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$\mathcal{H}_\infty$ Control for Differential-Algebraic Systems

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Abstract: For a differential-algebraic system, we construct the class of proper suboptimal $\mathcal{H}_\infty$ stabilizing controllers and give formulas in terms of realizations and solutions to appropriate Riccati equations.

Keywords: Robust control, singular systems, generalized state space, dynamic output feedback, algebraic Riccati equations.

1. INTRODUCTION

The $\mathcal{H}_\infty$ control problem was introduced at the beginning of the 1980s in the seminal paper Zames [1981] as a novel approach to closed-loop disturbance attenuation. The standard formulation of the problem was established in Doyle [1984] before definitive state-space formulas were given for the proper, continuous-time case in Doyle et al. [1989]. These were then expanded for more general cases such as: nonlinear (Van der Schaft [1992]), time-varying (Limebeer et al. [1992]) or discrete-time (Ionescu et al. [1999]) systems. As an alternative to the pure state-space approach, the original frequency-domain interpretation of the problem introduced in Zames [1981] yielded the J-lossless factorization proposed in Green [1992]. Very recently, a variant of this technique was employed in Stefanovski [2015] to provide a solution for the $\mathcal{H}_\infty$ problem where the plant may be improper in addition to having zeros along the extended imaginary axis. Although powerful, the algorithm presented in said paper is rather computationally demanding and does not provide a simpler routine for systems that are only improper. Our approach in this paper offers just such an alternative, for the case in which the improper plant's zeros are well-behaved.

It is in this context that we offer a solution for the set of MIMO (multiple-input multiple-output) linear time-invariant continuous-time systems described by a combination of both differential and algebraic equations

$$E\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + Bu(t), \tag{1}$$

$$y(t) = Cx(t) + Du(t), \tag{2}$$

with $A - sE$ a square pencil and E a singular matrix. The singularity of E represents the cause for which the system (1)-(2) features both algebraic and differential components. Alternatively, one may describe these systems in the frequency domain through the use of their transfer function matrices which are possibly improper, or even polynomial. These differential-algebraic systems are alternatively known in literature as generalized or descriptor (see, for example, Verghese et al. [1981]). They prove essential in modeling systems which exhibit physical algebraic constraints or impulsive behavior (see Dai [1989]). A large number of practical applications in the various fields of engineering may be surveyed in Pasqualetti et al. [2013], Offner et al. [2016], Campbell et al. [2019] and their respective references.

The paper is organized as follows. Preliminary results, as well as general definitions and notation are given in Section 2. The paper's main result, the suboptimal $\mathcal{H}_\infty$ problem for descriptor systems, is formulated and solved in Section 3. An illustrative numerical example is then showcased in Section 4. Finally, a set of concluding remarks are elaborated upon in Section 5.

2. PRELIMINARIES

2.1 Definitions and general notation

Let $\mathbb{C}$ bee the complex plane and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ the latter's extension at infinity. We denote by $\mathbb{C}^-$ and $\mathbb{C}^+$ the open left-half and right-half planes, respectively, of $\mathbb{C}$. Let the imaginary axis be labeled as $j\mathbb{R}$, along with its own extension at infinity, $j\overline{\mathbb{R}} := j\mathbb{R} \cup \{\infty\}$. For a matrix $A \in \mathbb{C}^{p\times m}$, A^* denotes its conjugate transpose and $\sigma_{\max}(A)$ its maximum singular value. If A is square, $\rho(A)$ represents its spectral radius. If $A - sE$ is a regular pencil, $\Lambda(A - sE)$ denotes the set of its generalized eigenvalues (both finite and infinite, counting multiplicity). Throughout the paper, we primarily describe a linear time-invariant dynamical system, with m inputs and p outputs, by its *transfer function matrix* (TFM)

$$\mathbf{H}(s) = \left[\frac{a_{ij}(s)}{b_{ij}(s)}\right]_{i=\overline{1:p},\ j=\overline{1:m}}, \tag{3}$$

where $a_{ij}(s)$ and $b_{ij}(s)$ are scalar polynomials with coefficients in $\mathbb{C}$. As stated, we aim to investigate differential-algebraic systems, whose TFMs may be *improper* (see Dai [1989]), $\deg[a_{ij}(s)] > \deg[b_{ij}(s)]$, or even *polynomial*, $b_{ij}(s) \equiv 1$, for some i, j. Let $\mathbb{C}^{p\times m}(s)$ be the set of all $p\times m$ complex TFMs.

If all the poles of a system, $\mathbf{H}(s)$, are located in $\mathbb{C}^-$, we deem it to be *stable*. Consequently, a stable system is also proper ($\deg[a_{ij}(s)] \leq \deg[b_{ij}(s)]$, $\forall i = \overline{1:p}$, $j = \overline{1:m}$) since, by our definition, stability precludes the presence of poles at infinity. Let the Banach space of complex $p\times m$ TFM bounded on $j\overline{\mathbb{R}}$, having the $\mathcal{H}_\infty$ norm defined as $\|\mathbf{H}\|_\infty := \sup_{\omega\in\overline{\mathbb{R}}} \sigma_{\max}\Big(\mathbf{H}(j\omega)\Big)$, be denoted by $\mathcal{RL}_\infty$. Let $\mathcal{RH}_\infty (\subset \mathcal{RL}_\infty)$ be the subset of stable TFMs. Additionally, denote by $\mathcal{BH}_\infty^{(\gamma)}$ the subset of $\mathcal{RH}_\infty$ such that any TFM $\mathbf{H}(s)$ belonging to it satisfies $\|\mathbf{H}\|_\infty < \gamma$.

2.2 Realizations for differential-algebraic systems

In literature, differential-algebraic systems are usually represented in the time-domain by *descriptor realizations* of type (1)-(2) (see Dai [1989], Verghese et al. [1981]), with $A - sE \in \mathbb{C}^{n\times n}(\lambda)$ a regular $n\times n$ pencil and E a singular matrix, along with all the associated constant complex matrices $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}^{p\times m}$ having appropriate dimensions. The connection between these matrices and the TFM representation is made through

$$\mathbf{H}(s) = C(sE - A)^{-1}B + D =: \left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right]. \qquad (4)$$

Descriptor realizations are an extension of the standard state-space ones, the former being introduced to represent systems which may have poles at ∞. Although (4) may be used to describe any TFM model (3), it does introduce a set of inconveniences when used to describe the type of problem under investigation. For example, the order n of the realization (4) is strictly greater than the McMillan degree of $\mathbf{H}(s)$ (see Definition 3.12 in Zhou et al. [1996]) if the latter system has any poles at ∞.

Moreover, the classical interpretations of controllability and observability do not amount to (4) being a minimal realization. Furthermore, any two such minimal realizations are not necessarily related by an equivalence transformation. Even worse, one cannot generally obtain a minimal realization using exclusively unitary transformations when starting from an arbitrary one (see Dai [1989], Van Dooren [1981], Verghese et al. [1981, 1979]).

These inconveniences can be overcome by employing *centered* realizations, first introduced in Rakowski [1992]. To define such a representation, one must first fix $\alpha, \beta \in \mathbb{C}$ and $s_0 \in \overline{\mathbb{C}}$ such that

$$\begin{cases} \alpha = 1, \quad \beta = 0, \text{ if } s_0 = \infty, \\ \alpha = s_0, \quad \beta = 1, \text{ if } s_0 \in \mathbb{C}. \end{cases} \qquad (5)$$

We say that

$$\mathbf{H}(s) = D + C(sE - A)^{-1}B(\alpha - \beta s) =: \left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right]_{s_0} \qquad (6)$$

is *realization centered at* s_0 and we assume the implicit choice of α and β in (5). If $s_0 = \infty$, we merely drop the index s_0 from (6) and get the representation in (4).

We say that the pair $(A - sE, B)$ or, alternatively, the realization (6) is *controllable at* $s \in \mathbb{C}$ if rank $[A - sE\; B] = n$. Furthermore, we affirm *controllability at* ∞ if rank $[E\; B] = n$. A realization or matrix pair are deemed *controllable* if they are controllable $\forall s \in \overline{\mathbb{C}}$. Relaxing this condition, they are deemed *stabilizable* if they are controllable $\forall s \in \overline{\mathbb{C}}\backslash\mathbb{C}^-$. By duality, the pair $(C, A - sE)$ or the realization (6) is *observable* if $(A^* - sE^*, C^*)$ is controllable. Similarly, we relax the notion of observability into *detectability*, which is equivalent to the pair $(A^* - sE^*, C^*)$ being stabilizable. Finally, a realization whose order is smallest among all others centered at s_0 is called *minimal*.

By choosing s_0 (the point at which we center our realization) as different from any pole of $\mathbf{H}(s)$, we recover all the pleasant features of standard state-space realizations (see Rakowski [1992] for a full discussion on the topic), while eliminating all the inconveniences introduced by realizations like (4). Moreover, the ease with which one obtains centered realizations (surprisingly similar to the standard ones) makes them all the more appealing for use in investigating the central problem of this paper.

Starting from the expression of its TFM (3), a simple method is given in Campbell et al. [2019] for directly obtaining a centered realization (6). Alternatively, one can use the procedures given in Pasqualetti et al. [2013] to switch between a descriptor realization (4) and a centered one (6), or vice-versa.

2.3 Algebraic Riccati equations

For systems described by centered realizations, there exists a particular type of Riccati equation which will prove instrumental in our investigation of the problem and which is treated at length in Dinicu and Oară [2015]. Consider the collection of matrices denoted by

$$\Sigma := (A - sE, B; Q, L, R), \qquad (7)$$

where $A, E \in \mathbb{C}^{n\times n}$, $B, L \in \mathbb{C}^{n\times m}$, $Q = Q^* \in \mathbb{C}^{n\times n}$, and $R = R^* \in \mathbb{C}^{m\times m}$ are invertible. Σ is used to summarily represent a quadratic performance index

$$\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t, \qquad (8)$$

with x and u being the state and input vectors, respectively, from (1). Thus, the optimization problem described by Σ is constrained to the system's dynamics, as given by its centered realization. Let the latter be fixed by $s_0 \in j\mathbb{R}$, as in (5), then we may associate with it the equation

$$\begin{aligned} A^*XE + E^*XA + Q - ((A - s_0E)^*XB + L)R^{-1} \\ \times(L^* + B^*X(A - s_0E)) = 0, \end{aligned} \qquad (9)$$

called the *descriptor continuous-time algebraic Riccati equation* and denoted by DCTARE(Σ, s_0). The matrix $X = X^* \in \mathbb{C}^{n\times n}$ is called the unique *stabilizing solution* of the DCTARE(Σ, s_0) if $\Lambda(A - sE + BF(\alpha - \beta s)) \subset \mathbb{C}^-$, where

$$F := -R^{-1}(L^* + B^*X(\beta A - \alpha E)) \qquad (10)$$

is the *stabilizing Riccati feedback*. Theorem 2 in Dinicu and Oară [2015] gives necessary and sufficient existence conditions for the unique stabilizing solution, together with numerically sound prototype algorithms for its computation.

Remark 1. In the standard (proper) case, when $s_0 = \infty$ and $E = I$, Σ is written in the form of a so-called *Popov triplet* $\Sigma := (A, B; Q, L, R)$ and DCTARE is replaced by the (standard) continuous-time algebraic Riccati equation associated with Σ, denoted CTARE(Σ), respectively

$$A^*X + XA + Q - (XB + L)R^{-1}(B^*X + L^*) = 0, \quad (11)$$

as defined in (3.19) of Ionescu et al. [1999]. In this case, $X = X^*$ is the unique stabilizing solution to the CTARE(Σ) if $\Lambda(A + BF) \subset \mathbb{C}^-$, where now $F := -R^{-1}(B^*X + L^*)$ is the stabilizing Riccati feedback.

2.4 Möbius transformations

We present here several technical results related to Möbius transformations, extracted from Section 7.4 of Tudor [2018] and instrumental in proving the main result of this paper.

Consider throughout this subsection that $s_0 = j\omega_0 \in j\mathbb{R}$ is fixed such that $s_0 \notin \Lambda(A - sE)$. Let the Möbius (omographic) transformation $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be defined as

$$f(\lambda) = \frac{a\lambda + b}{c\lambda + d}, \quad a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0, \quad (12)$$

with $a = j\omega_0,\ b = c = 1,\ d = 0$. For this particular choice, we have that $ad - bc = 1 \neq 0$. Using (12), one may define $\mathfrak{f} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ as

$$\begin{aligned} \mathfrak{f}(\lambda) &= j\omega_0 + \frac{1}{\lambda} =: s,\ \forall \lambda \in \mathbb{C}\backslash\{0\}, \\ \mathfrak{f}(0) &= \infty,\ \mathfrak{f}(\infty) = j\omega_0. \end{aligned} \quad (13)$$

Lemma 2. Consider $\mathfrak{f} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ as given in (13). Then, we have that:

(i) $\mathfrak{f}$ is continuous, bijective, conformal (it preserves angles locally) and its inverse map, $\mathfrak{f}^{-1}$, is given by

$$\begin{aligned} \lambda = \mathfrak{f}^{-1}(s) &= \frac{1}{s - j\omega_0},\ \forall s \in \mathbb{C}\backslash\{j\omega_0\}, \\ \mathfrak{f}^{-1}(j\omega_0) &= \infty,\ \mathfrak{f}^{-1}(\infty) = 0. \end{aligned} \quad (14)$$

(ii) $\mathfrak{f}$ maps the following subsets of $\mathbb{C}$ onto themselves:

$$\mathfrak{f}(j\mathbb{R}) = j\mathbb{R},\ \mathfrak{f}(\mathbb{C}^-) = \mathbb{C}^-,\ \mathfrak{f}(\mathbb{C}^+) = \mathbb{C}^+. \quad (15)$$

The next result showcases the preservation of key system properties under the mapping given by (13).

Lemma 3. Let $\mathbf{H} \in \mathbb{C}^{p\times m}(s)$ be a continuous-time differential-algebraic system having a realization centered at $s_0 = j\omega_0 \in j\mathbb{R}\backslash\Lambda(A - sE)$, as defined in (6). We may now introduce the new linear system

$$\widetilde{\mathbf{H}}(\lambda) := \mathbf{H}(s)\Big|_{s=\mathfrak{f}(\lambda)} = \mathbf{H}\Big(\mathfrak{f}(\lambda)\Big), \quad (16)$$

where $\mathfrak{f}$ is defined in (13). The following statements hold:

(i) $\widetilde{\mathbf{H}}(\lambda)$ is stable if and only if $\mathbf{H}(s)$ is stable.

(ii) $\widetilde{\mathbf{H}} \in \mathbb{C}^{p\times m}(\lambda)$ is a proper continuous-time system having the standard realization

$$\begin{aligned} \widetilde{\mathbf{H}}(\lambda) &= \left[\begin{array}{c|c} (A - j\omega_0 E)^{-1}E - \lambda I & (A - j\omega_0 E)^{-1}B \\ \hline C & D \end{array}\right] \\ &=: \left[\begin{array}{c|c} \widetilde{A} - \lambda I & \widetilde{B} \\ \hline C & D \end{array}\right]. \end{aligned} \quad (17)$$

(iii) $\Lambda(\widetilde{A}) \subset \mathbb{C}^-$ if and only if $\Lambda(A - sE) \subset \mathbb{C}^-$.

(iv) The realization (17) of $\widetilde{\mathbf{H}}(\lambda)$ is stabilizable and detectable if and only if the realization (6) of $\mathbf{H}(s)$ is stabilizable and detectable. Moreover, (17) is a minimal realization if and only if (6) is also minimal.

(v) $\left\|\widetilde{\mathbf{H}}\right\|_\infty = \|\mathbf{H}\|_\infty$.

Finally, we present a connection, under the mapping $\mathfrak{f}$ from (13), between the stabilizing solutions to DCTARE and its standard counterpart, CTARE (see Remark 1 and Ionescu et al. [1999]).

Lemma 4. Let $s_0 = j\omega_0 \in j\mathbb{R}\backslash\Lambda(A-sE)$, $\mathbf{H}(s)$ in (6), and $\widetilde{\mathbf{H}}(\lambda)$ in (16) given by the realization (17). Let

$$\begin{aligned} \Sigma_c^{rs} &:= (A - sE, B;\ Q_c, L_c, R_c), \\ \Sigma_o^{rs} &:= (A^* - sE^*, C^*;\ Q_o, L_o, R_o), \end{aligned} \quad (18)$$

and their proper counterparts

$$\begin{aligned} \widetilde{\Sigma}_c^{rs} &:= (\widetilde{A}, \widetilde{B};\ Q_c, L_c, R_c), \\ \widetilde{\Sigma}_o^{rs} &:= (\widetilde{A}^*, C^*;\ \widetilde{Q}_o, \widetilde{L}_o, R_o), \\ \widetilde{Q}_o &:= (A - s_0E)^{-1}Q_o(A - s_0E)^{-*}, \\ \widetilde{L}_o &:= (A - s_0E)^{-1}L_o. \end{aligned} \quad (19)$$

Then the DCTARE(Σ_c^{rs}, s_0) and DCTARE(Σ_o^{rs}, s_0) have stabilizing solutions $X = X^*$ and $Y = Y^*$ if and only if the CTARE($\widetilde{\Sigma}_c^{rs}$) and CTARE($\widetilde{\Sigma}_o^{rs}$) have stabilizing solutions $\widetilde{X} = \widetilde{X}^*$ and $\widetilde{Y} = \widetilde{Y}^*$, respectively. Furthermore, one obtains:

$$\begin{aligned} \widetilde{X} &= (A - j\omega_0E)^*X(A - j\omega_0E), & \widetilde{Y} &= Y, \\ \widetilde{K}_{rs} &= (A - j\omega_oE)^{-1}K_{rs}, & \widetilde{F}_{rs} &= F_{rs}, \end{aligned} \quad (20)$$

where F_{rs}, K_{rs}, $\widetilde{F}_{rs}$, $\widetilde{K}_{rs}$ are the corresponding stabilizing Riccati feedbacks of DCTARE(Σ_c^{rs}, s_0), DCTARE(Σ_o^{rs}, s_0), CTARE($\widetilde{\Sigma}_c^{rs}$), and CTARE($\widetilde{\Sigma}_o^{rs}$), respectively.

3. $\mathcal{H}_\infty$ SUBOPTIMAL CONTROL

Let $\mathbf{H} \in \mathbb{C}^{p\times m}(s)$ be described by a set of differential-algebraic equations and assume $\mathbf{H}$ is written in partitioned form as

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{H}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \left[\begin{array}{c|c} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \hline \mathbf{H}_{21} & \mathbf{H}_{22} \end{array}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \left[\begin{array}{c|c|c} A - sE & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right]_{s_0}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \end{aligned} \quad (21)$$

with $s_0 = j\omega_0 \in j\mathbb{R}\backslash\Lambda(A - sE)$, $A \in \mathbb{R}^{n\times n}$, $\mathbf{H}_{ij} \in \mathbb{C}^{p_i\times m_j}(s)$, and $i, j = 1, 2$, $m = m_1 + m_2$, $p = p_1 + p_2$. Let $\mathbf{K} \in \mathbb{C}^{m_2\times p_2}(s)$ be an output feedback controller, which is to say that $u_2 = \mathbf{K}y_2$.

The $\mathcal{H}_\infty$ suboptimal control problem consists in finding $\mathbf{K}$ such that the lower linear fractional transformation (abbreviated LLFT), as defined below:

$$\mathbf{H}_{CL} = \text{LLFT}(\mathbf{H}, \mathbf{K}) = \mathbf{H}_{11} + \mathbf{H}_{12}\mathbf{K}(I - \mathbf{H}_{22}\mathbf{K})^{-1}\mathbf{H}_{21} \quad (22)$$

is well-posed, internally stable (for the precise definition, see Section 5.1 of Vidyasagar [1985]) and $\|\mathbf{H}_{CL}\|_\infty < \gamma$, for any feasible γ. In short, internal stability implies that all rational matrices that model the transfers from exogenous inputs to closed-loop signals are stable, and thus implicitly proper (by our definition of stability).

For the above-stated objective to be feasible, the following hypotheses have been inherited from the standard, proper formulation of the control problem (see Theorem 10.3.1 in Ionescu et al. [1999]):

(H_1) The pair $(A - sE, B_2)$ is stabilizable and the pair $(A - sE, C_2)$ is detectable;

(H_2) rank $\begin{bmatrix} A - j\omega E & B_2(j\omega_0 - j\omega) \\ C_1 & D_{12} \end{bmatrix} = n + m_2, \forall \omega \in \mathbb{R};$

(H_3) rank $\begin{bmatrix} A - j\omega E & B_1(j\omega_0 - j\omega) \\ C_2 & D_{21} \end{bmatrix} = n + p_2 \ , \forall \omega \in \mathbb{R};$

(H_4) $\left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & I \\ \hline 0 & I & 0 \end{array}\right].$

We remark that only hypotheses $(H_1)-(H_3)$ are necessary for the existence of a Riccati-based solution. (H_1) is fundamental, as it implies that there exists a class of stabilizing controllers for our system. $(H_2) - (H_3)$ are made so as to ensure the solvability of the dual Riccati equation approach and may be relaxed, if one opts for the linear matrix inequality formulation of the problem. (H_4) is *not at all necessary*, and has been made only to simplify the formulas given in the main result, through the so-called "normalizing conditions". Indeed, a system with an arbitrary feedthrough matrix can be made compliant with (H_4) through a series of reversible transformations (see section 17.2 of Zhou et al. [1996]).

The following theorem represents the main result of this paper, and can be seen as an extension of Theorem 10.3.1 in Ionescu et al. [1999] to systems described by differential-algebraic equations.

Theorem 5. Let $\mathbf{H}$ be given by a realization partitioned as in (21), that satisfies hypotheses $(H_1) - (H_4)$ and with s_0 chosen on $j\mathbb{R}$ such that $s_0 = j\omega_0 \in j\mathbb{R}\backslash\Lambda(A - sE)$.

Consider

$$\begin{aligned} \Sigma_c &:= (A - sE, B_c;\ Q_c, L_c, R_c), \\ \Sigma_o &:= (A^* - sE^*, C_o^*;\ Q_o, L_o, R_o), \end{aligned} \tag{23}$$

where

$$B_c = \begin{bmatrix} \frac{1}{\sqrt{\gamma}}B_1 & \sqrt{\gamma}B_2 \end{bmatrix},\ C_o = \begin{bmatrix} \frac{1}{\sqrt{\gamma}}C_1 \\ \sqrt{\gamma}C_2 \end{bmatrix},$$

$$Q_c = \tfrac{1}{\gamma}C_1^*C_1, \qquad Q_o = \tfrac{1}{\gamma}B_1B_1^*,$$

$$L_c = \begin{bmatrix} 0 & \frac{1}{\sqrt{\gamma}}C_1^*D_{12} \end{bmatrix}, \quad L_o = \begin{bmatrix} 0 & \frac{1}{\sqrt{\gamma}}B_1D_{21}^* \end{bmatrix},$$

$$R_c = \begin{bmatrix} -I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix}, \quad R_o = \begin{bmatrix} -I_{p_1} & 0 \\ 0 & I_{p_2} \end{bmatrix}.$$

Then we have that:

I. There exists a feedback controller $\mathbf{K} \in \mathbb{C}^{m_2 \times p_2}(s)$ that solves the suboptimal $\mathcal{H}_\infty$ control problem if and only if the following conditions hold simultaneously:

(C_1) The DCTARE$(\Sigma_c, j\omega_0)$ has a stabilizing positive semidefinite solution $X = X^* \geq 0$.

(C_2) The DCTARE$(\Sigma_o, j\omega_0)$ has a stabilizing positive semidefinite solution $Y = Y^* \geq 0$.

(C_3) $\rho[(A - j\omega_0E)^*X(A - j\omega_0E)Y] < 1$.

II. Assume that conditions $(C_1) - (C_3)$ hold and let

$$\begin{aligned} F_c &:= \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{\gamma}}B_1^*X(A - j\omega_0E) \\ -\left(\sqrt{\gamma}B_2^*X(A - j\omega_0E) + \frac{1}{\sqrt{\gamma}}D_{12}^*C_1\right) \end{bmatrix} \end{aligned} \tag{24}$$

be the stabilizing feedback corresponding to the DCTARE$(\Sigma_c, j\omega_0)$, and define

$$Z := Y(I - (A - j\omega_0E)^*X(A - j\omega_0E)Y)^{-1}. \tag{25}$$

The class of all controllers that solve the suboptimal $\mathcal{H}_\infty$ control problem is $\mathbf{K} = \text{LLFT}(\mathbf{K}_g, \mathbf{\Phi})$, where $\mathbf{\Phi} \in \mathcal{BH}_\infty^{(\gamma)}$ is arbitrary and

$$\mathbf{K}_g(s) = \left[\begin{array}{c|cc} A_g - sE_g & B_{g1} & B_{g2} \\ \hline C_{g1} & 0 & I \\ C_{g2} & I & 0 \end{array}\right]_{j\omega_0}, \tag{26}$$

with

$$\begin{aligned} A_g &= A + j\omega_0(B_cF_c + B_{g1}C_{g2}), \\ E_g &= E + B_cF_c + B_{g1}C_{g2}, \\ B_{g1} &= -B_1D_{21}^* - (A - j\omega_0E)Z(\gamma C_2^* + \sqrt{\gamma}F_1^*D_{21}^*), \\ B_{g2} &= -B_2 + \frac{1}{\sqrt{\gamma}}(A - j\omega_0E)ZF_2^*, \\ C_{g1} &= -\sqrt{\gamma}F_2, \\ C_{g2} &= C_2 + \frac{1}{\sqrt{\gamma}}D_{21}F_1. \end{aligned}$$

Proof.

I. It can be easily shown that, if $\mathbf{H}(s)$ satisfies hypotheses $(H_1) - (H_4)$, then $\widetilde{\mathbf{H}}(\lambda)$ is eligible for Theorem 10.3.1 in Ionescu et al. [1999], which gives a solution to the suboptimal $\mathcal{H}_\infty$ problem for proper systems. (H_1) and (iv) of Lemma 4 ensure that there exists a class of stabilizing controllers for $\widetilde{\mathbf{H}}(\lambda)$. $(H_2) - (H_3)$ and (ii) of Lemma 3 applied to the zeros of $\mathbf{H}_{12}(s)$ and $\mathbf{H}_{21}(s)$ enable the two CTAREs associated with $\widetilde{\mathbf{H}}(\lambda)$ to have stabilizing solutions. Finally, by (H_4) and (17), "normalizing conditions" also apply to $\widetilde{\mathbf{H}}(\lambda)$.

Partition first

$$\widetilde{\mathbf{H}}(\lambda) := \left[\begin{array}{c|cc} \widetilde{A} & \widetilde{B}_1 & \widetilde{B}_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \tag{27}$$

and recall from (17) that

$$\begin{aligned} \widetilde{A} &= (A - j\omega_0E)^{-1}E, \\ \begin{bmatrix} \widetilde{B}_1 & \widetilde{B}_2 \end{bmatrix} &= (A - j\omega_0E)^{-1}\begin{bmatrix} B_1 & B_2 \end{bmatrix}. \end{aligned}$$

Let

$$\begin{aligned} \widetilde{\Sigma}_c &:= (\widetilde{A}, \widetilde{B}_c;\ Q_c, L_c, R_c), \\ \widetilde{\Sigma}_o &:= (\widetilde{A}^*, C_o^*;\ \widetilde{Q}_o, \widetilde{L}_o, R_o), \end{aligned} \tag{28}$$

where

$$\begin{aligned} \widetilde{B}_c &= \begin{bmatrix} \frac{1}{\sqrt{\gamma}}\widetilde{B}_1 & \sqrt{\gamma}\widetilde{B}_2 \end{bmatrix}, \\ \widetilde{Q}_o &= \tfrac{1}{\gamma}\widetilde{B}_1\widetilde{B}_1^*, \\ \widetilde{L}_o &= \begin{bmatrix} 0 & \frac{1}{\sqrt{\gamma}}\widetilde{B}_1D_{21}^* \end{bmatrix}. \end{aligned} \tag{29}$$

Using statement I of Theorem 10.3.1 in Ionescu et al. [1999], we get that, for a feasible γ, CTARE($\widetilde{\Sigma}_c$) and CTARE($\widetilde{\Sigma}_o$) have positive semidefinite stabilizing solutions and therefore, by Lemma 4, so do DCTARE($\Sigma_c, j\omega_o$) and DCTARE($\Sigma_o, j\omega_o$) for the same γ, thus enabling (C_1) and (C_2) to hold.

Furthermore, let $\widetilde{X}$ be the above-mentioned solution of CTARE($\widetilde{\Sigma}_c$), just as $\widetilde{Y}$ is for CTARE($\widetilde{\Sigma}_o$). From Lemma 4, we have that $\widetilde{Y} = Y$ and that $\widetilde{X} = (A - j\omega_0 E)^* X (A - j\omega_0 E)$. Condition (C_3) of statement I in Theorem 10.3.1 in Ionescu et al. [1999] holds for the previously selected γ, and thus

$$\rho(\widetilde{X}\widetilde{Y}) < 1, \tag{30}$$

from where we retrieve the inequality given by (C_3) from the main result and enable said condition to hold for the same feasible γ.

II. We apply (v) from Lemma 3 to $\mathbf{H}_{CL}$ and get that

$$\|\mathbf{H}_{CL}\|_\infty = \left\|\mathrm{LLFT}(\widetilde{\mathbf{H}}, \widetilde{\mathbf{K}})\right\|_\infty, \tag{31}$$

where

$$\widetilde{\mathbf{K}}(\lambda) := \mathbf{K}(\mathfrak{f}(\lambda)). \tag{32}$$

Additionally, we impose $\widetilde{\mathbf{K}}$ be written as

$$\widetilde{\mathbf{K}} = \mathrm{LLFT}(\widetilde{\mathbf{K}}_g, \widetilde{\mathbf{\Phi}}), \tag{33}$$

with $\widetilde{\mathbf{K}}_g \in \mathbb{C}^{(m_2+p_2)\times(m_2+p_2)}(\lambda)$ and $\widetilde{\mathbf{\Phi}} \in \mathbb{C}^{m_2\times p_2}(\lambda)$. Thus, the controller that enforces $\mathbf{H}_{CL} \in \mathcal{BH}_\infty^{(\gamma)}$ is

$$\mathbf{K} = \mathrm{LLFT}(\mathbf{K}_g, \mathbf{\Phi}), \tag{34}$$

having

$$\mathbf{K}_g(s) = \widetilde{\mathbf{K}}_g(\mathfrak{f}^{-1}(s)), \tag{35}$$
$$\mathbf{\Phi}(s) = \widetilde{\mathbf{\Phi}}(\mathfrak{f}^{-1}(s)). \tag{36}$$

The suboptimal $\mathcal{H}_\infty$ control problem of $\mathbf{H}$ is equivalent to that of $\widetilde{\mathbf{H}}$, with their solutions being linked by (32). We have proven previously that $\widetilde{\mathbf{H}}$ is eligible for Theorem 10.3.1 in Ionescu et al. [1999]. If, additionally, conditions $(C_1) - (C_3)$ pertaining to $\widetilde{\mathbf{H}}$ are satisfied for a chosen γ, then we can define

$$\widetilde{F}_c := \begin{bmatrix} \widetilde{F}_1 \\ \widetilde{F}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\gamma}}\widetilde{B}_1^*\widetilde{X} \\ -\left(\sqrt{\gamma}\widetilde{B}_2^*\widetilde{X} + \frac{1}{\sqrt{\gamma}}D_{12}^*C_1\right) \end{bmatrix} \tag{37}$$

and

$$\widetilde{Z} := \widetilde{Y}(I - \widetilde{X}\widetilde{Y})^{-1}, \tag{38}$$

along with an arbitrary $\widetilde{\mathbf{\Phi}}(\lambda) \in \mathcal{BH}_\infty^{(\gamma)}$ and

$$\widetilde{\mathbf{K}}_g(\lambda) = \left[\begin{array}{c|c:c} \widetilde{A}_g & \widetilde{B}_{g1} & \widetilde{B}_{g2} \\ \hline \widetilde{C}_{g1} & 0 & I \\ \hdashline \widetilde{C}_{g2} & I & 0 \end{array}\right], \tag{39}$$

where

$$\begin{aligned}
\widetilde{A}_g &= \widetilde{A} + \widetilde{B}_c\widetilde{F}_c + \widetilde{B}_{g1}\widetilde{C}_{g2}, \\
\widetilde{B}_{g1} &= -\widetilde{B}_1 D_{21}^* - \widetilde{Z}(\gamma C_2^* + \sqrt{\gamma}\widetilde{F}_1^* D_{21}^*), \\
\widetilde{B}_{g2} &= -\widetilde{B}_2 + \frac{1}{\sqrt{\gamma}}\widetilde{Z}\widetilde{F}_2^*, \\
\widetilde{C}_{g1} &= -\sqrt{\gamma}\widetilde{F}_2, \\
\widetilde{C}_{g2} &= C_2 + \frac{1}{\sqrt{\gamma}}D_{21}\widetilde{F}_1.
\end{aligned}$$

From Lemma 4, we have that $\widetilde{F}_c = F_c$. Given that $\widetilde{B}_c^*\widetilde{X} = B_c^* X(A - j\omega_0 E)$ by (20) and (29), one retrieves (24). From the same result, we can further deduce the following three identities:

$$\widetilde{Z} = Y(I - (A - j\omega_0 E)^* X (A - j\omega_0 E)Y) \equiv Z, \tag{40}$$

$$\widetilde{C}_{g1} = -\sqrt{\gamma}F_2 \equiv C_{g1}, \tag{41}$$

$$\widetilde{C}_{g2} = C_2 + \frac{1}{\sqrt{\gamma}}D_{21}F_1 \equiv C_{g2}. \tag{42}$$

Using (40)-(42), we notice that

$$\widetilde{A}_g = (A - j\omega_0 E)^{-1}E_g, \tag{43}$$
$$\widetilde{B}_{g1} = (A - j\omega_0 E)^{-1}B_{g1}, \tag{44}$$
$$\widetilde{B}_{g2} = (A - j\omega_0 E)^{-1}B_{g2}. \tag{45}$$

Denoting

$$B_g = [\, B_{g1} \;\; B_{g2} \,],\; C_g = \begin{bmatrix} C_{g1} \\ C_{g2} \end{bmatrix},\; D_g = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

and plugging (41)-(45) into (39), we may perform a change of variable $\lambda = \frac{1}{s - j\omega_0}$ and group up terms as in (17) to obtain that $\widetilde{\mathbf{K}}_g(\mathfrak{f}^{-1}(s))$ has the centered realization from (26).

Finally, we need only prove that, if $\widetilde{\mathbf{\Phi}}(\lambda) \in \mathcal{BH}_\infty^{(\gamma)}$, then $\widetilde{\mathbf{\Phi}}(\mathfrak{f}^{-1}(s)) \in \mathcal{BH}_\infty^{(\gamma)}$. This follows directly from (i) and (v) of Lemma 3.

□

Remark 6. For most practical applications, the central controller, which is obtained by taking $\mathbf{\Phi} = 0$, is sufficient in providing an adequate solution. However, upon inspecting (26), one notices that the matrix E_g is not guaranteed to be invertible, which means that the central controller may be improper depending on the target plant. Therefore, it is fortunate that we possess an entire class of controllers, as parametrized by the pair $(\mathbf{K}_g, \mathbf{\Phi})$, which may be used to guarantee that the obtained controller is always proper. Recall (34) and the structure of $\mathbf{K}_g$ from (26). Then, by Theorem 12.19 from Zhou et al. [1996], we may alternatively express the class of suboptimal $\mathcal{H}_\infty$ controllers through

$$\mathbf{K}(\mathbf{\Phi}) = (\mathbf{K}_{11}\mathbf{\Phi} + \mathbf{K}_{12})(\mathbf{K}_{21}\mathbf{\Phi} + \mathbf{K}_{22})^{-1} \equiv \mathrm{LLFT}(\mathbf{K}_g, \mathbf{\Phi}), \tag{46}$$

where

$$\mathbf{K}_R(s) := \left[\begin{array}{c:c} \mathbf{K}_{11}(s) & \mathbf{K}_{12}(s) \\ \hdashline \mathbf{K}_{21}(s) & \mathbf{K}_{22}(s) \end{array}\right] = \left[\begin{array}{c|c:c} A_g - sE_g - B_{g1}C_{g2}(j\omega_0 - s) & B_{g2} & B_{g1} \\ \hline C_{g1} & I & 0 \\ \hdashline C_{g2} & 0 & I \end{array}\right]_{j\omega_0} \tag{47}$$

is a right coprime factorization, with all partitions proper and stable. Assume the central controller, given by

$$\mathbf{K}(0) = \mathbf{K}_{12}\mathbf{K}_{22}^{-1}, \tag{48}$$

is improper. Then, it must be that $\mathbf{K}_{22}$ has zeros at ∞. However, both $\mathbf{K}_{21}$ and $\mathbf{K}_{22}$ may not simultaneously have zeros at ∞ as, being partitions of a right coprime factorization, that would invalidate the Bézout identity that said partitions must satisfy for any s, including the point at ∞. Therefore, any $\mathbf{\Phi}(s) \in \mathcal{BH}_\infty^{(\gamma)}$ with $\mathrm{rank}\,[\mathbf{\Phi}(\infty)] = \mathrm{rank}_n\,[\mathbf{\Phi}(s)]$ guarantees that $\mathbf{K}(\mathbf{\Phi})$ is proper when $\mathbf{K}(0)$ is not.

4. NUMERICAL EXAMPLE

Consider the following example. Let

$$\mathbf{H}(s) = \begin{bmatrix} \dfrac{s}{s+2} & s \\ \dfrac{s}{s+2} & s+1 \\ \dfrac{s^3+5s^2+4}{s^2+4s+4} & s^2+s \end{bmatrix}.$$

Notice that $\mathbf{H}$ has both proper and improper elements. The system possesses two double poles, at ∞ and at -2, thus having a McMillan degree of $\delta(\mathbf{H}) = 4$. By centering at $s_0 = 0 \in j\mathbb{R}$, one obtains a minimal realization

$$\mathbf{H}(s) = \left[\begin{array}{cccc|c:c} 1 & 0.5633s & 0 & 0 & -1.1945 & -0.9646 \\ 0 & 1 & 0 & 0 & 0 & 2.1206 \\ 0 & 0 & 1+0.5s & 0 & 1.1067 & 0 \\ 0 & 0 & -0.9036s & 1+0.5s & -1 & 0 \\ \hline 0 & 0.4716 & 0.4518 & 0 & 0 & 0 \\ 0 & 0.4716 & 0.4518 & 0 & 0 & 1 \\ \hdashline -0.8372 & 0.0907 & -0.9036 & 1 & 1 & 0 \end{array}\right]_0 .$$

The order of the above realization is $n = \delta(\mathbf{H}) = 4$ and $\mathbf{H}(0) = D$. We now obtain the class of suboptimal $\mathcal{H}_\infty$ controllers for $\gamma = 0.7$ using Theorem 5. The DCTAREs have stabilizing solutions (see Dinicu and Oară [2015] for computational details):

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.1374 & 0.0973 & 0 \\ 0 & 0.0973 & 0.0895 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} 2.0006 & 0 & -1.3290 & 0.0868 \\ 0 & 0 & 0 & 0 \\ -1.3290 & 0 & 1.0367 & -0.3733 \\ 0.0868 & 0 & -0.3733 & 0.6511 \end{bmatrix}.$$

The class of all suboptimal $\mathcal{H}_\infty$ controllers, $\mathbf{K}(\mathbf{\Phi})$, that ensure $\|\mathbf{H}_{CL}\|_\infty < 0.7 = \gamma$ is given by

$$\mathbf{K}_g(s) = \left[\begin{array}{cccc|c:c} -0.5765 & 0.6923 & 0.5105 & 0.2456 & -0.5318 & -1.1390 \\ -1.1396 & -0.6657 & -0.2723 & 0.3141 & -0.6442 & -0.6736 \\ 0.4139 & 0.8594 & -0.6829 & -0.4765 & -0.1770 & 0.8021 \\ 0.3667 & 0.3482 & -0.5949 & -0.4760 & -0.0575 & 0.5331 \\ \hline 0.6043 & 0.7263 & -0.6937 & -0.5228 & 0 & 1 \\ \hdashline 1.1023 & -0.5842 & -0.4399 & -0.1191 & 1 & 0 \end{array}\right]_0 ,$$

and $\mathbf{\Phi}(s) \in \mathcal{BH}_\infty^{(\gamma)}$. The central controller, for $\mathbf{\Phi} = 0$, is

$$\mathbf{K}(s) = \frac{-2.019s^2 - 4.038s}{s^4 + 12.8s^3 + 15.72s^2 + 15.24s + 6.346}.$$

Note that $\mathbf{K}$ is proper, and thus readily available for implementation in any practical control scheme. The closed-loop system's poles:

$$\Lambda\left[\mathbf{H}_{CL}(s)\right] = \{-0.2156 \pm 0.7534j, -0.6645, -2\},$$

indicate that $\mathbf{K}$ is indeed a stabilizing controller, but also a suboptimal $\mathcal{H}_\infty$ controller, since

$$\|\mathbf{H}_{CL}\|_\infty = 0.6795 < 0.7 = \gamma.$$

5. CONCLUSIONS

For a differential-algebraic system, we have solved the suboptimal $\mathcal{H}_\infty$ control problem and offered a viable alternative to the approach in Stefanovski [2015] for improper systems with zeros not on the extended imaginary axis. For these systems, our chief contribution is that we can always obtain a proper controller through a less computationally expensive procedure (see the conclusions of Stefanovski [2015]), based upon a Möbius transformation and a pair of algebraic Riccati equations. In real-life applications, it is beneficial to employ a proper controller due to its ease of implementation and lack of impulsive behavior.

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