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Formulation of min-max model predictive control as a box-constrained robust least squares estimation problem

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Abstract: The purpose of this paper is to propose a new approach to the Min-Max Model Predictive Control (MMMPC) of Linear Time-Invariant Discrete-time Polytopic (LTIDP) systems. The purpose is to simplify the treatment of complex issues like stability and feasibility analysis of robust MPC as well as to reduce the complexity of the relative optimization procedure. The new approach is based on a two Degrees Of Freedom (2DOF) control scheme where the output $r(k)$ of the feedforward Input Estimator (IE) is used as input forcing a stable closed-loop system Σ_f. Σ_f is the feedback connection of an LTIDP plant Σ_p with an LTI dynamic controller Σ_g. The task of Σ_g is to guarantee the quadratic stability of Σ_f, as well as the fulfillment of hard constraints on some physical variables of Σ_f for any input $r(k)$ satisfying an "a priori" determined admissibility condition. The input $r(k)$ is computed by the feedforward IE through the on-line minimization of a worst case finite-horizon quadratic cost functional and is applied to Σ_f according to the usual receding horizon strategy. Rather than resorting to an "ad hoc" software, the numerical complexity issue is here addressed reducing the number of both decision variables and constraints involved in the on-line constrained optimization procedure. This is obtained modeling $r(k)$ as a B-spline function, which is known to be a universal approximator which also admits a parsimonious parametric representation. This allows us to reformulate the minimization of the worst case cost functional as a box-constrained Robust Least Squares (RLS) estimation problem which can be efficiently solved using Second Order Cone Programming (SOCP).

Keywords: Model Predictive Control, Receding Horizon, Constrained Optimization, Second Order Cone Programming.

1. INTRODUCTION

A common approach to robust MPC is its formulation in terms of a closed-loop min-max constrained optimization problem. The usually proposed approaches (see e.g. Kothare et al. (1996); Scokaert and Mayne (1998); Bemporad et al. (2003); Wan and Khotare (2003);Kerrigan and Maciejowki (2004); Sakizlis et al. (2004); Munoz et al. (2006); Raminez et al. (2006); Gao and Chong (2012) and references therein) inherit in a considerably increased way the major issues of MPC for exactly known plants: more complicated stability and feasibility conditions and, especially, much more computationally demanding procedures for the numerical solution of the on line optimization problem. In fact the MMMPC requires minimizing the worst case of a cost functional which is computed as the maximum with respect to all the possible uncertainties over the prediction horizon.

The twofold purpose of this paper is: 1) to propose a novel MMMPC strategy characterized by greatly simplified stability and feasibility analysis, 2) to significantly reduce the complexity of the on line constrained optimization procedure.

The basic point of the alternative approach proposed here is the adoption of an MPC strategy in a 2DOF control scheme to exploit the advantages of feedback prediction and of the degrees of freedom introduced by the feedforward IE. In practice, the present MMMPC works according to the following two-step procedure:

- Step 1. Given an LTIDP plant Σ_p, a LTI dynamic controller Σ_g is designed to guarantee the quadratic stability of the closed-loop system Σ_f and the fulfillment of hard constraints on some physical variables in correspondence of any admissible input (i.e. $\|r(k)\|_2^2 \leq \gamma, \ \forall k \geq 0$, for a suitably computed γ) forcing Σ_f.
- Step 2. An admissible input sequence $r(k)$ is applied to Σ_f according to a receding horizon control strategy. This sequence is computed searching for the minimum of a "worst case" quadratic cost functional over each prediction interval in the linear space generated by B-spline functions of a fixed degree. This second step is executed by the feedforward IE.

Decomposing the MMMPC problem in the above two distinct steps entails the following remarkable advantages: 1) Stability and recursive feasibility of the adopted MMMPC strategy are guaranteed in advance, regardless

the chosen prediction horizon. In fact, the internal stability of Σ_f and the admissibility condition on $r(k)$ assure both the uniform boundedness of any internal variable of the 2DOF control scheme and the fulfillment of all constraints at any time instant.
2) If Σ_g also contains an internal model of the desired (and admissible) reference to be tracked, an exact asymptotic tracking can be directly achieved even in the case of plant-model mismatch, (Desoer and Wang (1980)). This greatly simplifies the alternative solutions which are mostly based on augmenting the model of the plant, which in turn implies an increase of the decision variables involved in the optimization problem (see e.g. Maeder and Morari (2010)). The internal model also yields a Σ_f with a diagonal static gain matrix, so that it guarantees the noticeable advantage of an exact static decoupling, (Jetto and Orsini (2018)).
3) Modeling $r(k)$ as a B-spline decreases the number of decision variables because these functions are universal approximators which admit a parsimonious parametric representation and belong to the convex hull defined by the relative control points, De Boor (1978). This property allows the transfer of any amplitude constraint defined on a B-spline function to its control points. As a consequence (see Section 5), the constrained minimization of the cost functional can be formulated as a box-constrained RLS estimation problem with only box constraints on the unknowns (the control points defining the admissible B-spline function $r(k)$). Approaching this problem by SOCP allows the application of numerically efficient primal-dual interior-point methods (El Ghaoui and Lebret (1997); Lobo et al. (1998)).
The paper is organized in the following way. Some mathematical preliminaries are recalled in Section 2, the problem setting is defined in Section 3, the design of the internal feedback controller is illustrated in Section 4. The constrained on line estimation of the input $r(k)$ is explained in Section 5. A numerical example is reported in Section 6. Some concluding remarks are given in Section 7.

2. MATHEMATICAL BACKGROUND

2.1 B-spline functions (De Boor (1978))

Analytic B-splines are defined in the following way:

$$s(v) = \sum_{i=1}^{\ell} c_i B_{i,d}(v), \qquad v \in [\hat{v}_1, \hat{v}_{\ell+d+1}] \subseteq \mathbb{R}, \qquad (1)$$

where the c_i's are real numbers representing the control points of $s(v)$, d is the degree of the spline, the $(\hat{v}_i)_{i=1}^{\ell+d+1}$ are the non decreasing knot points, and the $B_{i,d}(v)$ are given by the Cox-de Boor recursion formula.

Convex hull property. Any value assumed by $s(v)$, $\forall v \in [\hat{v}_j, \hat{v}_{j+1}]$, $j > d$, lies in the convex hull of its $d+1$ control points $c_{j-d}, \cdots, c_j$.

Smoothness property. Suppose that $\hat{v}_i < \hat{v}_{i+1} = \cdots = \hat{v}_{i+m} < \hat{v}_{i+m+1}$, with $1 \leq m \leq d+1$ then the B-spline function $s(v)$ has continuous derivative up to order $d-m$ at knot $\hat{v}_{i+1}$. This property implies that the spline smoothness can be changed using multiple knot points. It is common choice to set $m = d+1$ multiple knot points for the initial and the last knot points and to evenly distribute the other ones. In this way (1) assumes the first and the final control points as initial and final values.

Identifying the parameter v of (1) with the time instant t, the sampled B-spline $s(k\,T_c)$ is obtained by direct uniform sampling of the corresponding analytic B-spline.

The discrete B-spline $s(k)$ (omitting the explicit dependence on T_c) can be used to represent a scalar discrete time signal. Defining

$$\mathbf{c} \triangleq [c_1 \cdots c_\ell]^T, \quad \mathbf{B}_d(k) \triangleq [B_{1,d}(k) \cdots B_{\ell,d}(k)], \qquad (2)$$

where each $B_{i,d}(k)$ is obtained setting $v = k$ and $\hat{v}_i = \hat{k}_i$, $i = 1, \cdots, d+\ell+1$, the sampled B-spline $s(k)$ can be represented as

$$s(k) = \mathbf{B}_d(k)\mathbf{c}, \quad k \in [\hat{k}_1, \hat{k}_{\ell+d+1}]. \qquad (3)$$

For a q-component vector $\mathbf{s}(k) = [s_1(k) \cdots s_q(k)]^T$, a compact B-splines representation can be used

$$\mathbf{s}(k) = \bar{\mathbf{B}}_d(k)\bar{\mathbf{c}}, \quad k \in [\hat{k}_1, \hat{k}_{\ell+d+1}], \qquad (4)$$

where: $\bar{\mathbf{c}} \triangleq \left[\mathbf{c}_1{}^T \cdots \mathbf{c}_q{}^T\right]^T$, $\bar{\mathbf{B}}_d(k) \triangleq \text{diag}\,[\mathbf{B}_d(k) \cdots \mathbf{B}_d(k)]$. Each $\mathbf{c}_i \triangleq [c_{i,1} \cdots c_{i,\ell}]^T$, $i = 1, \cdots, q$, is defined as in (2). The dimensions of $\bar{\mathbf{c}}$ are $(q\ell \times 1)$. The dimensions of the block diagonal matrix $\bar{\mathbf{B}}_d(k)$ are $(q \times q\ell)$.

Remark 1. From (3) it is apparent that, once the degree d and the knot points $\hat{k}_i$ have been fixed, the scalar B-spline $s(k)$, $k \in [\hat{k}_1, \hat{k}_{\ell+d+1}]$, is completely determined by the corresponding vector $\mathbf{c}$ of ℓ control points. As, in general, $\ell << k_M$, where k_M is the number of sampled instants of $[\hat{k}_1, \hat{k}_{\ell+d+1}]$, B-splines are said to admit a parsimonious parametric representation.

2.2 SOCP formulation of the RLS problem (El Ghaoui and Lebret (1997),Lobo et al. (1998))

Given an overdetermined set of linear equations $Df \approx g$, with $D \in \mathbb{R}^{r\times s}$, $g \in \mathbb{R}^r$, subject to unknown but bounded errors: $\|\delta D \;\; \delta g\|_F \leq \rho$, $(\rho > 0)$, the robust least squares estimate $\hat{f} \in \mathbb{R}^s$ is the value of f minimizing the following L_2 norm

$$\phi(D, g, \rho) \triangleq \min_f \max_{\|\delta D\, \delta g\|_F \leq \rho} \|(D+\delta D)f - (g+\delta g)\|_2, \; (5)$$

where $\|\cdot\|_F$ denotes the Frobenius norm.
Assuming $\rho = 1$, in (El Ghaoui and Lebret (1997)) it is shown that problem (5) can be formulated as the following SOCP

$$\begin{aligned} &\text{minimize} \qquad \lambda \\ &\text{subject to } \|(Df-g)\|_2 \leq \lambda - \tau, \|[f^T, 1]^T\|_2 \leq \tau, \end{aligned}$$

which can be efficiently solved using interior point methods. Possible constraints on f of the kind $f_{min} \leq f \leq f_{max}$, can be taken into account by imposing all the scalar linear inequalities deriving from the above vector constraint.
The solution of (5) can be directly extended to the case $\rho \neq 1$, using the fact: $\rho\phi(D/\rho, g/\rho, 1) = \phi(D, g, \rho)$.

2.3 System constraints and invariant sets

Consider a generic LTIDP plant of the form

$$x(k+1) = A(\alpha)x(k) + Br(k), \quad y(k) = Cx(k), \tag{6}$$

where $x(k) \in \mathbb{R}^n$, $r(k) \in \mathbb{R}^q$, $y(k) \in \mathbb{R}^q$, $A(\alpha)$ belongs to the polytopic matrix family $A(\alpha) \triangleq \sum_{i=1}^{l} \alpha_i A_i$ and the vector $\alpha = [\alpha_1, \cdots, \alpha_l]^T$, belongs to the unit simplex (denoted by Λ_l).

An invariant γ-feasible set of (6) is a convex compact set $\mathcal{X}$ containing the origin, such that, for every input $r(k)$, $k \geq 0$ satisfying the following admissibility condition

$$\|r(k)\|_2^2 \leq \gamma, \tag{7}$$

one has $x(k) \in \mathcal{X} \Rightarrow Ax(k) + Br(k) \in \mathcal{X}$ and the following constraints are satisfied

$$|z_i(k)| = \|z_i(k)\|_2 \leq \bar{z}_i, \; i = 1, \cdots, h, \tag{8}$$

where $z_i(k)$ is the i-th element of the h-vector $z(k) = C_z x(k)$, and $\bar{z}_i$ is the corresponding pre-specified hard constraint.

Here $\mathcal{X}$ is assumed to be an ellipsoid set defined as $\mathcal{E}(P, \gamma) = \{x(k) \mid x^T(k)Px(k)\} \leq \gamma$, where $P \triangleq Q^{-1}$ is a symmetric positive definite matrix.

3. PROBLEM SETUP

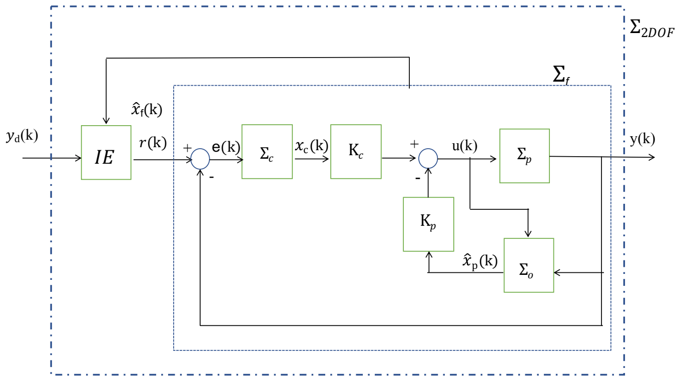

Fig. 1. 2DOF control scheme

The MMMPC strategy proposed in this paper is realized through the 2DOF control scheme shown in Fig. 1, where: $y_d(k) \in \mathbb{R}^q$ is the piecewise constant desired reference to be tracked and $y(k) \in \mathbb{R}^q$ is the controlled output. The output of the feedforward IE is the input $r(k) \in \mathbb{R}^q$ forcing Σ_f.

The block $\Sigma_f \equiv (C_f, A_f(\alpha), B_f)$ is the feedback connection of a LTIDP plant $\Sigma_p \equiv (C_p, A_p(\alpha), B_p)$, $\alpha \in \Lambda_l$, like (6) with a dynamic LTI output controller Σ_g which includes the internal model of constant signals Σ_c and a full state observer Σ_o. The state vectors of Σ_c, Σ_p and Σ_f are denoted by $x_c(k)$, $x_p(k)$ and $x_f(k)$ respectively. The vectors $\hat{x}_p(k)$ and $\hat{x}_f(k)$ are the estimates of $x_p(k)$ and $x_f(k)$. The control input forcing the LTIDP plant Σ_p is denoted as $u(k) \in \mathbb{R}^m$.

In view of the tracking requirement the following assumptions on Σ_p are made: A1) $m \geq q$, A2) for no value of $\alpha \in \Lambda_l$, Σ_p has a transmission zero at $z = 1$ of Z plane. The explicit expressions of Σ_c and Σ_o will be given in the next section. As Σ_g is LTI and independent of α, also $\Sigma_f \equiv (C_f, A_f(\alpha), B_f)$ results to be an LTIDP system of the same kind of Σ given by (6).

The purpose of Σ_g is to guarantee the fulfillment of the following requirements:

- r1) quadratic stability of Σ_f;
- r2) the existence of an invariant γ-feasible set $\mathcal{X}$ for Σ_f, such that $x_f(k) \in \mathcal{X} \Rightarrow A_f(\alpha)x_f(k) + B_f r(k) \in \mathcal{X}$, $\forall \alpha \in \Lambda_l$, and constraints like (8) are satisfied by each component of the vector $z_f(k) = C_{z_f} x_f(k)$, for any admissible input $r(k)$ of Σ_f satisfying (7).

Vector $z_f(k)$ defines the constrained variables corresponding to some suitably defined C_{z_f}. Asymptotic tracking of a fixed set point $y_d(k) = y_d$, can be obtained as a consequence of r1, of assumptions A1) and A2) and of the introduction of Σ_c (Desoer and Wang; 1980), provided that $r(k)$ converge to y_d.

The inputs of IE are $y_d(k)$ and $\hat{x}_f(k)$. This information is exploited by the IE to compute $r(k)$ solving the following Min-Max Constrained Optimization Problem (MMCOP) at each $k = jN_r$, for some $N_r > 0$, $j = 0, 1, 2 \cdots$,

$$\textbf{MMCOP}: \quad \min_{[r(k), \cdots, r(k+N_y-1)]} \max_{\alpha \in \Lambda_l} J_\alpha,$$

$$\begin{aligned} J_\alpha &\triangleq \sum_{i=1}^{N_y} e_y^T(k+i/k) Q_y(k+i/k) e_y(k+i/k) \\ &+ \lambda_1(k) \sum_{i=0}^{N_y-1} e_r^T(k+i/k) Q_r(k+i/k) e_r(k+i/k) \\ &+ \lambda_2(k) \sum_{i=1}^{N_u} e_u^T(k+i/k) Q_u(k+i/k) e_u(k+i/k), \end{aligned} \tag{9}$$

where $Q_y(k+i/k)$, $Q_r(k+i/k)$ and $Q_u(k+i/k)$ are positive definite matrices and

$$N_y \geq N_u, \; N_y > N_r, \; \lambda_1(k) \geq 0, \lambda_2(k) \geq 0, k \geq 0 \tag{10}$$

$$e_y(k+i/k) \triangleq y_d(k) - y(k+i/k), \tag{11}$$

$$e_r(k+i/k) \triangleq y_d(k) - r(k+i), \tag{12}$$

$$e_u(k+i/k) \triangleq u(k+i/k) - \tilde{u}(k), \tag{13}$$

subject to

$$r_{min} \leq r(k+i) \leq r_{max}, \;\; i = 0, \cdots, N_y - 1. \tag{14}$$

In the above equations $\tilde{u}(k)$ is the steady-state value of $u(k)$ corresponding to a suitably defined nominal plant, $y(k+i/k)$, $u(k+i/k)$ and $r(k+i)$ are the predicted output, control effort and B-spline respectively, r_{min} and r_{max} are q-vectors computed so as to satisfy (7).

Note that in equations (11)-(12), the reference trajectory is evaluated at time instant k to avoid undesired anticipative effects on $y(k)$ due to possible set point changes inside the prediction horizon N_y.

The MMCOP is solved at each time instant $k = jN_r$ and only the first N_r samples of the whole sequence $[r(k), \cdots, r(k+N_y-1)]$ are applied to Σ_f according to the receding horizon control policy.

Remark 2. The considerations developed in this section clearly show the idea underlying the present approach and the relative advantages of the resulting MMMPC procedure. Designing Σ_g according to r1 guarantees the uniform boundedness of $x_f(k)$ for any uniformly bounded $r(k)$, independently of N_y, N_u, N_r, $\lambda_1(\cdot)$, $\lambda_2(\cdot)$, $Q_u(\cdot)$, $Q_r(\cdot)$ and $Q_y(\cdot)$. This releases the stability issue from the prediction horizon and other tuning parameters. Requirement r2 allows us to transfer any constraint on $z_f(k)$ of the kind (8) on a corresponding upper bound γ on $\|r(k)\|_2^2$. This bound is explicitly taken into account in the MMCOP through (14). As it will be formally stated in Theorem 2 of Section 5, the above implies that the proposed two-step procedure yields an MMMPC strategy with guarantee of internal stability of Σ_{2DOF} and recursive feasibility.

Remark 3. The presence of the internal model Σ_c guarantees exact asymptotic tracking if $r(k)$ exactly converges to the desired set point value. The penalty term $\lambda_1(k)\sum_{i=0}^{N_y-1} e_r^T(k+i/k)Q_r(k+i/k)e_r(k+i/k)$ is useful to speed up such a convergence. This is particularly important in the case of piecewise constant signals $y_d(k)$ which are not frozen on a fixed set point for a sufficiently long time interval and tracking precision is the dominant criterion.

4. STEP 1: DESIGN OF Σ_G

The feedback controller is designed here using ellipsoidal robust invariant sets because of their closed relation to quadratic Lyapunov functions leading to an LMI-based optimization problem.
The controller Σ_g includes the internal model of constant signals Σ_c, whose state-space representation is $x_c(k+1) = A_c x_c(k) + B_c(r(k) - y(k))$ $(A_c = B_c = I_q)$ and a full state observer Σ_o of the form

$$\hat{x}_p(k+1) = \bar{A}_p\hat{x}_p(k) + B_p u(k) + L(y(k) - C_p\,\hat{x}_p(k)), \quad (15)$$

where: $\bar{A}_p \triangleq (\sum_{i=1}^{l} A_{p_i})/l$ is the assumed nominal dynamical matrix of the plant.
The output $u(k) \in \mathbb{R}^m$, of Σ_g forcing the polytopic plant Σ_p is given by

$$u(k) = -K_p\hat{x}_p(k) + K_c x_c(k). \quad (16)$$

The state space representation $(C_f, A_f(\alpha), B_f)$ of the square closed loop system Σ_f with $x_f \triangleq [\hat{x}_p^T,\, x_c^T,\, x_p^T - \hat{x}_p^T]^T \in \mathbb{R}^n$ and $n \triangleq 2n_p + n_c$ is

$$x_f(k+1) = \begin{bmatrix} \bar{A}_p - B_pK_p & B_pK_c & LC_p \\ -B_cC_p & A_c & -B_cC_p \\ \Delta A_p(\alpha) & 0 & A_p(\alpha) - LC_p \end{bmatrix} x_f(k) + \begin{bmatrix} 0 \\ B_c \\ 0 \end{bmatrix} r(k) \quad (17)$$

$$y(k) = [C_p\ 0\ C_p]\, x_f(k) \quad (18)$$

where $\Delta A_p(\alpha) \triangleq A_p(\alpha) - \bar{A}_p$.
The constrained variables are

$$z_f(k) \triangleq [z_u^T(k), z_{x_f}^T(k)]^T \quad (19)$$

where the respective components $z_{u,r}(k)$ and $z_{x_f,w}(k)$ have to satisfy (8) for some given $\bar{z}_{u,r}$ and $\bar{z}_{x_f,w}$ respectively. Typically $z_u(k) = C_{z_u}x_f(k) = u(k)$, so that, by (16), $C_{z_u} = [-K_p\ K_c\ 0] \triangleq \hat{K}$ while $z_{x_f}(k) = C_{z_{x_f}}x_f(k) \in \mathbb{R}^{n_{x_f}}$ represents any vector of variables linearly depending on the state. For example if $z_{x_f}(k) = y(k)$ then by (18) $C_{z_{x_f}} = C_f = [C_p\ 0\ C_p]$ and $n_{x_f} = q$.
It is remarked that the above distinction between $z_u(k)$ and $z_{x_f}(k)$, is necessary because, unlike $z_{x_f}(k)$, $z_u(k) \triangleq u(k)$, depends on $x_f(k)$ through of a matrix which is a design parameter. Such a matrix has to be determined imposing the fulfillment of the control specifications.

Once Σ_c has been designed according to the internal model principle, the controller gain matrices are computed as specified in the next section.

4.1 Design of the controller gains

For any fixed matrix L of the observer (15), the gain matrix $[-K_p\ K_c\ 0] \triangleq \hat{K}$ can be computed observing that by (17) the polytopic closed loop dynamical matrix $A_f(\alpha)$ can be rewritten as $A_f(\alpha) \triangleq \hat{A}(\alpha) + \hat{B}\hat{K}$, where

$$\hat{A}(\alpha) = \begin{bmatrix} \bar{A}_p & 0 & LC_p \\ -B_cC_p & A_c & -B_cC_p \\ \Delta A_p(\alpha) & 0 & A_p(\alpha) - LC_p \end{bmatrix}, \hat{B} = \begin{bmatrix} B_p \\ 0 \\ 0 \end{bmatrix} \quad (20)$$

Equations (20) are used to design Σ_g according to the following procedure, which can be devised as a sort of separation principle working for sufficiently small parametric uncertainty:

i) The observer gain matrix L is chosen such that $(A_p(\alpha) - LC_p)$ is quadratically stable $\forall\alpha \in \Lambda_l$.
ii) Once the observer Σ_o has been designed, the gain matrix $\hat{K}$ is computed as solution of the following problem.
P1 Given the polytopic plant $(\hat{A}(\alpha), \hat{B})$ in (20), find a matrix $\hat{K}$ and the maximum invariant γ-feasible set $\mathcal{X}$ (where also γ is maximized), such that the following conditions are satisfied:
  · $\Sigma_f \equiv (C_f, \hat{A}(\alpha)+\hat{B}\hat{K}, B_f)$ is quadratically stable $\forall\alpha \in \Lambda_l$,
  · constraints on $z_f(k)$ are fulfilled for every initial condition $x_f(0) \in \mathcal{X}$, $\forall\alpha \in \Lambda_l$ and every admissible input $r(k)$ satisfying (7).

Remark 4. Since in the augmented state x_f only the plant state x_p, is of interest, instead of maximizing the entire ellipsoid volume only the ellipsoid projection on x_p subspace is maximized.The projection of $\mathcal{X}$ onto x_p is given by $\mathcal{X}_{x_p} \triangleq \mathcal{E}_{x_p}(P,\gamma) = \{x_p(k)\,|\,x_p^T(k)(T_{x_p}QT_{x_p}^T)^{-1}x_p(k) \leq \gamma\}$ with T_{x_p} defined by $x_p = T_{x_p}x_f$.

Theorem 1. Consider the plant $(\hat{A}(\alpha), \hat{B})$ in (20) and define η as $\eta \triangleq \gamma^{-1}$. Quadratic stability and the invariant γ feasible set $\mathcal{X}$ (where both $\mathcal{X}_{x_p}$ and γ are maximized) for $\Sigma_f \equiv (C_f, \hat{A}(\alpha) + \hat{B}\hat{K}, B_f)$ subject to (8) and forced by any $r(k)$ satisfying (7), are obtained by solving following the semidefinite programming problem:

$$\text{minimize}\quad (\text{-log}(\det(T_{x_p}QT_{x_p}^T) + \eta)\quad \text{subject to:}$$

$$\begin{bmatrix} Q & 0 & \beta Q & Q\hat{A}_i^T + Y^T\hat{B}^T \\ 0 & \beta I & 0 & B_f^T \\ \beta Q & 0 & \beta Q & 0 \\ \hat{A}_i Q + \hat{B}Y & B_f & 0 & Q \end{bmatrix} \geq 0, \quad i = 1, \cdots, l \tag{21}$$

$$\begin{bmatrix} Q & Y^T I_r^T \\ I_r Y & \bar{z}_{u,r}^2 \eta \end{bmatrix} \geq 0, \quad r = 1, \cdots, m \tag{22}$$

$$\begin{bmatrix} Q & (Q\hat{A}_i^T + Y^T\hat{B}^T)C_{z_{x_f}}^T I_w^T \\ I_w C_{z_{x_f}}(\hat{A}_i Q + \hat{B}Y) & \bar{z}_{x_f,w}^2 \eta - I_w C_{z_{x_f}} B_f B_f^T C_{z_{x_f}}^T I_w^T \end{bmatrix} \geq 0$$

$$i = 1, \cdots, l \text{ and } w = 1, \cdots, n_{x_f}. \tag{23}$$

in the variables $\eta > 0$, $\beta \in [0, 1)$, $Q = Q^T = \text{diag}[Q_1, Q_2] \in \mathbb{R}^{n \times n}$, $n = 2n_p + n_c$ and $Y = [Y_1\ 0] \in \mathbb{R}^{m \times n}$, $Y_1 \in \mathbb{R}^{m \times (n_p + n_c)}$ and in the vertices

$$\hat{A}_i \triangleq \begin{bmatrix} \bar{A}_p & 0 & LC_p \\ -B_c C_p & A_c & -B_c C_p \\ A_{p_i} - \bar{A}_p & 0 & A_{p_i} - LC_p \end{bmatrix}.$$

The row vector I_r (I_w) is composed of all null elements save the element 1 in the r-th (w-th) position. If the set of inequalities admits a solution then the quadratically stabilizing feedback gain $\hat{K} = YQ^{-1} = [Y_1 Q_1^{-1}\ 0_{m \times n_p}]$ is found. The maximum admissible value $\gamma = \eta^{-1}$ is found for $r(k)$ and the invariant γ-feasible set $\mathcal{X} \equiv \mathcal{E}(P, \gamma)$ with $P = Q^{-1}$ for Σ_f is obtained.

Proof of Theorem 1. For sake of brevity, the proof is not reported. It follows along the line provided in Anamaria et al. (2009) (see Theorem 1) with some substantial modifications due to : 1) in Anamaria et al. (2009) $\|r\|_2^2$ is overbounded by 1, here $\|r\|_2^2$ is overbounded by a scalar γ, which is maximized including $\eta = \gamma^{-1}$ in the functional to be minimized; 2) in Anamaria et al. (2009) an euclidean norm bound is imposed to the constrained variables, here component-wise bounds are considered. △

Remark 5. The presence of $\beta \in [0, 1)$ makes inequality (21) a Bilinear Matrix Inequality (BMI). As β is a scalar, an optimal β can be found by executing a simple search line.

Once $\hat{K}$ and $\mathcal{X}$ are determined, the idea underlying the preliminary results can be summarized as follows. The stabilizing feedback gain $\hat{K}$ guarantees that, $\forall x_f(0) \in \mathcal{X}$, $\forall \alpha \in \Lambda_l$, the fulfillment of all hard constraints (8) is ensured " a priori" provided the obtained quadratically stable Σ_f is forced by an admissible control input $r(k)$, namely an $r(k)$ satisfying (7) with $\gamma = \eta^{-1}$.
Next step will be determining the trajectory of the admissible input $r(k)$ driving Σ_f. As detailed in the next section, this step is performed modeling $r(k)$ as a vector of sampled B-spline functions whose control points are iteratively estimated, and then applying the computed $r(k)$ according to the usual receding horizon strategy.

5. STEP 2: COMPUTATION OF THE B-SPLINE INPUT FORCING Σ_F

This section shows how the MMCOP stated in Section 3 can be reformulated as an RLS estimation problem which can be solved using the SOCP approach of Section 2.2.

To this purpose the closed loop dynamical matrix $A_f(\alpha)$ of Σ_f is rewritten as $A_f(\alpha) \triangleq \bar{A}_f + \Delta A_f(\alpha)$ where $\bar{A}_f$ is the nominal closed loop dynamical matrix (obtained putting $A_p(\alpha) = \bar{A}_p$ in $A_f(\alpha)$) and $\Delta A_f(\alpha) \triangleq A_f(\alpha) - \bar{A}_f$. Consequently, any term of the kind $A_f^k(\alpha)$ can be written as $A_f^k(\alpha) \triangleq \bar{A}_f^k + \Delta A_{f,k}(\alpha)$, where $\Delta A_{f,k}(\alpha)$ is a suitably defined matrix.
Expressing the input $r(j)$ as $r(j) = \bar{\mathbf{B}}_d\ (j)\bar{\mathbf{c}}$ according to (4) and recalling that $u(k) = C_{z_u}(k)x_f(k)$ and $\hat{x}_f(k)$ is the current state estimate, the predicted output and control effort are given by

$$y(k+i/k) = C_f A_f^i(\alpha)\hat{x}_f(k) + \sum_{j=k}^{k+i-1} C_f A_f^{k+i-j-1}(\alpha) B_f \bar{\mathbf{B}}_d\ (j)\bar{\mathbf{c}}, \quad i = 1, \cdots, N_y, \tag{24}$$

$$u(k+i/k) = C_{z_u} A_f^i(\alpha)\hat{x}_f(k) + \sum_{j=k}^{k+i-1} C_{z_u} A_f^{k+i-j-1}(\alpha) B_f \bar{\mathbf{B}}_d\ (j)\bar{\mathbf{c}}, \quad i = 1, \cdots, N_u. \tag{25}$$

By (24),(25) and $r(k+i) = \bar{\mathbf{B}}_d\ (k+i)\bar{\mathbf{c}}$, $e_y(k+i/k)$, $e_r(k+i/k)$ and $e_u(k+i/k)$ given by (11)-(13) respectively, can be rewritten as

$$e_y(k+i/k) = (g_y(k+i/k) + \delta g_y(k+i/k)) - (D_y(k+i/k) + \delta D_y(k+i/k))f \tag{26}$$

$$e_r(k+i/k) = g_r(k+i/k) - D_r(k+i/k)f \tag{27}$$

$$e_u(k+i/k) = (g_u(k+i/k) + \delta g_u(k+i/k)) + (D_u(k+i/k) + \delta D_u(k+i/k))f \tag{28}$$

where

$$g_y(k+i/k) \triangleq y_d(k) - C_f \bar{A}_f^i \hat{x}_f(k),$$
$$\delta g_y(k+i/k) \triangleq -C_f \Delta A_{f,i}(\alpha)\hat{x}_f(k),$$
$$D_y(k+i/k) \triangleq \sum_{j=k}^{k+i-1} C_f \bar{A}_f^{k+i-j-1} B_f \bar{\mathbf{B}}_d(j),$$
$$\delta D_y(k+i/k) \triangleq \sum_{j=k}^{k+i-1} C_f \Delta A_{f,k+i-j-1}(\alpha) B_f \bar{\mathbf{B}}_d(j),$$
$$g_r(k+i/k) \triangleq y_d(k), \quad D_r(k+i/k) \triangleq \bar{\mathbf{B}}_d(k+i),$$
$$g_u(k+i/k) \triangleq C_{z_u} \bar{A}_f^i \hat{x}_f(k) - \tilde{u}(k),$$
$$\delta g_u(k+i/k) \triangleq C_{z_u} \Delta A_{f,i}(\alpha)\hat{x}_f(k),$$
$$D_u(k+i/k) \triangleq \sum_{j=k}^{k+i-1} C_{z_u} \bar{A}_f^{k+i-j-1} B_f \bar{\mathbf{B}}_d(j)$$
$$\delta D_u(k+i/k) \triangleq \sum_{j=k}^{k+i-1} C_{z_u} \Delta A_{f,k+i-j-1}(\alpha) B_f \bar{\mathbf{B}}_d(j)$$
$$f \triangleq \bar{\mathbf{c}}$$

Define the following vectors $e \triangleq [e_y^T\, e_r^T\, e_u^T]^T$, $g \triangleq [g_y^T\, g_r^T\, g_u^T]^T$, $\delta g \triangleq [\delta g_y^T \;\; 0^T \;\; \delta g_u^T]^T$ and matrices $D \triangleq \begin{bmatrix} D_y \\ D_r \\ -D_u \end{bmatrix}$, $\delta D \triangleq \begin{bmatrix} \delta D_y \\ 0 \\ -\delta D_u \end{bmatrix}$, $Q_e \triangleq \begin{bmatrix} Q_y & 0 & 0 \\ 0 & Q_r & 0 \\ 0 & 0 & Q_u \end{bmatrix}$ where:

$$e_y \triangleq \left[e_y^T(k+1/k) \cdot\cdot\, e_y^T(k+N_y/k)\right]^T,$$
$$g_y \triangleq \left[g_y^T(k+1/k) \cdot\cdot\, g_y^T(k+N_y/k)\right]^T,$$
$$\delta g_y \triangleq \left[\delta g_y^T(k+1/k) \cdot\cdot\, \delta g_y^T(k+N_y/k)\right]^T,$$
$$D_y \triangleq \left[D_y^T(k+1/k) \cdot\cdot\, D_y^T(k+N_y/k)\right]^T,$$
$$\delta D_y \triangleq \left[\delta D_y^T(k+1/k) \cdot\cdot\, \delta D_y^T(k+N_y/k)\right]^T,$$
$$Q_y \triangleq \mathrm{diag}\{Q_y(k+i/k)\}, \;\; i=1,\cdots,N_y.$$
$$Q_r \triangleq \lambda_1(k)\mathrm{diag}\{Q_r(k+i/k)\}, \;\; i=0,\cdots,N_y-1.$$
$$Q_u \triangleq \lambda_2(k)\mathrm{diag}\{Q_u(k+i/k)\}, \;\; i=1,\cdots,N_u.$$

An analogous definition applies to vectors e_r, e_u, g_r, g_u, δg_u and matrices D_r, D_u and δD_u. From the above definitions, it is evident that only δg and δD are depending on α. This dependence is now explicitly reintroduced to better clarify the formulation of MMCOP as an RLS estimation problem.

Exploiting the above defined vectors and matrices, the $2qN_y + mN_u$ scalar equations (26)-(28) can be expressed in the compact form $e(\alpha) = (g + \delta g(\alpha)) - (D + \delta D(\alpha))f$ and functional (9) can be written as $J_\alpha \triangleq J(e'(\alpha)) = e'^T(\alpha)e'(\alpha)$, where $e'(\alpha) \triangleq Q_e^{1/2}e(\alpha)$. Also defining $g' + \delta g'(\alpha) \triangleq Q_e^{1/2}(g+\delta g(\alpha))$ and $D' + \delta D'(\alpha) \triangleq Q_e^{1/2}(D + \delta D(\alpha))$, it is evident that MMCOP is equivalent to the constrained minimization of the squared L_2 norm of the worst-case weighted residual $e'(\alpha)$. Namely MMCOP is equivalent to solve the following box-constrained RLS problem

$$\min_f \max_{\|\delta D'(\alpha)\delta g'(\alpha))\|_F \le \rho} \|(D' + \delta D'(\alpha))f - (g' + \delta g'(\alpha)\|_2 \quad (29)$$

$$\text{subject to } \; f_{min} \le f \le f_{max}. \quad (30)$$

The bounds f_{min} and f_{max} relative to the vector $\bar{\mathbf{c}} \triangleq f$ of control points are determined on the basis of condition (14) (and hence (7)).

At each $k = jN_r$ the bound ρ such that $\|\delta D'(\alpha)\; \delta g'(\alpha)\|_F \le \rho$ is computed by performing a gridding on the parameter vector $\alpha \in \Lambda_l$. Next, the parameter vector $\bar{\mathbf{c}} \triangleq f$ of control points is estimated through an SOCP as explained in Section 2.2. The corresponding B-spline input $r(k)$ results to be known over $[k, k+N_y-1]$, but only the first N_r samples are applied to Σ_f according to the usual receding horizon strategy.

Feasibility of the MMMPC strategy and stability of Σ_{2DOF} can be now formally stated in the following theorem.

Theorem 2. Assume that the problem P1 stated in Section 4 is solvable and that the input $r(k)$ of Σ_f is computed as the solution of the box-constrained RLS problem (29),(30), then the resulting 2-step MMMPC strategy explained in the above sections is recursively feasible and yields an asymptotically internally stable Σ_{2DOF}.

Proof of Theorem 2. Recursive feasibility is a direct consequence of computing $r(k)$ as the solution of an optimization problem where the feasible box-constraints (30) are imposed on a vector of variables which is the same one with respect to the optimization problem has to be solved. Moreover, by Theorem 1, the fulfillment of (30) directly implies that also the components of $z_f(k)$ satisfy constraints like (8). Internal asymptotic stability of the resulting overall control system Σ_{2DOF} is a direct consequence of the internal asymptotic stability of Σ_f and of the uniform boundedness of $r(k)$ resulting from (30).

Remark 6. Some comments on the claimed simplification of the constrained optimization problem involved in the new MMMPC strategy are in order. The B-spline parametrization of $r(k)$ allowed us to formulate the MMCOP as a box-constrained RLS estimation problem of a parameter vector f. This problem can be solved through an SOCP for which numerically efficient primal-dual interior point methods can be used see e.g. (El Ghaoui and Lebret (1997),Lobo et al. (1998), and references therein). The vector f to be estimated is composed of $q\ell$ elements, where q is the dimension of $r(k)$ and ℓ is the number of control points of each scalar B-spline function composing $r(k)$. The well known approximation properties of B-splines allow choosing a value $\ell << N_y$, thus obtaining a greatly reduced number of decision variables with respect to qN_y, as required by the usual MMMPC methods. Moreover, as shown in Section 4, all constraints on $z_f(k)$ can be transferred on the surely feasible interval type inequalities (14), whose number is $q\,N_y$. Nevertheless, by the convexity property of B-splines, these constraints must only concern the control points, so that their number reduces to $q\ell$. Following the usual approaches, the constraints to be satisfied (provided they can be satisfied) would be $n_u\,N_u + n_{x_f}\,N_y$ where n_u and n_{x_f} are the dimensions of $z_u(k)$ and $z_{x_f}(k)$ respectively. It is recalled that $z_u(k) \triangleq u(k)$ and hence $n_u = m$. Moreover, if $z_{x_f}(k) \equiv y(k)$ then $n_{x_f} = q$.

6. NUMERICAL RESULTS

The example concerns the angular positioning system considered in Kothare et al. (1996). The system consists of a rotating antenna at the origin of the plane, driven by an electric motor. Unlike Kothare et al. (1996), the state is here assumed to be unmeasurable. Denoting by θ (rad) and $\dot{\theta}$ $(rad\,s^{-1})$, the angular position and the angular velocity of the antenna respectively, and by setting $x_p \triangleq [\theta,\, \dot{\theta}]^T$, the following discretized time equations are obtained from their continuous time counterparts using a sampling time T_c of 0.1s and Euler's first-order approximation for the derivative

$$x_p(k+1) = \begin{bmatrix} 1 & 0.1 \\ 0 & 1-0.1\omega \end{bmatrix} x_p(k) + \begin{bmatrix} 0 \\ 0.1\,\kappa \end{bmatrix} u(k) \quad (31)$$

$$y(k) = [1\;\, 0]\, x_p(k), \quad (32)$$

where $\kappa = 0.787\; rad^{-1}V^{-1}$, $u \in \mathbb{R}^m$, $m = 1$, $y \in \mathbb{R}^q$, $q = 1$. The parameter ω is proportional to the coefficient of viscous friction in the rotating parts of the antenna and is assumed to be unknown over the range $0.1s^{-1} \le \omega \le 10s^{-1}$. Consequently, the dynamical matrix of Σ_p belongs to the following polytopic matrix family

$$A_p(\alpha) = \sum_{i=1}^{2} \alpha_i A_{p_i} = \alpha_1 \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 0.1 \\ 0 & 0 \end{bmatrix}, \, \alpha \in \Lambda_2.$$

The control problem consists of using the input voltage (μV) to the motor to rotate the antenna so that it point in the direction of an object in the plane whose angular position is denoted by $y_d(k)(rad)$. The desired piece-wise constant reference signal $y_d(k)$ to be tracked is: $y_d(k) = y_{d_1} = 0.1$(rad), $0 \le k < 300(30s)$; $y_d(k) = y_{d_2} = -0.1$ (rad), $300 \le k < 600$ (60s) and $y_d(k) = y_{d_3} = 0.05$ (rad), $600 \le k \le 900$ (90s).
The control effort is required to satisfy the constraint: $|u(k)| <= 2V, \, k > 0$. Hence, according to (19), the constrained variable $z_u(k)$ is assumed to be given by the control effort $u(k)$ with bound $\bar{z}_{u,1} = 2$.
The first step of the proposed MMMPC strategy is to design a LTI output controller Σ_g. According to the procedure described in Section 4, the observer gain L of Σ_o is first computed. The gain matrix $L = [1.2076, \, 1.0387]$ is found. Considering the pair $(\hat{A}(\alpha), \hat{B})$ given by (20), the feedback gain $\hat{K} = [-K_p \quad K_c \quad 0]$ and the invariant γ-feasible set $\mathcal{X}$ for $\Sigma_f \equiv (C_f, \hat{A}(\alpha) + \hat{B}\hat{K}, B_f)$ are determined solving the semidefinite programming problem defined by (21)-(23). As the constraints only concern $u(k)$, by (19) one has $z_f(k) \equiv z_u(k)$ and the set of inequalities (23) concerning z_{x_f} are not considered. According to Remark 5, (21) has been transformed in an LMI executing a search line for $\beta \in [0, 1)$. For $\beta = 0.01$, the feedback gain $\hat{K} = [-8.0219 \; -7.8762 \; 0.0499 \; 0 \; 0]$ is found. The invariant γ-feasible set $\mathcal{X} \equiv \mathcal{E}(P, \gamma)$ with $P = Q^{-1} =$
$$\begin{bmatrix} 1.1971 & 0.7379 & -0.0075 & 0 & 0 \\ 0.7379 & 1.3586 & -0.0046 & 0 & 0 \\ -0.0075 & -0.0046 & 0.0001 & 0 & 0 \\ 0 & 0 & 0 & 31.5090 & -2.3552 \\ 0 & 0 & 0 & -2.3552 & 0.4772 \end{bmatrix}$$
and $\gamma = \eta^{-1} = 0.0632$ is obtained for the resulting closed loop system $\Sigma_f \equiv (C_f, \hat{A}(\alpha) + \hat{B}\hat{K}, B_f)$.
The second step is to determine the trajectory of the input $r(k)$, subject to (7) with $\gamma = 0.0632$, and optimally driving the output transition between two consecutive set points of the given switching sequence. This step is performed modeling $r(k) \in \mathbb{R}^q$, $q = 1$, as a sampled B-spline function. The control points defining the B-spline $r(k)$ over a moving prediction horizon are iteratively estimated by the SOCP as explained in Section 2.2. At each $k = jN_r$, $j = 0, 1, 2, \cdots$ and $N_r = 1$, the bound ρ such that $\|\delta D'(\alpha) \; \delta g'(\alpha)\| \le \rho$ is computed by performing a gridding for $\alpha \in \Lambda_2$. The obtained sequence of ρ ranges in the interval [0, 0.2292]. The computed $r(k)$ is applied according to the usual receding horizon control strategy.
The following parameters are chosen: $d = 3$ (order of B-spline), $\ell = 5$ (number of control points of the scalar B spline over each prediction horizon N_y), $9 \overset{\triangle}{=} \ell + d + 1$ (number of knot points $\hat{k}_i$ over each N_y) and $N_y = 40$. All the weight matrices are set to identity matrix. An S-shaped membership function is chosen for $\lambda_1(k)$ for the following motivations. In correspondence of the transient response following any set point change, a null initial value of $\lambda_1(k)$ allows $r(k)$ to vary over all the admissible range. After the transition period has elapsed, $\lambda_1(k)$ should tend to a suitable positive value $\bar{\lambda}$ to speed up the convergence of $r(k)$ to the desired set point value y_d. In this case the value $\bar{\lambda} = 1$ has been chosen. A null $\lambda_2(k)$, has been fixed $\forall k \ge 0$, because the feedback controller has been designed to guarantee that, for any $r(k)$ satisfying (7), the control effort $u(k) \overset{\triangle}{=} z_u(k)$ obey constraint (8). The vector $f \overset{\triangle}{=} \bar{\mathbf{c}} = \mathbf{c}_1$ of decision variables to be determined at each $k = jN_r$ is composed by $\ell = 5$ control points. As $\gamma = 0.0632$ and $r(k)$ is a scalar, the bounds of inequalities (30) are $|f_{min}| = f_{max} = \sqrt{\gamma} = 0.2565$. The simulation has been performed starting from $x_f(0) = [\hat{x}_p^T(0), \, x_c^T(0), \, x_p^T(0) - \hat{x}_p^T(0)] = [-0.05, \, 0, \, 0, \, 0, \, 0.001]^T \in \mathcal{X}$ and choosing $A_p(\bar{\alpha}) \overset{\triangle}{=} \bar{\alpha}_1 A_1 + \bar{\alpha}_2 A_2 = 0.2A_1 + 0.8A_2$. The obtained input $r(k)$ is depicted in figure 2. The actual controlled output of Σ_f yielded by $r(k)$ is given in figure 3 (solid line). The behavior of the constrained control effort is shown in figure 4.
As for the computational complexity, the following considerations hold. According to Remark 6, at each $k = jN_r$ the number of decision variables (control points of $r(k)$) involved in the proposed on-line optimization procedure is $q\ell = 5 << qN_y = 40$, and the total number of interval type inequalities (30) to be imposed is $q\ell = 5 << qN_y = 40$.

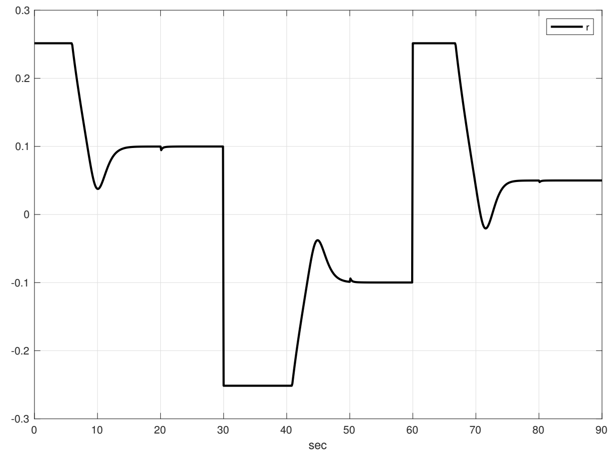

Fig. 2. The trajectory of the computed input $r(k)$ (a scalar B-spline of order $d = 3$ with $\ell = 5$ control points and $9 \overset{\triangle}{=} \ell + d + 1$ knots points)

7. CONCLUSIONS

The advantage of using a 2DOF control scheme to deal with the MMMPC consists in the possibility of decomposing the problem in two distinct steps: the first one is the off-line design of a feedback controller which stabilizes the uncertain plant and guarantees in advance the fulfillment of hard constraints for any input $r(k)$ satisfying the admissibility condition; the second step consists in the on-line computation of the input $r(k)$ forcing the closed-loop system Σ_f. Modeling $r(k)$ as a B-spline decreases the number of decision variables and reduces the constrained optimization of the quadratic cost functional to a much simpler RLS estimation problem with box-constraints on the unknowns (the control points of $r(k)$). Computing the box constraints is a straightforward consequence of the admissibility condition established off line at step 1 and of the membership of B-splines to the convex hull defined by

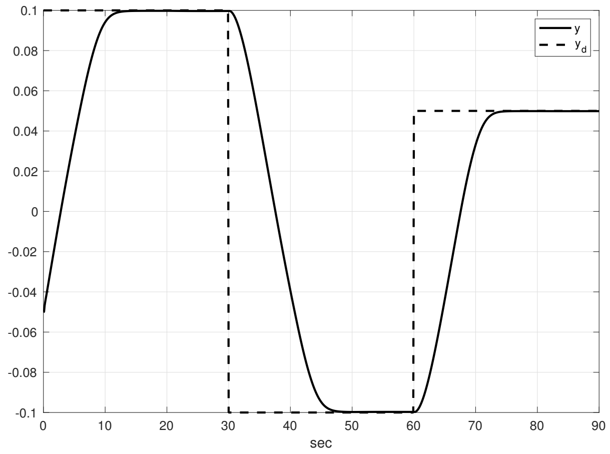

Fig. 3. The desired piece-wise output (dashed line) and the actual controlled output (solid line) of Σ_f forced by $r(k)$

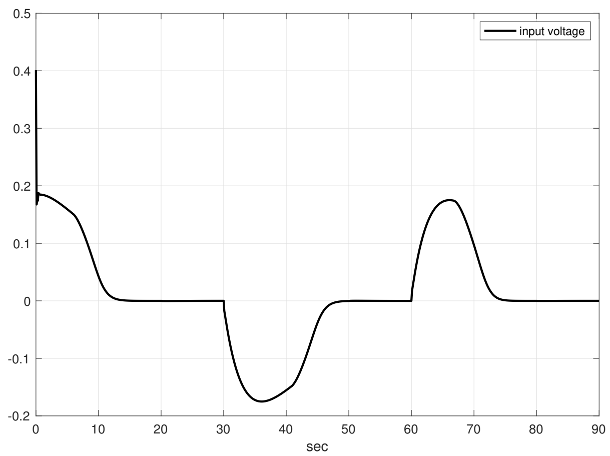

Fig. 4. The behavior of constrained control effort $u(k)$ forcing Σ_p.

their control points. The robust estimation problem can be formulated as a SOCP, for which numerically efficient interior point methods exist.

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