2^2+\|y_{t+j}-y^{s*}(t+k)\|_2^2).\end{aligned}\tag{21}$$

Combining (21) with (16) and (4) concludes the proof of Lemma 8. □

It follows from (12) and (17) that $j \neq k$, as long as $\gamma_1 < \gamma_3$, which will be assumed in the following. At time $t+j$, we define now a different candidate solution as a convex combination between the optimal solution and the candidate solution from Case 1, i.e.,

$$\bar{u}''(t+j)=\beta\bar{u}'(t+j)+(1-\beta)\bar{u}^*(t+j),$$

for some $\beta \in [0,1]$, with $\bar{u}'(t+j)$ as in the beginning of Part (c.3) of the proof. The other variables $\bar{y}''(t+j), u^{s\prime\prime}(t+j), y^{s\prime\prime}(t+j), \alpha''(t+j)$ are defined analogously. By convexity, this is a feasible solution to (6). Hence,

$$\begin{aligned}&J_L^*(x_{t+j},u^T,y^T) \\ \leq&J_L(x_{t+j},u^T,y^T,u^{s\prime\prime}(t+j),y^{s\prime\prime}(t+j),\alpha''(t+j)) \\ \leq&\beta J_L(x_{t+j},u^T,y^T,u^{s\prime}(t+j),y^{s\prime}(t+j),\alpha'(t+j)) \\ &+(1-\beta)J_L^*(x_{t+j},u^T,y^T) \\ &-2\bar{c}\beta(1-\beta)\|u^{s*}(t+j)-u^{s\prime}(t+j)\|_2^2 \\ &-2\bar{c}\beta(1-\beta)\|y^{s*}(t+j)-y^{s\prime}(t+j)\|_2^2, \\ \overset{(8)}{\leq}&\beta J_L^*(x_{t+j-1},u^T,y^T)+(1-\beta)J_L^*(x_{t+j},u^T,y^T) \\ &-2\bar{c}\beta(1-\beta)\|u^{s*}(t+j)-u^{s\prime}(t+j)\|_2^2 \\ &-2\bar{c}\beta(1-\beta)\|y^{s*}(t+j)-y^{s\prime}(t+j)\|_2^2,\end{aligned}$$

where the second inequality follows from strong convexity of (2) for some $\bar{c} > 0$. Fixing $\beta = \frac{1}{2}$ and dividing by β, we obtain

$$\begin{aligned}&J_L^*(x_{t+j},u^T,y^T)-J_L^*(x_{t+j-1},u^T,y^T) \\ \leq&-\bar{c}\|u^{s*}(t+j)-u^{s*}(t+j-1)\|_2^2 \\ &-\bar{c}\|y^{s*}(t+j)-y^{s*}(t+j-1)\|_2^2.\end{aligned}$$

Suppose now $j > k$. Then, by defining similar candidate solutions at time instances $t+k, \ldots, t+j$, and applying $a^2+b^2 \geq \frac{1}{2}(a+b)^2$ repeatedly $j-k-1$ times, we obtain

$$\begin{aligned}&J_L^*(x_{t+j},u^T,y^T)-J_L^*(x_{t+k},u^T,y^T)\leq-\frac{\bar{c}}{2^{j-k-1}}\cdot \\ &\left(\|u^{s*}(t+j)-u^{s*}(t+k)\|_2^2+\|y^{s*}(t+j)-y^{s*}(t+k)\|_2^2\right).\end{aligned}$$

Conversely, if $k > j$, we arrive at

$$\begin{aligned}&J_L^*(x_{t+k},u^T,y^T)-J_L^*(x_{t+j},u^T,y^T)\leq-\frac{\bar{c}}{2^{k-j-1}}\cdot \\ &\left(\|u^{s*}(t+j)-u^{s*}(t+k)\|_2^2+\|y^{s*}(t+j)-y^{s*}(t+k)\|_2^2\right).\end{aligned}$$

Combining the two cases and noting that $j \neq k$ (cf. Lemma 8), it follows from (8) that

$$\begin{aligned}&J_L^*(x_{t+n},u^T,y^T)-J_L^*(x_t,u^T,y^T) \\ \leq&-\frac{\bar{c}}{2^{|j-k|-1}}\|u^{s*}(t+j)-u^{s*}(t+k)\|_2^2 \\ &-\frac{\bar{c}}{2^{|j-k|-1}}\|y^{s*}(t+j)-y^{s*}(t+k)\|_2^2.\end{aligned}\tag{22}$$

Now, we bound the right-hand side of (22) by using

$$a+b=\sqrt{(a+b)^2}\geq\sqrt{a^2+b^2},\tag{23a}$$

$$a+b=\sqrt{(a+b)^2}\leq\sqrt{2a^2+2b^2},\tag{23b}$$

which hold for any $a, b \geq 0$. In the following, let γ_1 be sufficiently small such that $\gamma_1 < \min\{1, \frac{1}{8c_6}\}\gamma_3$, where $c_6 = \frac{\lambda_{\max}(S,T)}{\lambda_{\min}(Q,R)} > 0$. Note that this implies $\gamma_1 < \gamma_3$ and hence $j \neq k$, as is required above. Moreover,

$$\begin{aligned}&\|u^{s*}(t+j)-u^{s*}(t+k)\|_2+\|y^{s*}(t+j)-y^{s*}(t+k)\|_2 \\ \geq&\|u_{t+j}-u^{s*}(t+k)\|_2-\|u_{t+j}-u^{s*}(t+j)\|_2 \\ &+\|y_{t+j}-y^{s*}(t+k)\|_2-\|y_{t+j}-y^{s*}(t+j)\|_2 \\ \overset{(17),(23a)}{\geq}&\frac{1}{2}\|u_{t+j}-u^{s*}(t+k)\|_2+\frac{1}{2}\|y_{t+j}-y^{s*}(t+k)\|_2 \\ &+\frac{\sqrt{\gamma_3}}{2}\sqrt{\sum_{i=0}^{n-1}\|u^{s*}(t+i)-u^{sr}\|_2^2+\|y^{s*}(t+i)-y^{sr}\|_2^2} \\ &-\|u_{t+j}-u^{s*}(t+j)\|_2-\|y_{t+j}-y^{s*}(t+j)\|_2 \\ \overset{(12),(23b)}{\geq}&\frac{1}{2}\|u_{t+j}-u^{s*}(t+k)\|_2+\frac{1}{2}\|y_{t+j}-y^{s*}(t+k)\|_2 \\ &+\left(\frac{\sqrt{\gamma_3}}{2}-\sqrt{2\gamma_1c_6}\right) \\ &\cdot\sqrt{\sum_{i=0}^{n-1}\|u^{s*}(t+i)-u^{sr}\|_2^2+\|y^{s*}(t+i)-y^{sr}\|_2^2} \\ \overset{(20)}{\geq}&\frac{1}{2}\sqrt{\frac{1}{n}\sum_{i=0}^{n-1}\|u_{t+i}-u^{s*}(t+k)\|_2^2+\|y_{t+i}-y^{s*}(t+k)\|_2^2} \\ &+\left(\frac{\sqrt{\gamma_3}}{2}-\sqrt{2\gamma_1c_6}\right) \\ &\cdot\sqrt{\|u^{s*}(t+k)-u^{sr}\|_2^2+\|y^{s*}(t+k)-y^{sr}\|_2^2} \\ \geq&c_7\sum_{i=0}^{n-1}\|u_{t+i}-u^{sr}\|_2+\|y_{t+i}-y^{sr}\|_2,\end{aligned}$$

for a suitable $c_7 > 0$, where the last inequality follows from an inequality similar to (23b). This, together with (22) implies the existence of a constant $c_8 > 0$ such that

$$\begin{aligned} &J_L^*(x_{t+n}, u^T, y^T) - J_L^*(x_t, u^T, y^T) \\ &\leq -c_8 \sum_{i=0}^{n-1} \left(\|u_{t+i} - u^{sr}\|_2^2 + \|y_{t+i} - y^{sr}\|_2^2\right). \end{aligned}$$

Combination: Combining all cases, there exists some $c_9 > 0$ such that

$$\begin{aligned} &J_L^*(x_{t+n}, u^T, y^T) - J_L^*(x_t, u^T, y^T) \\ &\leq -c_9 \sum_{i=0}^{n-1} \|u_{t+i} - u^{sr}\|_2^2 + \|y_{t+i} - y^{sr}\|_2^2. \end{aligned} \tag{24}$$

Furthermore, the IOSS Lyapunov function W satisfies

$$\begin{aligned} &W(x_{t+1} - x^{sr}) - W(x_t - x^{sr}) \\ =&W(A(x_t - x^{sr}) + B(u_t - u^{sr})) - W(x_t - x^{sr}) \\ \leq& -\frac{1}{2}\|x_t - x^{sr}\|_2^2 + c_1\|u_t - u^{sr}\|_2^2 + c_2\|y_t - y^{sr}\|_2^2. \end{aligned} \tag{25}$$

Applying this inequality recursively, we arrive at

$$\begin{aligned} &W(x_{t+n} - x^{sr}) - W(x_t - x^{sr}) \\ &\leq \sum_{i=0}^{n-1} -\frac{1}{2}\|x_{t+i} - x^{sr}\|_2^2 + c_1\|u_{t+i} - u^{sr}\|_2^2 \\ &\qquad + c_2\|y_{t+i} - y^{sr}\|_2^2. \end{aligned} \tag{26}$$

Thus, choosing $\gamma = \frac{c_9}{\max\{c_1, c_2\}} > 0$, the bounds (24) and (26) can be used to bound the Lyapunov function candidate $V(x_t, u^T, y^T)$ as

$$V(x_{t+n}, u^T, y^T) - V(x_t, u^T, y^T) \leq -\frac{\gamma}{2} \sum_{i=0}^{n-1} \|x_{t+i} - x^{sr}\|_2^2.$$

Due to Parts (c.1) and (c.2) of the proof, V is locally quadratically lower and upper bounded. Thus, the equilibrium x^{sr} is exponentially stable by standard Lyapunov arguments. □

The proof of Theorem 7 follows the lines of Köhler et al. (2019). The main difference lies in the fact that an n-step analysis of the closed loop is performed. Intuitively, this can be explained by noting that the input-output behavior over n steps allows to draw conclusions on the behavior of the internal state, which is relevant for stability. Further, by making an additional case distinction in Case 2b of the proof, we show a decay in the Lyapunov function for the scenario that the actual input and output values (u_{t+i}, y_{t+i}) are close to their respective artificial equilibria $(u^{s*}(t+i), y^{s*}(t+i))$ for $i \in \mathbb{I}_{[0,n-1]}$, but at least one internal state x_{t+k} is not close to $x^{s*}(t+k)$ with $k \in \mathbb{I}_{[0,n-1]}$. As is shown in Lemma 8, this implies that, along n time steps, at least one input or output must be distant from the artificial equilibrium $(u^{s*}(t+k), y^{s*}(t+k))$. By using this insight and defining a new candidate solution as a convex combination of a simpler candidate solution and the optimal solution, a suitable decay of the optimal cost can be shown. Finally, an IOSS Lyapunov function is employed to translate this decay, which is in terms of input-output values, to a decay in the state.

Theorem 7 requires compact constraints for Case 2a of the proof, which applies a local controllability argument to treat the case that the state is close to the current artificial steady-state. The proof is readily extendable to non-compact constraints, when considering initial states within some compact sublevel set of the Lyapunov function $V(x_t, u^T, y^T) \leq V_{\max}$, for a given level $V_{\max}$. The only difference in this scenario is that the size of γ_2, and hence also the exponential decay rate of V, depends on $V_{\max}$. That is, for larger initial values, the convergence rate derived in the proof decreases.

4. NUMERICAL EXAMPLE

In this section, we apply the developed tracking MPC scheme to a four tank system, which has been considered in Raff et al. (2006). This example is well-known as an open-loop stable system which can be rendered unstable by an MPC without terminal constraints if the horizon is chosen too short. It was also considered by Berberich et al. (2019) for data-driven MPC with a fixed terminal equality constraint. In the following, we show that the present tracking MPC scheme with online optimization of an artificial equilibrium admits a significantly larger region of attraction, without requiring knowledge of the equilibrium input corresponding to a desired output target setpoint. The linearized dynamics of the system are

$$x_{k+1} = \begin{bmatrix} 0.921 & 0 & 0.041 & 0 \\ 0 & 0.918 & 0 & 0.033 \\ 0 & 0 & 0.924 & 0 \\ 0 & 0 & 0 & 0.937 \end{bmatrix} x_k + \begin{bmatrix} 0.017 & 0.001 \\ 0.001 & 0.023 \\ 0 & 0.061 \\ 0.072 & 0 \end{bmatrix} u_k,$$

$$y_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_k.$$

We assume that the system matrices are unknown, but one input-output trajectory $\{u_k^d, y_k^d\}_{k=0}^{N-1}$ of length $N = 100$ is available, which is generated by sampling u_k^d uniformly from $[-1, 1]^2$. This trajectory is used via the proposed MPC scheme in order to track the desired target output $y^T = [1\ 1]^\top$, without specification of an input setpoint, i.e., the input weight is set to $S = 0$. Further, we impose constraints on the input and output as $\mathbb{U} = [-1.2, 1.2] \times [-2, 2], \mathbb{Y} = [0, 1.2]^2$. The equilibrium constraints are chosen as $\mathbb{U}^s = 0.99\mathbb{U}, \mathbb{Y}^s = 0.99\mathbb{Y}$.

The prediction horizon is set to $L = 25$, which is the maximal prediction horizon such that an input trajectory of length $N = 100$ can be persistently exciting of order[1] $L + 2n$. Further, the cost matrices are defined as $Q = 5I_2, R = I_2, T = 200I_2$. For a given trajectory, there exist infinitely many (arbitrarily large) vectors $\alpha(t)$ satisfying (6a) and, therefore, a direct implementation of the proposed scheme can be numerically ill-conditioned. Therefore, we include a norm-penalty of the form $10^{-4} \cdot \|\alpha(t)\|_2^2$ in the stage cost, whose utility was thoroughly analyzed in Berberich et al. (2019) for a robust data-driven MPC scheme in the presence of noise. Figure 1 illustrates the closed-loop behavior of the first component of the output as well as multiple exemplary open-loop predictions. It can be seen that the artificial equilibrium $y^{s*}(t)$ is updated continuously and converges to the desired target setpoint, which is in this case equal to the optimal reachable equilibrium, i.e., $y^{sr} = y^T$. Thus, also the closed-loop output converges to the target setpoint.

[1] Note that, for the matrix $H_{L+2n}(u^d)$ to have full row rank, it must hold that $N \geq (m+1)(L+2n) - 1$.

Compared to the scheme of Berberich et al. (2019), which relied only on terminal equality constraints, but not on an online optimization of the setpoint, the present scheme exhibits several advantages. First of all, to apply the scheme of Berberich et al. (2019), the optimal reachable equilibrium input $u^{sr} = [1\ 1.8]^\top$ needs to be computed explicitly, which is non-trivial without model knowledge, whereas the present scheme computes u^{sr} automatically. Further, optimizing the setpoint online increases the size of the region of attraction, since the terminal equality constraints do not need to be satisfied already in the first iteration. In particular, for the above example, the scheme of Berberich et al. (2019) is only initially feasible for *significantly* larger prediction horizons. This requires a) more computational power and b) a significantly longer data trajectory. Furthermore, the present scheme leads to smoother closed-loop trajectories with less overshoot, compared to the scheme without artificial equilibrium.

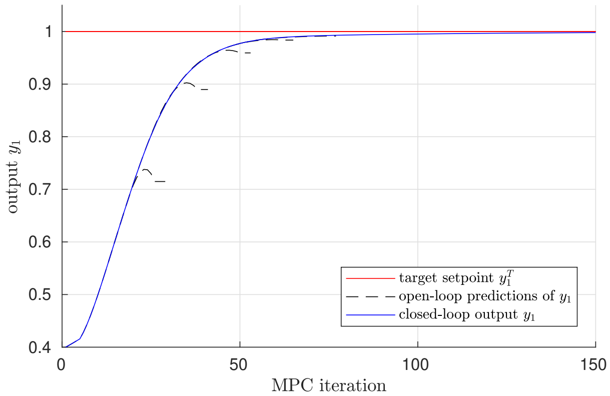

Fig. 1. First component of the target setpoint y^T, of the open-loop predictions $\bar{y}^*(t)$ at multiple time instances $t \in \{0, 12, 24, 36, 48\}$, and of the closed-loop trajectory $\{y_t\}_{t=0}^{150}$.

5. CONCLUSION

We presented a novel data-driven tracking MPC scheme which can cope with unreachable setpoints, which may potentially change online. The scheme is purely data-driven and does not require any model knowledge. Various desirable properties of the closed loop were proven, and the practical applicability of the scheme was illustrated for a realistic example. Our results are also an important extension of model-based tracking MPC to the case of positive semidefinite stage costs, which could be dealt with in the present paper by showing a cost decay over n consecutive time steps. Although the presented scheme is in most scenarios more robust than the one with simple terminal equality constraints used in Berberich et al. (2019), an important issue for future research is to give robust stability *guarantees* for the proposed tracking MPC scheme in the case of measurement noise.

REFERENCES

Berberich, J., Allgöwer, F., 2020. A trajectory-based framework for data-driven system analysis and control. In: Proc. European Control Conference. To appear, preprint online: arXiv:1903.10723.

Berberich, J., Koch, A., Scherer, C. W., Allgöwer, F., 2020. Robust data-driven state-feedback design. In: Proc. American Control Conference. To appear, preprint online: arXiv:1909.04314.

Berberich, J., Köhler, J., Müller, M. A., Allgöwer, F., 2019. Data-driven model predictive control with stability and robustness guarantees. arXiv:1906.04679.

Cai, C., Teel, A. R., 2008. Input–output-to-state stability for discrete-time systems. Automatica 44 (2), 326–336.

Coulson, J., Lygeros, J., Dörfler, F., 2019a. Data-enabled predictive control: in the shallows of the DeePC. In: Proceedings of the 18th European Control Conference. pp. 307–312.

Coulson, J., Lygeros, J., Dörfler, F., 2019b. Regularized and distributionally robust data-enabled predictive control. arXiv:1903.06804.

De Persis, C., Tesi, P., 2019. Formulas for data-driven control: Stabilization, optimality and robustness. arXiv:1903.06842.

Köhler, J., Müller, M. A., Allgöwer, F., 2019. A nonlinear tracking model predictive control scheme for dynamic target signals. Automatica, submitted, preprint online: arXiv:1911.03304.

Limón, D., Alvarado, I., Alamo, T., Camacho, E. F., 2008. MPC for tracking piecewise constant references for constrained linear systems. Automatica 44 (9), 2382–2387.

Limón, D., Ferramosca, A., Alvarado, I., Alamo, T., 2018. Nonlinear MPC for tracking piece-wise constant reference signals. IEEE Transactions on Automatic Control 63 (11), 3735–3750.

Raff, T., Huber, S., Nagy, Z. K., Allgöwer, F., 2006. Nonlinear model predictive control of a four tank system: An experimental stability study. In: Proceedings of the IEEE International Conference on Control Applications. pp. 237–242.

Romer, A., Berberich, J., Köhler, J., Allgöwer, F., 2019. One-shot verification of dissipativity properties from input-output data. IEEE Control Systems Letters 3 (3), 709–714.

Salvador, J. R., Ramirez, D. R., Alamo, T., de la Pena, D. M., Garcia-Marin, G., 2019. Data driven control: an offset free approach. In: Proceedings of the 18th European Control Conference. pp. 23–28.

van Waarde, H. J., Eising, J., Trentelman, H. L., Camlibel, M. K., 2019. Data informativity: a new perspective on data-driven system analysis and control. arXiv:1908.00468.

Willems, J. C., Rapisarda, P., Markovsky, I., De Moor, B., 2005. A note on persistency of excitation. Systems & Control Letters 54, 325–329.

Yang, H., Li, S., 2015. A data-driven predictive controller design based on reduced hankel matrix. In: Proceedings of the 10th Asian Control Conference. pp. 1–7.