Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

L_2-Gain Performance Analysis for Discrete-Time Systems with Input Saturation: an LPV Approach

Hoai-Nam Nguyen *

* *Control, Signal and System Department - IFP Energies Nouvelles, Rond-point de l'echangeur de Solaize BP3, 69360 Solaize, France (e-mail:hoai-nam.nguyen@ifp.fr).*

Abstract: In this paper, we consider the problem of estimating the L_2-gain under a given feedback law for linear discrete-time systems subject to actuator saturation. The basic idea is to use the linear parameter varying system framework to model the saturation nonlinearity. It is shown that the conditions can be expressed as a set of linear matrix inequalities. Furthermore, it is proved that the conditions are guaranteed to be less conservative than several existing solutions in the literature. One numerical example is presented to illustrate the effectiveness of the proposed method.

Keywords: Linear Discrete-time System, Actuator Saturation, LPV Modeling, L_2-gain, Linear Matrix Inequalities (LMI)

1. INTRODUCTION

In control engineering, saturation is one of the most commonly encountered non-linearities. It is well known that the input saturation can degrade the system performance, can create limit cycles, and even can cause instability. Therefore, analyzing the system performance that can be achieved under the input saturation is of great importance, see for example, Hu and Lin (2001), Kapila and Grigoriadis (2002). In this paper, we are particularly interested in estimating the L_2-gain, because it is one of the most important performance indices of the control system.

Using the circle and Popov criteria, for the L_2-gain analysis of linear discrete-time systems in the presence of input saturation, several methods have been considered in the literature, see Hindi and Boyd (1998), Paim et al. (2002). In Hu and Lin (2001) by combining a quadratic Lyapunov function, and a linear differential inclusion (LDI) to model the saturation nonlinearity, it is shown that the condition is less conservative than that based on the circle criterion. In addition, it is also shown that the condition obtained based on the LDI framework can be converted into linear matrix inequalities (LMIs) constraints, while based on the circle criterion, the conditions are bilinear matrix inequalities, which as non-convex.

In Wada et al. (2004), Ma and Yang (2008), the LDI approach is improved by using more general Lyapunov function. In particular, a saturation-dependent Lyapunov function that captures the real-time information on the severity of saturation was proposed. The conditions are expressed as LMI constraints. However, the existing results were obtained by using an auxiliary linear state feedback law.

The objective of this paper is to further reduce the conservativeness in the estimation of the L_2-gain performance by using an auxiliary nonlinear saturation-dependent state feedback law. The resulting system is modeled as an LPV system. It is proved that the conditions are guaranteed to be less conservative than the existing results in the literature. Furthermore, it is shown that the proposed conditions can be reduced to a set of LMIs.

The paper is organized as follows. Section 2 describes the problem formulation and some preliminaries concerning the LMIs constrains. Section 3 is dedicated to the main result of the paper. In Section 4, one numerical example is presented. Finally, some conclusions are drawn in Section 5.

Notation: A positive-definite (semi-definite) matrix P is denoted by $P \succ 0$ ($P \succeq 0$). $\mathbf{0}$, $\mathbf{I}$ are, respectively, the zero matrix and the identity matrix of appropriate dimensions. For a given $P \succeq 0$, $\mathcal{E}(P,\alpha)$ represents the following ellipsoid

$$\mathcal{E}(P,\alpha) = \{x \in \mathbb{R}^n : x^T P^{-1} x \leq \alpha\}$$

For a given vector $f \in \mathbb{R}^{1\times n}$, $\mathcal{L}(f)$ is used to denote the following set

$$\mathcal{L}(f) = \{x \in \mathbb{R}^n : -1 \leq fx \leq 1\}$$

For symmetric matrices, the symbol $(*)$ denotes each of its symmetric block.

2. PROBLEM FORMULATION AND PRELIMINARIES

2.1 Problem Formulation

Consider the following linear discrete-time system subject to input saturation

$$\begin{cases} x(k+1) &= Ax(k) + Bsat(u(k)) + Ew(k), \\ z(k) &= Cx(k) + Dw(k) \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the measured state, $u \in \mathbb{R}$ is the control input, $w \in \mathbb{R}^q$ is the disturbance, $z \in \mathbb{R}^p$ is the performance output. For simplicity, only the single input case is considered. However the approach in the paper can be straightforwardly extended to the multi-input case.

The saturation function is defined as

$$sat(u) = \begin{cases} 1, & \text{if } u \leq 1, \\ u, & \text{if } -1 \leq u \leq 1, \\ -1 & \text{if } u \leq -1, \end{cases} \quad (2)$$

In this paper, we are interested in estimating the L_2- gain performance of the system (1) under a given linear state feedback law

$$u(k) = Kx(k) \quad (3)$$

It is assumed that $A + BK$ is a Schur matrix, i.e., all eigenvalues of $A+BK$ are in the interior of the unit circle. Furthermore, it is also assumed that

$$w \in W = \{w \in \mathbb{R}^q : \|w\|_2^2 \leq \beta\} \quad (4)$$

where $\beta > 0$ is a given constant.

For a given $\gamma > 0$, the system (1),(2),(3) is said to be with a L_2 performance gain less than γ, if for the zero initial condition, the following condition

$$\sum_{k=0}^{\infty} z^T(k)z(k) - \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k) \leq 0 \quad (5)$$

is satisfied $\forall w(k) \in W$.

2.2 Preliminaries

The following lemma is taken from Hu et al. (2002). It will be used to model the saturation non-linearity (2).

Lemma 1: For any $u \in \mathbb{R}$, the exist $-1 \leq v \leq 1$, and $\lambda_1 \geq 0, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, such that the saturation function (2) is written as

$$sat(u) = \lambda_1 u + \lambda_2 v \quad (6)$$

The following double sum negativity problem of the form

$$x^T(\sum_{j=1}^{s}\sum_{l=1}^{s} \lambda_j \lambda_l \Gamma_{jl})x \geq 0 \quad (7)$$

will be dealt several times in this paper, where the coefficients λ_j satisfy

$$\sum_{j=1}^{s} \lambda_j = 1, \lambda_j \geq 0, \forall j = \overline{1,s}$$

Lemma 2: The double sum (7) is positive, if, see Wang et al. (1996),

$$\begin{cases} \Gamma_{jj} \succeq 0, \\ \Gamma_{jl} + \Gamma_{lj} \succeq 0, \ \forall j, \forall l = \overline{1,s}, j < l \end{cases} \quad (8)$$

Lemma 3: The double sum (7) is positive, if, see Tuan et al. (2001),

$$\begin{cases} \Gamma_{jj} \succeq 0, \\ \dfrac{2}{s-1}\Gamma_{jj} + \Gamma_{jl} + \Gamma_{lj} \succeq 0, \ \forall j, \forall l = \overline{1,s}, j \neq l \end{cases} \quad (9)$$

Lemma 4: Given matrices P, G of appropriate dimension with $P \succeq 0$. Then, see De Oliveira et al. (1999)

$$(G-P)^T P^{-1}(G-P) \succeq 0 \Leftrightarrow G^T P^{-1} G \succeq G^T + G - P \quad (10)$$

Lemma 5: For a vector $f \in \mathbb{R}^{1\times n}$ and a matrix $P \succeq 0$, $\mathcal{E}(P, \alpha) \subseteq \mathcal{L}(f_0)$ if and only if, see Hu and Lin (2001)

$$f_0 P f_0^T \leq \frac{1}{\alpha} \quad (11)$$

In what follows, we will make use of the following results, concerning the LMIs.

Property 1 (Congruence): Let P and Q are matrices of appropriate dimension, where $P = P^T$, and Q is a full rank matrix. It holds that

$$P \succeq 0 \Leftrightarrow Q^T P Q \succeq 0 \quad (12)$$

Property 2: (Schur complement): Consider a matrix M, with

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$$

and M_{11}, M_{22} being square matrices. Then, see Boyd et al. (1994)

$$M \succeq 0 \Leftrightarrow \begin{cases} M_{11} \succeq 0, \\ M_{22} - M_{12}^T M_{11} M_{12} \succeq 0 \end{cases} \Leftrightarrow \begin{cases} M_{22} \succeq 0, \\ M_{11} - M_{12} M_{22} M_{12}^T \succeq 0 \end{cases} \quad (13)$$

3. MAIN RESULTS

Substituting (3) to (1), one gets

$$x(k+1) = Ax(k) + Bsat(Kx(k)) + Ew(k)$$

Thus, using Lemma 1, there exist $\lambda_1(k) \geq 0, \lambda_2(k) \geq 0$, $\lambda_1(k) + \lambda_2(k) = 1$, and $-1 \leq v(k) \leq 1$ such that

$$x(k+1) = Ax(k) + B(\lambda_1(k)Kx(k) + \lambda_2(k)v(k)) + Ew(k)$$

Thus, with $A = (\lambda_1(k) + \lambda_2(k))A$

$$x(k+1) = \mathcal{A}(\lambda(k))x(k) + \mathcal{B}(\lambda(k))v(k) + Ew(k) \quad (14)$$

where

$$\begin{cases} \mathcal{A}(\lambda(k)) = \lambda_1(k)A_1 + \lambda_2(k)A_2 \\ \mathcal{B}(\lambda(k)) = \lambda_1(k)B_1 + \lambda_2(k)B_2 \end{cases} \quad (15)$$

with

$$A_1 = A + BK, A_2 = A,$$
$$B_1 = \mathbf{0}_{n\times 1}, B_2 = B$$

Hence the linear system (1) subject to input saturation is rewritten as a linear parameter varying (LPV) system (14), where $v(k)$ is considered as a control input. The problem of computing the L_2-gain for (1) becomes the problem of selecting the control law v such that the L_2-gain for (14) is minimal.

Consider the following control law

$$v(k) = F(\lambda(k))G(\lambda(k))^{-1}x(k) \quad (16)$$

with

$$\begin{cases} F(\lambda(k)) = \lambda_1(k)F_1 + \lambda_2(k)F_2, \\ G(\lambda(k)) = \lambda_1(k)G_1 + \lambda_2(k)G_2 \end{cases}$$

where $F_1 \in \mathbb{R}^{1\times n}, F_2 \in \mathbb{R}^{1\times n}, G_1 \in \mathbb{R}^{n\times n}$ and $G_2 \in \mathbb{R}^{n\times n}$ are unknown vectors and matrices that will be treated as decision variables.

Remark 1: In the literature, see for example Hu et al. (2002), Wada et al. (2004), Ma and Yang (2008), only a linear control law $v(k) = FG^{-1}x(k)$ was considered. Clearly, this is a particular case of (16) with $F_1 = F_2 = F$ and $G_1 = G_2 = G$. The control law (16) takes the real time information of the saturation into account. Hence as shown in the example, a less conservative estimate of the L_2-gain is obtained.

Substituting (16) into (14), one obtains the following closed-loop system

$$x(k+1) = \mathcal{A}_c(\lambda(k))x(k) + Ew(k) \tag{17}$$

where

$$\mathcal{A}_c(\lambda(k)) = \mathcal{A}(\lambda(k)) + \mathcal{B}(\lambda(k))F(\lambda(k))G(\lambda(k))^{-1}$$

Define the following parameter-dependent Lyapunov function

$$\begin{aligned} V(k, x(k)) &= x(k)^T(P(\lambda(k)))^{-1}x(k) \\ &= x(k)^T(\lambda_1(k)P_1 + \lambda_2(k)P_2)^{-1}x(k) \end{aligned} \tag{18}$$

where $P_1 \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n}$ are unknown positive definite matrix.

With small abuse of notation, $\mathcal{A}_c(k), \mathcal{A}(k), \mathcal{B}(k), G(k), P(k)$ will be used to denote $\mathcal{A}_c(\lambda(k)), \mathcal{A}(\lambda(k)), \mathcal{B}(\lambda(k)), G(\lambda(k))$, $P(\lambda(k))$.

The following theorem provides the main result of the paper. It establishes the theoretical support of the algorithm proposed to obtain an estimation of the L_2-gain performance for the system (14), (16).

Theorem 1: Consider the system (14), (16). For given scalars $\alpha > 0, \beta > 0$, assume that there exist positive define matrices P_1, P_2, matrices F_1, F_2, G_1, G_2 and a positive scalar γ satisfying the following matrix inequalities

$$\begin{bmatrix} P(k+1) & \mathcal{A}(k)G(k) + \mathcal{B}(k)F(k) & E & \mathbf{0} \\ (*) & G(k) + G(k)^T - P(k) & \mathbf{0} & G(k)^T C^T \\ (*) & (*) & \mathbf{I} & D^T \\ (*) & (*) & (*) & \gamma^2 \mathbf{I} \end{bmatrix} \succeq 0 \tag{19}$$

$$\begin{bmatrix} \dfrac{1}{\alpha+\beta} & F(k) \\ F(k)^T & G(k) + G(k)^T - P(k) \end{bmatrix} \succeq 0 \tag{20}$$

then, $\forall x(0)$ such that $x(0)^T P(0)^{-1} x(0) \leq \alpha$, one has $x(k)^T P(k)^{-1} x(k) \leq \alpha + \beta$, $\forall k \geq 1$, and the following inequality holds

$$\sum_{k=0}^{\infty} z^T(k)z(k) \leq \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k) + \alpha. \tag{21}$$

Proof: Consider the Lyapunov function (18). For the L_2-gain performance, it is required that

$$\begin{aligned} & V(k+1, x(k+1)) - V(k, x(k)) \leq \\ & \leq -\frac{1}{\gamma^2} z(k)^T z(k) + w(k)^T w(k) \end{aligned} \tag{22}$$

for all $x(k), x(k+1)$ satisfying (17), and for all $w(k) \in W$.

If (22) holds, then it follows that

$$\begin{aligned} & V(\infty, x(\infty)) - V(0, x(0)) \leq \\ & \leq -\frac{1}{\gamma^2} \sum_{k=0}^{\infty} z(k)^T z(k) + \sum_{k=0}^{\infty} w(k)^T w(k) \end{aligned} \tag{23}$$

Note that the system (17) is asymptotically stable for states near the origin. It follows that $\lim_{k\to\infty} x(k) = 0$. Hence $\lim_{k\to\infty} V(k, x(k)) = 0$. With the zero initial condition, i.e., $x(0) = \mathbf{0}$, inequality (23) becomes

$$0 \leq -\frac{1}{\gamma^2} \sum_{k=0}^{\infty} z(k)^T z(k) + \sum_{k=0}^{\infty} w(k)^T w(k)$$

or equivalently

$$\sum_{k=0}^{\infty} z(k)^T z(k) \leq \gamma^2 \sum_{k=0}^{\infty} w(k)^T w(k)$$

It is concluded that the system 14), (16) has L_2-gain performance γ.

Using (18), (17), the left hand side of (22) can be rewritten as

$$\begin{aligned} & V(k+1, x(k+1)) - V(k, x(k)) = \\ & = x(k+1)^T P(k+1)^{-1} x(k+1) - x(k)^T P(k)^{-1} x(k) \\ & = \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} \mathcal{A}_c(k)^T \\ E^T \end{bmatrix} P(k+1)^{-1} \left[\mathcal{A}_c(k) \;\; E \right] \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \\ & \quad - \left[x(k)^T \;\; w(k)^T \right] \begin{bmatrix} P(k)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \end{aligned} \tag{24}$$

Using (1), the right hand side of (22) can be rewritten as

$$\begin{aligned} & -\frac{1}{\gamma^2} \sum_{k=0}^{\infty} z(k)^T z(k) + \sum_{k=0}^{\infty} w(k)^T w(k) = \\ & = -\frac{1}{\gamma^2} \left[x(k)^T \;\; w(k)^T \right] \begin{bmatrix} C^T \\ D^T \end{bmatrix} \left[C \;\; D \right] \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + \\ & \quad + \left[x(k)^T \;\; w(k)^T \right] \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \end{aligned} \tag{25}$$

Using (24), (25), it follows that the inequality (22) holds if and only if

$$\begin{aligned} & \begin{bmatrix} P(k)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathcal{A}_c(k)^T \\ E^T \end{bmatrix} P(k+1)^{-1} \left[\mathcal{A}_c(k) \;\; E \right] \\ & \succeq \frac{1}{\gamma^2} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \left[C \;\; D \right] - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \end{aligned}$$

or equivalently

$$\begin{aligned} & \begin{bmatrix} P(k)^{-1} - \frac{1}{\gamma^2} C^T C & -\frac{1}{\gamma^2} C^T D \\ -\frac{1}{\gamma^2} D^T C & -\frac{1}{\gamma^2} D^T D + \mathbf{I} \end{bmatrix} - \\ & \begin{bmatrix} \mathcal{A}_c(k)^T \\ E^T \end{bmatrix} P(k+1)^{-1} \left[\mathcal{A}_c(k) \;\; E \right] \succeq 0 \end{aligned} \tag{26}$$

Using Schur complement, equation (26) can be rewritten as

$$\begin{bmatrix} P(k+1) & \mathcal{A}_c(k) & E \\ (*) & P(k)^{-1} - \frac{1}{\gamma^2} C^T C & -\frac{1}{\gamma^2} C^T D \\ (*) & (*) & -\frac{1}{\gamma^2} D^T D + \mathbf{I} \end{bmatrix} \succeq 0$$

or equivalently

$$\begin{bmatrix} P(k+1) & \mathcal{A}_c(k) & E \\ (*) & P(k)^{-1} & \mathbf{0} \\ (*) & (*) & \mathbf{I} \end{bmatrix} - \frac{1}{\gamma^2} \begin{bmatrix} \mathbf{0} \\ C^T \\ D^T \end{bmatrix} \left[\mathbf{0} \;\; C \;\; D \right] \succeq 0$$

Thus, with Schur complement

$$\begin{bmatrix} P(k+1) & \mathcal{A}_c(k) & E & \mathbf{0} \\ (*) & P(k)^{-1} & \mathbf{0} & C^T \\ (*) & (*) & \mathbf{I} & D^T \\ (*) & (*) & (*) & \gamma^2 \mathbf{I} \end{bmatrix} \succeq 0 \tag{27}$$

Pre- and post-multiplication of (27) by

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (*) & G(k)^T & \mathbf{0} & \mathbf{0} \\ (*) & (*) & \mathbf{I} & \mathbf{0} \\ (*) & (*) & (*) & \mathbf{I} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (*) & G(k) & \mathbf{0} & \mathbf{0} \\ (*) & (*) & \mathbf{I} & \mathbf{0} \\ (*) & (*) & (*) & \mathbf{I} \end{bmatrix}$$

one gets

$$\begin{bmatrix} P(k+1) & \mathcal{A}(k)G(k)+\mathcal{B}(k)F(k) & E & \mathbf{0} \\ (*) & G(k)^T P(k)^{-1}G(k) & \mathbf{0} & G(k)C^T \\ (*) & (*) & \mathbf{I} & D^T \\ (*) & (*) & (*) & \gamma^2\mathbf{I} \end{bmatrix} \succeq 0 \tag{28}$$

Using Lemma 4, one has

$$G(k)^T P(k)^{-1}G(k) \succeq G(k)^T + G(k) - P(k) \tag{29}$$

Substituting (29) into (28), one obtains (19).

For all initial conditions $x(0)$ such that $x(0)^T P(0)^{-1}x(0) \leq \alpha$, using (22), one gets

$$x(k)P(k)^{-1}x(k) \leq x(0)^T P(0)^{-1}x(0) - \\ -\frac{1}{\gamma^2}\sum_{j=0}^{k-1} z(j)^T z(j) + \sum_{j=0}^{k-1} w(j)^T w(j)$$

It follows that

$$x(k)P(k)^{-1}x(k) \leq x(0)^T P(0)^{-1}x(0) + \sum_{j=0}^{k-1} w(j)^T w(j)$$

Hence

$$x(k)P(k)^{-1}x(k) \leq \alpha + \beta \tag{30}$$

Recall that

$$-1 \leq v(k) = F(k)G(k)^{-1}x(k) \leq 1$$

Thus, using Lemma 5, and (30), one obtains

$$F(k)G(k)^{-1}P(k)(G(k)^{-1})^T F(k)^T \leq \alpha + \beta$$

Using Schur complement, this condition is equivalently rewritten as

$$\begin{bmatrix} \frac{1}{\alpha+\beta} & F(k) \\ F(k)^T & G(k)^T P(k)^{-1}G(k) \end{bmatrix} \succeq 0$$

Using Lemma 4, one gets

$$\begin{bmatrix} \frac{1}{\alpha+\beta} & F(k) \\ F(k)^T & G(k)+G(k)^T - P(k) \end{bmatrix} \succeq 0$$

The proof is complete. □

By setting $F_1 = F_2, G_1 = G_2$ in the conditions of Theorem 1 the following corollary is derived

Corollary 1: Consider the system (14), (16). For given scalars $\alpha > 0, \beta > 0$, assume that there exist positive define matrices P_1, P_2, matrices F_1, F_2, G_1, G_2 and a positive scalar γ satisfying the following matrix inequalities

$$\begin{bmatrix} P(k+1) & \mathcal{A}(k)G+\mathcal{B}(k)F & E & \mathbf{0} \\ (*) & G+G^T-P(k) & \mathbf{0} & G^T C^T \\ (*) & (*) & \mathbf{I} & D^T \\ (*) & (*) & (*) & \gamma^2\mathbf{I} \end{bmatrix} \succeq 0 \tag{31}$$

$$\begin{bmatrix} \frac{1}{\alpha+\beta} & F \\ F^T & G+G^T-P(k) \end{bmatrix} \succeq 0 \tag{32}$$

then, $\forall x(0)$ such that $x(0)^T P(0)^{-1}x(0) \leq \alpha$, one has $x(k)^T P(k)^{-1}x(k) \leq \alpha+\beta$, $\forall k \geq 1$, and the following inequality holds

$$\sum_{k=0}^{\infty} z^T(k)z(k) \leq \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k) + \alpha. \tag{33}$$

Corollary 1 is Theorem 1 in Wada et al. (2004). Hence Theorem 1 in Wada et al. (2004) is a particular case of our theorem. As a consequence, Theorem 1 in the paper can be used to obtain less conservative L_2-gain performance than that by using existing results in the literature.

Note that (20) holds if and only if

$$\begin{bmatrix} \frac{1}{\alpha+\beta} & F_j \\ F_j^T & G_j+G_j^T-P_j \end{bmatrix} \succeq 0, \forall j = \overline{1,2} \tag{34}$$

Define $\gamma_s = \gamma^2$. Rewrite (19) as

$$\sum_{j=1}^{2}\sum_{l=1}^{2} \lambda_j\lambda_l\Gamma_{jl}^m \succeq 0 \tag{35}$$

with $m = 1, 2$, and

$$\Gamma_{jl}^m = \begin{bmatrix} P_m & (A_jG_l+B_jF_l) & E & \mathbf{0} \\ (*) & (G_l+G_l-P_l) & 0 & G_lC^T \\ (*) & (*) & \mathbf{I} & D^T \\ (*) & (*) & (*) & \gamma_s\mathbf{I} \end{bmatrix} \tag{36}$$

Using (35), (34), relaxed LMI conditions can be easily be formulated using Lemmas 2, or 3 as follows

Corollary 2: The linear discrete-time system (1), (3) is with a L_2- performance index less than $\gamma = \sqrt{\gamma_s}$, if for given positive scalars α, β, there exist positive definite matrices P_j, matrices G_j, F_j, $j = 1, 2$, and positive scalar γ_s, such that (34) holds and

$$\begin{cases} \Gamma_{jj}^m \succeq 0, \ \forall m, \forall j = \overline{1,2} \\ \Gamma_{jl}^m + \Gamma_{lj}^m \succeq 0, \ \forall m, \forall j, \forall l = \overline{1,2}, j < l \end{cases} \tag{37}$$

or

$$\begin{cases} \Gamma_{jj}^m \succeq 0, \ \forall m = \overline{1,2}, \forall j = \overline{1,2} \\ 2\Gamma_{jj}^m + \Gamma_{jl}^m + \Gamma_{lj}^m \succeq 0, \ \forall m, \forall j, \forall l = \overline{1,2}, j \neq l \end{cases} \tag{38}$$

The proof of corollary 2 is straightforward.

Remark 2: Note that condition (38) is less conservative than (37), i.e., if there exist matrices P_j, G_j, F_j satisfying (37), they also satisfy (38). The main advantage of (37) with respect to (38) is that (37) has a fewer number of LMI constraints than (38). Hence the computational complexity is reduced.

A natural idea is to optimize the L_2-performance index γ. Using Corollary 2 and (34), this can be formulated as

$$\min_{P_j,G_j,F_j} \gamma \quad \text{subject to } (37), (34) \tag{39}$$

or

$$\min_{P_j,G_j,F_j} \gamma \quad \text{subject to } (38), (34) \tag{40}$$

Since the optimization problems (39) and/or (40) are a convex semi-definite (SDP) problem, they can be solved efficiently using free available LMI parser such as CVX, see Grant and Boyd (2014), or Yalmip, see Löfberg (2004). In the following, we refer to the optimization problems (39), and (40), respectively, as algorithm 1 and algorithm 2.

Remark 3: In the linear system case, it is well known, see Gahinet and Apkarian (1994), that the parameter β has no impact on the L_2-gain γ. However, in the presence of the saturation non-linearity, this is no longer the case, i.e., γ is a function of β. Using (34), it should be clear that this function is non-increasing, i.e., if $\beta_1 \geq \beta_2$, then $\gamma_1 \leq \gamma_2$.

4. EXAMPLE

This example is taken from Wada et al. (2004). The CVX toolbox was used to solve SDP optimization problems. Consider system (1), (3) with

$$A = \begin{bmatrix} 0 & 1 \\ -0.58 & -0.6 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$K = [\,0\ -0.7\,],\ C = [\,1\ 0\,],\ D = 0$$

For this example, the L_2-gain of the closed-loop system in the linear region is 4.5690.

By applying algorithm 1 and algorithm 2 with $\alpha = 0$, $\beta = 10$, one gets, respectively, $\gamma = 10.1770$, $\gamma = 10.1220$. The matrices obtained using algorithm 2 are

$$P_1 = \begin{bmatrix} 24.7751 & -12.7511 \\ -12.7511 & 12.8289 \end{bmatrix},\ P_2 = \begin{bmatrix} 16.3648 & -7.4523 \\ -7.4523 & 15.1787 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} 24.7751 & -12.7511 \\ -12.7511 & 12.8289 \end{bmatrix},\ G_2 = \begin{bmatrix} 16.3647 & -7.4524 \\ -7.4523 & 15.1787 \end{bmatrix},$$
$$F_1 = [\,1.0991\ -0.1498\,],\ F_2 = [\,1.2354\ -0.8445\,]$$

For comparison, Theorem 1 in Wada et al. (2004) was applied to estimate the L_2-gain. As a result $\gamma = 13.1844$ is obtained with $\alpha = 0, \beta = 10$. Clearly, both algorithms 1 and 2 give smaller γ than the existing solution in the literature. Note that for this example, Theorem 3.2.2 in Ma and Yang (2008) gives the same γ as Theorem 1 in Wada et al. (2004).

For different $\beta \in [0.5\ 20]$, Fig. 1 presents the L_2-gain γ using algorithm 1 (dashed red), and algorithm 2 (solid blue). As discussed by remark 2, it can be observed that algorithm 2 is slightly outperforms algorithm 1. For comparison, Fig. 1 presents the L_2- gain γ obtained by using Theorem 1 in Wada et al. (2004) (dash-dot yellow).

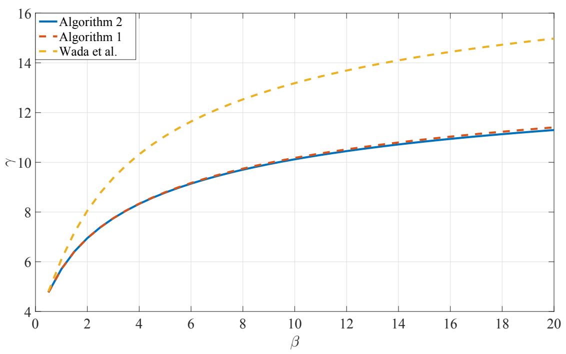

Fig. 1. L_2-gain performance index as a function of β for Algorithm 2 (solid blue), for Algorithm 1 (dashed red), and for Theorem 1 in Wada et al. (2004) (dash-dot yellow).

5. CONCLUSION

This paper considers the L_2-gain performance analysis for linear discrete-time systems subject to actuator saturation. The basic idea is to use an auxiliary nonlinear saturation-dependent state feedback law in conjunction with a saturation-dependent Lyapunov function. The closed-loop system is modeled as a linear parameter varying system. The obtained conditions are expressed as a set of LMIs constraints. It is proved that the proposed conditions are less conservative than the existing conditions in the literature. One numerical example illustrates that the proposed method improves existing results on the same problem.

REFERENCES

Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory*, volume 15. Siam.

De Oliveira, M.C., Bernussou, J., and Geromel, J.C. (1999). A new discrete-time robust stability condition. *Systems & control letters*, 37(4), 261–265.

Gahinet, P. and Apkarian, P. (1994). A linear matrix inequality approach to h control. *International journal of robust and nonlinear control*, 4(4), 421–448.

Grant, M. and Boyd, S. (2014). Cvx: Matlab software for disciplined convex programming, version 2.1.

Hindi, H. and Boyd, S. (1998). Analysis of linear systems with saturation using convex optimization. In *Proceedings of the 37th IEEE Conference on Decision and Control (Cat. No. 98CH36171)*, volume 1, 903–908. IEEE.

Hu, T. and Lin, Z. (2001). *Control systems with actuator saturation: analysis and design.* Springer Science & Business Media.

Hu, T., Lin, Z., and Chen, B.M. (2002). Analysis and design for discrete-time linear systems subject to actuator saturation. *Systems & Control Letters*, 45(2), 97–112.

Kapila, V. and Grigoriadis, K. (2002). *Actuator saturation control.* CRC Press.

Löfberg, J. (2004). Yalmip: A toolbox for modeling and optimization in matlab. In *Proceedings of the CACSD Conference*, volume 3. Taipei, Taiwan.

Ma, Y.M. and Yang, G.H. (2008). Performance analysis for linear discrete-time systems subject to actuator saturation. In *2008 American Control Conference*, 3608–3613. IEEE.

Paim, C., Tarbouriech, S., da Silva, J.G., and Castelan, E. (2002). Control design for linear systems with saturating actuators and/spl lscr//sub 2/-bounded disturbances. In *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*, volume 4, 4148–4153. IEEE.

Tuan, H.D., Apkarian, P., Narikiyo, T., and Yamamoto, Y. (2001). Parameterized linear matrix inequality techniques in fuzzy control system design. *IEEE Transactions on fuzzy systems*, 9(2), 324–332.

Wada, N., Oomoto, T., and Saeki, M. (2004). l/sub 2/-gain analysis of discrete-time systems with saturation nonlinearity using parameter dependent lyapunov function. In *2004 43rd IEEE Conference on Decision and Control (CDC)(IEEE Cat. No. 04CH37601)*, volume 2, 1952–1957. IEEE.

Wang, H.O., Tanaka, K., and Griffin, M.F. (1996). An approach to fuzzy control of nonlinear systems: Stability and design issues. *IEEE transactions on fuzzy systems*, 4(1), 14–23.