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# Stability of uncertain piecewise-affine systems with parametric dependence

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**Abstract:** This paper proposes a numerical approach to the stability analysis for a class of piecewise-affine systems with (possibly time-varying) parameter-dependent cells and dynamics. This class of model aims at allowing a better modelling of time-varying or parameter-varying nonlinearities of physical phenomena such as dry friction. We form the stability certification problem as the one of finding a Lyapunov function that is parameterised as a polynomial function of the variable parameter. The application of the well-known Lyapunov stability theorem together with the use of the generalised S-procedure reduces the problem to checking whether a certain set of matrices has the sum-of-squares property. The latter can be solved using well-documented numerical solvers, and we provide two examples of successful applications at the end of the paper.

*Keywords:* Hybrid systems, Piecewise-affine systems, Linear matrix inequalities.

## 1. INTRODUCTION

Finding a generic and systematic method for analyzing the properties of generic nonlinear system is a challenging problem in control theory. However, the seminal works in Johansson and Rantzer (1998); Hassibi and Boyd (1998) have shown that if we consider approximating the nonlinearities using a piecewise-affine (PWA) representation, then a systematic numerical approach can be developed. This approach is still an active topic of research in automatic control, as demonstrated by several recent works like Ameur et al. (2016), Waitman et al. (2017), Iervolino et al. (2017). A piecewise-affine representation consists of a polyhedral partition of the state-space of the system into region with disjoint interiors, each of which is associated with an affine time-invariant dynamical model. Whereas there are systems that can naturally be described by PWA models (e.g. linear systems interconnected by static nonlinearities such as saturations, dead zones, etc.), in most of the other situations, PWA systems can be regarded merely as convenient approximations in view of easier control design or stability analysis. From this latter perspective, one must account for the uncertainty associated to the approximation. This can be done through, for example, the construction of piecewise differential inclusions that embed the dynamics of the nonlinear system as in Johansson (1999) or more simply, by allowing the PWA model parameters to be parameter dependent.

We consider in this paper the problem of assessing the asymptotic stability for a class of PWA systems where the partition and the affine subsystem dynamical equations depend on a scalar time-varying parameter. The parameter in question, although unknown, is assumed to vary smoothly in a bounded interval. A comparable class of model has been defined in LeBel and Rodrigues (2008) under the name of piecewise-affine parameter-varying model (PWA-PV), where the cells were limited to time-invariant. The class of systems that we propose in this work typically arises when approximating a nonlinear time-varying system with a PWA model. Also, assuming a variable partition and letting the matrices of the affine subsystems be parameter dependent can account for the uncertainty associated with the approximation of nonlinear systems. These situations are illustrated through some motivating examples in Section 3.

The proposed analysis method consists in searching for a quadratic Lyapunov function on the state-space with a polynomial dependency on the scheduling parameter. We show that this can be reduced to a feasibility problem in the decision variables under matrix sum of squares (MSOS) constraints. The latter can be made convex and solved efficiently using available numerical tools.

The paper is organised as follows. Section 2 introduces all the preliminary notions, including the notation, piecewise-affine system, and sum of squares (SOS). Section 3 contains a discussion concerning the motivation for the work in this paper, whereas Section 4 explains the proposed approach, eventually leading to the main result of this paper. Section 5 shows the application of the main result to two cases, one derived from the motivating example and the second one derived from a famous example in the literature, and finally Section 6 draws the conclusions.

 

## 2. PRELIMINARIES

### 2.1 Notation

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of positive real numbers including zero, $\mathbb{Z}$ the set of the integers and $\mathbb{N}$ the set of positive integers. For a symmetric matrix $A$, the expression $A \succeq 0$ indicates that $A$ is positive semidefinite. We also use the notation $M(:)$ to indicate the entries of a matrix $M$ taken one at a time, and for imposing component-wise constraints; for example $M(:) \geqslant 0$ indicates that all the entries of $M$ are positive or equal to zero.

*Definition 1.* (Uncertain piecewise-affine system). In the context of this work, we call an autonomous dynamical system "uncertain piecewise-affine with parametric dependence" when it has a representation given by

$$\dot{x}(t) = A_i(\rho)x(t) + a_i(\rho) \ \text{ for } x(t) \in X_i(\rho) \tag{1}$$

where $\rho \in \mathcal{R} := [\rho_{\min}, \rho_{\max}]$ is a time-varying scalar parameter, continuous with respect to time and with bounded rate of change. The regions (or cells) $X_i(\rho)$, for $i \in \mathcal{I} := \{1, \ldots, L\}$, are closed convex polyhedral sets defined by $X_i = \{x \in \mathbb{R}^n \mid E_i(\rho)x + e_i(\rho) \geqslant 0\}$ with non-empty and pairwise disjoint interiors $\forall \rho \in \mathcal{R}$, such that $\bigcup_{i\in\mathcal{I}} X_i(\rho) = \mathbb{R}^n$. Then, $\{X_i(\rho)\}_{i\in\mathcal{I}}$ constitutes a finite partition of $\mathbb{R}^n$. From the geometry of $X_i(\rho)$, the intersection $X_i(\rho) \cap X_j(\rho)$ between two different regions is always contained in a hyperplane. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be the index set for cells that contain the origin for at least one value of $\rho \in \mathcal{R}$, and let $\mathcal{I}_1 = \mathcal{I} \setminus \mathcal{I}_0$. We assume that $a_i(\rho) = 0$ for $i \in \mathcal{I}_0$ and for all $\rho \in \mathcal{R}$ (this is a necessary and sufficient condition for equilibrium at the origin).

The representation above can be simplified by extending the state vector $x(t)$ as $\tilde{x}(t) = [x(t)^\top, \ 1]^\top$, which allows reformulating it as

$$\dot{\tilde{x}}(t) = \overline{A}_i(\rho)\tilde{x}(t) \ \text{ for } x(t) \in X_i(\rho) \tag{2}$$

and

$$X_i(\rho) = \{x \in \mathbb{R}^n \mid \overline{E}_i(\rho)\tilde{x} \geqslant 0\} \tag{3}$$

with opportune definitions of $\overline{A}_i(\rho)$ and $\overline{E}_i(\rho)$:

$$\overline{A}_i(\rho) = \begin{bmatrix} A_i(\rho) & a_i(\rho) \\ 0 & 0 \end{bmatrix}, \ \overline{E}_i(\rho) = [\, E_i(\rho) \ \ e_i(\rho) \,], \tag{4}$$

where $a_i(\rho) = 0$ and $e_i(\rho) = 0$ for $i \in \mathcal{I}_0$ (in the same way as in Johansson and Rantzer (1998)). In the remainder of the paper, we will assume that $\overline{A}_i(\rho)$ are rational functions of $\rho$, i.e. their entries can be expressed as ratios of polynomials in $\rho$, and that $\overline{E}_i(\rho)$ are polynomial functions of $\rho$. If $\overline{A}_i(\rho)$ is rational, then we add the additional hypothesis that $\overline{A}_i(\rho)$ is finite and defined for all $\rho \in \mathcal{R}$ (i.e., there is no denominator that is equal to zero for values of $\rho \in \mathcal{R}$); notice also that there is no loss of generality in assuming $\overline{E}_i(\rho)$ polynomial instead of rational: denominators in $\overline{E}_i(\rho)$ that are always defined never assume a value of 0 for $\rho \in \mathcal{R}$, so they always have the same sign and they can be simplified without changing the definitions of the cells. We also assume the absence of sliding modes, as it is the case for example if the right-hand side of (1) is continous across boundaries. In any case, sliding modes can be dealt with as explained in Section IIV of Johansson and Rantzer (1998).

### 2.2 Sum of squares

*Definition 2.* (Generic SOS problems, Parrilo (2003)). Let $p(t)$ be a polynomial of degree up to $2d \in \mathbb{N}$ in the variable $t \in \mathbb{R}$. We call a sum of squares problem (SOS) the problem of finding whether there exists a finite number $l$ of polynomials $\pi_i(t)$ such that

$$p(t) = \sum_{i=1}^{l} \pi_i(t)^2. \tag{5}$$

If such a decomposition into a sum of squares of polynomials exists, we say that $p(t)$ is sum of squares (SOS), which implies that $p(t) \geqslant 0$ for all $t$.

Let $\mathcal{P}(t)$ be a symmetric matrix of polynomials of degree up to $2d \in \mathbb{N}$ in the variable $t \in \mathbb{R}$. We call a matrix sum of squares problem (MSOS) the problem of finding whether there exists a finite number $l$ of symmetric matrices of polynomials $\Pi_i(t)$ such that

$$\mathcal{P}(t) = \sum_{i=1}^{l} \Pi_i(t)^\top \Pi_i(t). \tag{6}$$

If such a decomposition exists, we say that $P(t)$ is matrix sum of squares (MSOS), which implies that $P(t) \succeq 0$ for all $t$.

The definition above is restricted to univariate polynomials ($t \in \mathbb{R}$), as this is the case we consider in this paper. SOS and MSOS problems are convex problems. A derived class of problems is the one of feasibility problems under SOS or MSOS constraints, defined here.

*Definition 3.* (SOS constraints feasibility, Parrilo (2003)). Let $p_i(t, \theta)$, $i = 1, \ldots, q$, be a set of $q$ polynomials of degree up to $2d \in \mathbb{N}$ in the variable $t \in \mathbb{R}$, where $\theta \in \mathbb{R}^\rho$ is a vector of parameters or unknowns, with $p_i(t, \theta)$ affine with respect to the entries of $\theta$. A feasibility problem under SOS constraints consists in finding, if it exists, a value of $\theta = \theta^*$ for which

$$p_i(t, \theta^*) \text{ is SOS, for } i = 1, ..., q. \tag{7}$$

If such a $\theta^*$ exists, then the problem is feasible; otherwise it is unfeasible.

*Definition 4.* (MSOS constraints feasibility, Chesi (2010)). Let $\mathcal{P}_i(t, \theta)$, $i = 1, \ldots, q$, be a set of $q$ symmetric matrices of polynomials of degree up to $2d \in \mathbb{N}$ in the variable $t \in \mathbb{R}$, where $\theta \in \mathbb{R}^\rho$ is a vector of parameters or unknowns, and the matrices $P_i(t, \theta)$ are affine with respect to the entries of $\theta$. A feasibility problem under SOS constraints consists in finding, if it exists, a value of $\theta = \theta^*$ for which

$$P_i(t, \theta^*) \text{ is MSOS, for } i = 1, ..., q. \tag{8}$$

If such a $\theta^*$ exists, then the problem is feasible; otherwise it is unfeasible.

Feasibility problems under SOS and MSOS problems are convex optimisation problems, and they can be reformulated as linear matrix inequality (LMI) feasibility problems; this can either be done explicitly, or relying on automated procedures, like the one available in the Yalmip toolbox (Löfberg, 2009) under Matlab.

### 2.3 The general S-procedure

By the name S-procedure, one commonly identifies a lemma which allows restricting a certain class of inequalities to a certain subset (see Boyd et al. (1994)). In this article, we rely on a very simple but general expression, whose proof is obvious, which will be specialised according to the cases.

*Lemma 1.* (Generalised S-procedure). Let $F(x)$, $G(x)$ be (symmetric matrix) functions of the (vector) variable $x$. Let $g(x)$ be a scalar function of $x$. The following implications hold.

$$F(x) \succeq 0 \text{ for } G(x) \succeq 0 \Leftarrow F(x) - G(x)\lambda \succeq 0 \text{ for } \lambda \geqslant 0,\ \forall x \tag{9}$$

$$F(x) \succeq 0 \text{ for } g(x) \geqslant 0 \Leftarrow F(x) - g(x)\Lambda \succeq 0 \text{ for } \Lambda \succeq 0,\ \forall x \tag{10}$$

The terms $\Lambda$ and $\lambda$ are called multipliers, which can be chosen at one's convenience, i.e. they are decision variables subject to the positivity constraints above. When used in the context of polynomial problems as in the context of this paper, the lemma above is a direct consequence of a lemma known as Positivstellensatz (see Chesi (2010)) or p-satz, of which several versions exist in the literature. The multipliers can in this case have a polynomial dependence on $x$, which allows satisfying the positivity constraints by means of either SOS or MSOS constraints.

### *2.4 Lyapunov stability*

*Lemma 2.* (Adapted from Kalman and Bertram (1960)). Consider a parameter varying dynamical system

$$\dot{x} = g(\rho, x),$$

with $g$ continuous depending on a continuously differentiable function $\rho$ of time. If there exists a continuous and differentiable function $V(x,\rho) = x^\top P(\rho, x)x$ such that

$$\alpha||x||^2 \leqslant V(x,\rho) \leqslant \beta||x||^2 \tag{11}$$

and

$$\dot{V}(x,\rho) \leqslant -\varepsilon||x||^2 \tag{12}$$

with $\alpha > 0$, $\beta > 0$, $\varepsilon > 0$, then the system is asymptotically exponentially stable for any initial condition.

## 3. MOTIVATION

The motivation of this work, as stated in the introduction, is based on the need of an accurate description of nonlinearities that have some parametric dependence, for which in turn the parameter might be time-varying. Several examples of these kind of systems can be found in practical applications, we can for example mention varistors (see Eda (1989)) and thermistors (see Scarr and Setterington (1960)); these electrical components have nonlinear current-versus-voltage curves depending on parameters (the exponent $\alpha$ for varistors and the temperature for the thermistors). Besides these, the other main motivation for a more accurate description of nonlinearities comes from friction and specifically dry friction (see for example Karnopp (1985); Persson (2006)). Dry friction is a nonlinear phenomenon that is very difficult to model, and which depends on many factors and parameters. The dry friction coefficient $\mu$, i.e. the ratio between friction force and contact force, depends not only on the relative sliding velocity but also on several parameters such as the temperature, the contact force itself, and even the acceleration of the motion. For example, from Persson (2006) one can derive a model of $\mu$ with a parametric dependence as in Fig. 1.

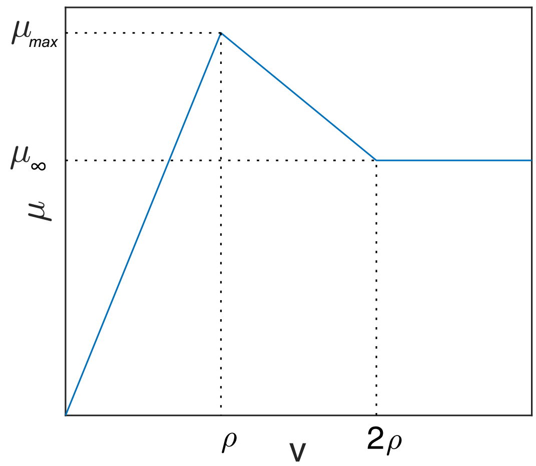


Fig. 1. Example of a curve of friction coefficient $\mu$ as a function of the sliding velocity $v$; the shape features a parametric dependence on $\rho$ (simplified and adapted from Persson (2006)).

A first attempt to deal with time-varying and parameter-varying friction coefficient has been done in Ameur et al. (2016), with very gross and inaccurate differential inclusions. On the other hand, a curve like the one in Fig. 1 can be accurately described with a piecewise-affine model, if one only allows the boundary themselves to change as a function of the parameter. This model of friction, presented here as a motivation for the work, will be featured again in the first numerical example of this paper, in Section 5.1.

## 4. MAIN RESULT

In this section, we state the main result of the paper which is a stability theorem for the uncertain PWA system defined in (1). To assess the global asymptotic stability of that system we will be searching for a piecewise quadratic Lyapunov function $V : \mathbb{R}^n \times \mathcal{R} \rightarrow \mathbb{R}_+$, which depends polynomially on the the same scalar parameter $\rho$ as the dynamics of the system (1). More precisely, inspired by the work reported in Johansson and Rantzer (1998) we set $V(x,\rho)$ to be of the form

$$V(x,\rho) = \tilde{x}^\top \overline{P}_i(\rho)\tilde{x} \text{ if } x \in X_i(\rho) \text{ for some } i \in \mathcal{I} \tag{14}$$

where $\tilde{x}$ is defined from $x$ as before and

$$\overline{P}_i(\rho) = \begin{bmatrix} P_i(\rho) & p_i(\rho) \\ p_i(\rho)^\top & r_i(\rho) \end{bmatrix}, \tag{15}$$

of any chosen degree in $\rho$, with the constraints that $p_i(\rho) = 0$ and $r_i(\rho) = 0$ for $i \in \mathcal{I}_0$. Observe that $V(x,\rho)$ is defined here on a partition of the state-space which coincides with that of the system to be analyzed, but in a more general setting one could consider a piecewise Lyapunov function candidate defined on a partition of the state-space which is completely different from that of the system to be analyzed.

*Continuity of $V$:* Similarly as in Johansson and Rantzer (1998) we parameterise the matrix functions $\overline{P}_i(\rho)$ in the

$$\begin{aligned}
&\text{for } i \in \mathcal{I}_0 \begin{cases} P_i(\rho) - E_i(\rho)^\top W_i(\rho) E_i(\rho) - G_i(\rho)\eta(\rho) - I\varepsilon \text{ is MSOS} \\ \nu_i(\rho)\left(-A_i(\rho)^\top P_i(\rho) - P_i(\rho)A_i(\rho) - \rho_d \frac{\partial P_i(\rho)}{\partial \rho} - I\varepsilon\right) - E_i(\rho)^\top U_i(\rho) E_i(\rho) - H_i(\rho)\eta(\rho) \text{ is MSOS} \\ \nu_i(\rho)\left(-A_i(\rho)^\top P_i(\rho) - P_i(\rho)A_i(\rho) + \rho_d \frac{\partial P_i(\rho)}{\partial \rho} - I\varepsilon\right) - E_i(\rho)^\top U_i'(\rho) E_i(\rho) - H_i'(\rho)\eta(\rho) \text{ is MSOS} \\ W_i(\rho)(:), U_i(\rho)(:), U_i'(\rho)(:) \text{ are SOS, } G_i(\rho), H_i(\rho), H_i'(\rho) \text{ are MSOS} \end{cases} \\
&\text{for } i \in \mathcal{I}_1 \begin{cases} \overline{P}_i(\rho) - \overline{E}_i(\rho)^\top \overline{W}_i(\rho) \overline{E}_i(\rho) - \overline{G}_i(\rho)\eta(\rho) - \overline{I}\varepsilon \text{ is MSOS} \\ \nu_i(\rho)\left(-\overline{A}_i(\rho)^\top \overline{P}_i(\rho) - \overline{P}_i(\rho)\overline{A}_i(\rho) - \rho_d \frac{\partial \overline{P}_i(\rho)}{\partial \rho} - \overline{I}\varepsilon\right) - \overline{E}_i(\rho)^\top \overline{U}_i(\rho) \overline{E}_i(\rho) - \overline{H}_i(\rho)\eta(\rho) \text{ is MSOS} \\ \nu_i(\rho)\left(-\overline{A}_i(\rho)^\top \overline{P}_i(\rho) - \overline{P}_i(\rho)\overline{A}_i(\rho) - \rho_d \frac{\partial \overline{P}_i(\rho)}{\partial \rho} - \overline{I}\varepsilon\right) - \overline{E}_i(\rho)^\top \overline{U}_i'(\rho) \overline{E}_i(\rho) - \overline{H}_i'(\rho)\eta(\rho) \text{ is MSOS} \\ \overline{W}_i(\rho)(:), \overline{U}_i(\rho)(:), \overline{U}_i'(\rho)(:) \text{ are SOS, } \overline{G}_i(\rho), \overline{H}_i(\rho), \overline{H}_i'(\rho) \text{ are MSOS} \end{cases}
\end{aligned} \tag{13}$$

form

$$\overline{P}_i(\rho) = \overline{F}_i(\rho)^\top T(\rho) \overline{F}_i(\rho) \tag{16}$$

where $T(\rho)$ is a parameterised polynomial matrix function of $\rho$ of a chosen arbitrary degree, and the $\overline{F}_i(\rho)$ are known matrix polynomial functions:

$$\overline{F}_i(\rho) = [\, F_i(\rho) \;\; f_i(\rho) \,], \tag{17}$$

of appropriate dimensions, satisfying

$$\overline{F}_i(\rho)\tilde{x} = \overline{F}_j(\rho)\tilde{x} \tag{18}$$

for all pairs $(i,j) \in \mathcal{I}^2$ of cells with non trivial intersection and all $\tilde{x}$ such that $x \in X_i(\rho) \cap X_j(\rho)$ (with $f_i(\rho) = 0$ for $i \in \mathcal{I}_0$, see again Johansson and Rantzer (1998)). With the parameterizations in (16) and the conditions (18), the Lyapunov function candidate $V(x,\rho)$ expressed in (14) is automatically ensured to be continuous on $\mathbb{R}^n$ when regarded as a function of $x$.

The Lyapunov functions candidate will need to satisfy three properties, of which a brief introductory account is given here, before stating the main theorem.

*Positive definiteness of $V(x,\rho)$:* We must require $V(x,\rho)$ to be positive-definite. For that it suffices that each piece of $V$ is positive in the corresponding cell, i.e., $\tilde{x}^\top \overline{P}_i(\rho)\tilde{x} > 0$ for all $\rho \in \mathcal{R}$ and all $x \in X_i(\rho)$, $x \neq 0$, $i \in \mathcal{I}$. These conditions can be ensured by making use of the device of the S-procedure recalled in Lemma 1. For this purpose note that a cell $X_i(\rho)$ is contained in any set represented by the inequality $\tilde{x}^\top \overline{E}_i(\rho)^\top \overline{W}_i(\rho)\overline{E}_i(\rho)\tilde{x} \geqslant 0$ provided that $\overline{W}_i(\rho)$ is a nonnegative matrix function, in the sense that all of its entries are nonnegative. Proceeding likewise the range $\mathcal{R}$ of $\rho$ can be embedded in the set defined by the matrix inequality $\overline{G}_i(\rho)\eta(\rho) \succeq 0$ for some positive semidefinite matrix function $\overline{G}_i(\rho)$ and with

$$\eta(\rho) = \frac{1}{4}(\rho_{\max} - \rho_{\min})^2 - \left(\rho - \frac{\rho_{\min} + \rho_{\max}}{2}\right)^2 \tag{19}$$

($\eta(\rho)$ is nonnegative when $\rho \in \mathcal{R}$). Wrapping it up, the existence for all $i \in \mathcal{I}$ of $\overline{W}_i(\rho)$, $\overline{G}_i(\rho)$ and $\varepsilon > 0$ such that

$$\overline{P}_i(\rho) - \overline{E}_i(\rho)^\top \overline{W}_i(\rho)\overline{E}_i(\rho) - \overline{G}_i(\rho)\eta(\rho) - \varepsilon\overline{I} \quad \text{is MSOS}$$

constitutes a sufficient condition for $V(x,\rho)$ to be positive-definite. Here

$$\overline{I} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \tag{20}$$

with $I_n$ the identity matrix of order $n$ (also written as $I$ for a lighter notation).

*Negativity of $\dot{V}(x,\rho)$:* The negativity of the time derivative $\dot{V}(x,\rho)$ along the system trajectories can also be enforced per cell using the same technique of S-procedure. The details are in the proof on the theorem that we are now ready to state.

*Theorem 1.* Consider an uncertain PWA system according to Definition 1, with matrices defined according to (4), (16), (17), and assume $|\dot{\rho}| \leqslant \rho_d$, with $\rho_d$ a given positive scalar.

The system is asymptotically exponentially stable for any initial conditions if for some $\varepsilon > 0$, the MSOS constraints in (13) are feasible (at the top of the page), for $i = 1, ..., N$. The decisions variables are the matrix coefficients of $T(\rho) = T_0 + \rho T_1 + \rho^2 T_2 ...$ and the polynomial multipliers $W_i(\rho)$, $G_i(\rho)$, $U_i(\rho)$, $U_i'(\rho)$, $H_i(\rho)$, $H_i'(\rho)$ or $\overline{W}_i(\rho)$, $\overline{G}_i(\rho)$, $\overline{U}_i(\rho)$, $\overline{U}_i'(\rho)$, $\overline{H}_i(\rho)$, $\overline{H}_i'(\rho)$, of fixed arbitrary degree. The function $\eta(\rho)$ is defined in (19), and $\nu_i(\rho)$, with $\nu_i(\rho) > 0$ for $\rho \in \mathcal{R}$, is a polynomial for which $\nu_i(\rho)A_i(\rho)$ (if $i \in \mathcal{I}_0$) or $\nu_i(\rho)\overline{A}_i(\rho)$ (if $i \in \mathcal{I}_i$) are pure polynomials (in case there are denominators in $A_i(\rho)$ or $\overline{A}_i(\rho)$, it simplifies them).

**Proof.** We show that $V(x,\rho)$ is a valid Lyapunov function in each cell. By definition, $V(x,\rho)$ is continuous with respect to $x$ and $\rho$. Subsequently, we show that the conditions in (13) imply the conditions in Lemma 2; the proof is shown in details only for cells with $i \in \mathcal{I}_0$, but the extension to $i \in \mathcal{I}_1$ follows exactly the same scheme. This proof is inspired by the approach to parameter-dependent systems that can be found in Wu and Prajna (2005). Consider a cell $X_i$ with $i \in \mathcal{I}_0$, and consider the first condition in (13). If the condition is satisfied, it means that the expression on the left hand side is always positive semidefinite; the terms $E_i(\rho)^\top W_i(\rho)E_i(\rho)$ and $G_i(\rho)\eta(\rho)$ are general S-procedure terms with $W_i(\rho)$ and $G_i(\rho)$ nonnegative multipliers (entry-wise for $W_i(\rho)$, positive semidefinite for $G_i(\rho)$, as implied by the fourth line of (13)); this implies that $x^\top(P_i(\rho) - I\varepsilon)x \geqslant 0$ when $E_i(\rho)x \geqslant 0$ and when $\eta(\rho) \geqslant 0$, i.e. $V(x,\rho)$ satisfies the left hand side inequality in (11) when $x \in X_i$ and $\rho \in \mathcal{R}$. The right hand side of (11) is proven by the definition of $V(x,\rho)$ as a quadratic function of $x$, with $\rho$ bounded. Going now to the second and third inequalities in (13): by the same S-procedure mechanisms explained above and subsequently simplifying the positive term $\nu_i(\rho)$, they imply that

$$x^\top \left(-A_i(\rho)^\top P_i(\rho) - P_i(\rho)A_i(\rho) - \rho_d \frac{\partial P_i(\rho)}{\partial \rho} - I\varepsilon\right) x \geqslant 0$$

and

$$x^\top \left(-A_i(\rho)^\top P_i(\rho) - P_i(\rho)A_i(\rho) + \rho_d \frac{\partial P_i(\rho)}{\partial \rho} - I\varepsilon\right) x \geqslant 0$$

for $x \in X_i$ and $\rho \in \mathcal{R}$. Notice that any convex combination of the two inequalities above will still hold, so the two imply that

$$x^\top \left( -A_i(\rho)^\top P_i(\rho) - P_i(\rho) A_i(\rho) - \dot{\rho}\frac{\partial P_i(\rho)}{\partial \rho} - I\varepsilon \right) x \geqslant 0$$

for any $\dot{\rho} \in [-\rho_d, \rho_d]$. This last inequality implies $\dot{V}(x,\rho) \leqslant -\varepsilon||x||^2$, i.e. it implies (12), concluding the proof.

*Remark 1.* (On moving boundaries). It might surprise some readers that no specific condition is set for the case when the border is moving, compared to the main result in Johansson and Rantzer (1998). In fact, there is no difference whether the state crosses a boundary because it is moving itself, or because the boundary is moving: consider that crossing a boundary is an action that requires some time, whereas the conditions in Theorem 1 assure that instantly, at each time, the Lyapunov function is decreasing; this is true also across the boundary (see Figure 2, point $x_b$), regardless of how the state arrived at the boundary and regardless of how the boundary moves. The time derivative of the Lyapunov function includes a term due to the derivative of the state, and a term due to the time-change of the expression of the Lyapunov function itself ($\dot{P}_i$); this last term includes also the effect of the boundaries, if any. The continuity of the Lyapunov function makes sure that there are no jumps of its value over time.

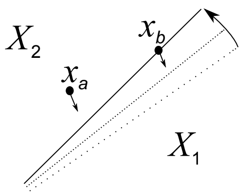


Fig. 2. Moving boundaries between two cells $X_1$ and $X_2$.

*Remark 2.* (On necessity). One might wonder if, at least in some cases, the conditions of Theorem 1 can be necessary as well; in fact we are in the case of univariate polynomials, for which several necessary and sufficient results exist, like for example the powerful Markov-Lukacs's theorem (Genin et al., 2000). Also in the matrix sum of square case the necessity would be obtained for a given (high) degree of the multipliers $G_i(\rho)$, (Briat, 2015; Scherer and Hol, 2006), but this degree cannot be known in advance and increasing it too much is often impossible once one implements the problem on a computer.

## 5. EXAMPLES

### *5.1 Motivating example (friction)*

Let us consider a feedback positioning system with both viscous friction and dry friction, of equation:

$$m\dot{v} = -ky - cv - N\mu(v,\rho) \tag{21}$$

where $y$ is the position, $v = \dot{y}$ is the velocity, $k$ is the force feedback gain, $c$ is a viscous friction coefficient, $\mu(v,\rho)$ is the static friction coefficient, $N$ is the contact force and $m$ is the mass. For $\mu(v,\rho)$, we consider a model as the one discussed in Section 3 (Fig. 1), which yields an uncertain PWA system depending on the time-varying parameter $\rho$. Taking the state vector as $x = [y,\, v]^\top$, the state-space is divided into 5 cells according to the value of the velocity $v$; the first cell is $X_1(\rho) = \{x \mid -\rho \leqslant v \leqslant \rho\}$ $(1 \in \mathcal{I}_0)$, where the dry friction coefficient is $\mu(v,\rho) = \frac{\mu_{\max}}{\rho}v$; the second is $X_2(\rho) = \{x \mid \rho \leqslant v \leqslant 2\rho\}$, where $\mu(v,\rho) = -\frac{\mu_{\max}-\mu_\infty}{\rho}v + (2\mu_{\max} - \mu_\infty)$; and the third is $X_3(\rho) = \{x \mid v \geqslant 2\rho\}$ $(2, 3 \in \mathcal{I}_1)$ where $\mu(v,\rho) = \mu_\infty$. The fourth and fifth cells are just the symmetric of the second and third; considering that the dynamical system is invariant to a central symmetry, we do not need to worry about them: if a Lyapunov function exists for the first three cells, then it exists for all the state-space.

The cells are described by matrices

$$A_1 = \left[\begin{array}{c|c} 0 & 1 \\ \hline -\frac{k}{m} & -\frac{c}{m} - \frac{\mu_{\max}N}{\rho m} \end{array}\right],\ E_1 = 0,\ F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{22}$$

$$\begin{gathered} \overline{A}_2 = \left[\begin{array}{c|c|c} 0 & 1 & 0 \\ \hline -\frac{k}{m} & \frac{c}{m} + \frac{(\mu_{\max}-\mu_\infty)N}{\rho m} & -\frac{(2\mu_{\max}-\mu_\infty)N}{m} \\ \hline 0 & 0 & 0 \end{array}\right], \\ \overline{E}_2 = \begin{bmatrix} 0 & 1 & -\rho \\ 0 & -1 & 2\rho \\ 0 & 0 & \rho \end{bmatrix}, \end{gathered} \tag{23}$$

$$\overline{A}_3 = \left[\begin{array}{c|c|c} 0 & 1 & 0 \\ \hline -\frac{k}{m} & -\frac{c}{m} & -\frac{\mu_\infty N}{m} \\ \hline 0 & 0 & 0 \end{array}\right],\ \overline{E}_3 = \begin{bmatrix} 0 & 1 & -\rho \\ 0 & 2 & -4\rho \\ 0 & 0 & \rho \end{bmatrix}, \tag{24}$$

$$\overline{F}_2 = \begin{bmatrix} 0 & 1 & -\rho \\ 0 & -1 & 2\rho \\ 0 & 0 & \rho \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ \overline{F}_3 = \begin{bmatrix} 0 & 1 & -\rho \\ 0 & 2 & -4\rho \\ 0 & 0 & \rho \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{25}$$

The numerical procedure in Theorem 1 is coded into Matlab with the help of Yalmip (Löfberg, 2009; Löfberg, 2004), and Mosek (ApS, 2017) is used as solver. Setting $m = 1$, $N = 1$, $\mu_{\max} = 2$, $\mu_\infty = 1$, $k = 0.1$, $c = 0.1$, $|\dot{\rho}| \leqslant 1$, $\rho \in [0.01, 2]$, and $\nu_i(\rho) = \rho$, the procedure is successful even for $T$ of zero degree in $\rho$, which yields a valid parameter-dependent Lyapunov function defined by the following matrices.

$$P_1(\rho) = \begin{bmatrix} 35.8 & 0.568 \\ 0.568 & 341.0 \end{bmatrix} \tag{26}$$

$$\overline{P}_2(\rho) = \begin{bmatrix} 35.8 & 0.204 & 0.364\,\rho \\ 0.204 & 359.0 & -13.8\,\rho \\ 0.364\,\rho & -13.8\,\rho & 9.14\,\rho^2 \end{bmatrix} \tag{27}$$

$$\overline{P}_3(\rho) = \begin{bmatrix} 35.8 & 0.585 & -0.398\,\rho \\ 0.585 & 358.0 & -9.22\,\rho \\ -0.398\,\rho & -9.22\,\rho & -6.51\,\rho^2 \end{bmatrix} \tag{28}$$

The stability of the positioning servo-system is then assured, even in presence of high uncertainty in the friction model.

### *5.2 Example from the literature*

We consider Example 1 on page 557 of Johansson and Rantzer (1998), and we add a time-varying uncertainty on the borders of the cells. The dynamics (second order) is the same as in the reference:

$$A_1 = A_3 = \begin{bmatrix} -\epsilon & \omega \\ -\alpha\omega & -\epsilon \end{bmatrix},\ A_2 = A_4 = \begin{bmatrix} -\epsilon & \alpha\omega \\ -\omega & -\epsilon \end{bmatrix}, \tag{29}$$

with $\alpha = 5$, $\omega = 1$ and $\epsilon = 3$. The difference is introduced in the definition of the cells:

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1+\rho \\ -1 & -1-\rho \end{bmatrix}, \tag{30}$$

$$E_2 = -E_4 = \begin{bmatrix} -1 & 1+\rho \\ 1 & 1+\rho \end{bmatrix}, \tag{31}$$

and $F_i^\top = [E_i^\top, I]$. We let $\rho \in [-0.5, 0.5]$ and code the numerical procedure of Theorem 1 with a second-degree matrix $T(\rho)$ (whereas $\nu_i(\rho) = 1$). The procedure is successful in finding a Lyapunov function for small values of $\rho_d$, i.e. it works roughly up to $\rho_d \approx 0.25$; for faster variations of the parameter, the system is not guaranteed to be stable, which was to be expected as a coordinated switching between the two different dynamics can lead the state to blow-up (i.e., there is no common Lyapunov function for both $A_1 = A_3$ and $A_2 = A_4$).

## 6. CONCLUSIONS

This work has demonstrated the possibility of systematically finding piecewise-quadratic Lyapunov function for piecewise-affine systems with parametric dependence, where not only the dynamics can change with respect to the parameter, but also the state partition itself. This work can be considered as preliminary and many possible extensions can be imagined, starting for example with piecewise polynomial Lyapunov function as in Ameur et al. (2016) or Samadi and Rodrigues (2011). In addition to this, although we have only considered here a scalar parameter dependency, the method can be generalized without many technical difficulties to dependency with respect to more than one parameter. Further research might also investigate the extension of the analysis from simple stability to several different indices of performance.

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