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Non-linear damping for scattering-passive systems in the Maxwell class ⋆

Shantanu Singh, * **George Weiss,** ** **Marius Tucsnak** ***

* *School of Electrical Eng., Tel Aviv University, Ramat Aviv 69978, Israel (e-mail: shantanu@tauex.tau.ac.il)*
** *School of Electrical Eng., Tel Aviv University, Ramat Aviv 69978, Israel (e-mail: gweiss@tauex.tau.ac.il)*
*** *Université de Bordeaux, Bordeaux INP, CNRS, F-33400 Talence, France (e-mail: marius.tucsnak@u-bordeaux.fr)*

Abstract: We start from a special class of scattering passive linear infinite-dimensional systems introduced in Staffans and Weiss (SIAM J. Control and Opt., 2012). This class is called the Maxwell class of systems, because it includes the scattering formulation of Maxwell's equations, as well as various wave and beam equations. We generalize this class by allowing a nonlinear damping term. While the system may have unbounded linear damping (for instance, boundary damping), the nonlinear damping term $\mathcal{N}$ is "bounded" in the sense that it defined on the whole state space (but no actual continuity assumption is made on $\mathcal{N}$). We show that this new class of nonlinear infinite dimensional systems is well-posed and scattering passive. Our approach uses the theory of maximal monotone operators and the Crandall-Pazy theorem about nonlinear contraction semigroups, which we apply to a Lax-Phillips type nonlinear semigroup that represents the whole system, with input and output signals.

Keywords: Well-posed linear system, operator semigroup, Lax Phillips semigroup, scattering passive system, Maxwell's equations, maximal monotone operator, Crandall-Pazy theorem.

1. MOTIVATION

In the modelling of physical systems, we often come across second order differential equations with a nonlinear damping term depending on the velocity, such as

$$\ddot{z}(t) + D\dot{z}(t) + \mathcal{N}(\dot{z}(t)) + A_0 z(t) \ni B_0 u(t), \tag{1}$$

$$y(t) = C_0 \dot{z}(t) + D_0 u(t). \tag{2}$$

Here $z(t) \in E$, where E is a finite-dimensional inner product space. The function $z \in C^2([0,\infty); E)$ usually represents a vector of displacements, $\dot{z}$ and $\ddot{z}$ are its first and second derivatives, $A_0,\ D \in \mathcal{L}(E)$ are strictly positive and $\mathcal{N} : E \to E$ is a monotone set-valued function. "Monotone" means that if $z_1, z_2 \in E$ and $w_1 \in \mathcal{N}(z_1)$, $w_2 \in \mathcal{N}(z_2)$, then

$$\langle w_1 - w_2, z_1 - z_2 \rangle_E \geqslant 0.$$

The signals u, y are the input and the output of the system, both with values in U, which is another finite-dimensional inner product space, while $B_0 \in \mathcal{L}(U,E)$, $C_0 \in \mathcal{L}(E,U)$ and $D_0 \in \mathcal{L}(U)$.

The addition of the damping term $\mathcal{N}$ may create a much more complex dynamic behaviour (compared to the linear case), as the following example illustrates.

Example 1.1. Let us assume that z represents the one-dimensional displacement of a rigid body with mass $m > 0$ along a straight line, under the influence of the external force u, while it is connected to the point denoted by zero on this straight line via a spring with constant $k > 0$, having viscous friction with coefficient $d > 0$, $A_0 = k/m$, $D = d/m$ and $B_0 = 1/m$. Suppose that the nonlinear function $\mathcal{N}$ is

$$\mathcal{N}(v) = \beta\, \mathrm{sign}(v), \tag{3}$$

where $m\beta > 0$ is the amplitude of the Coulomb (or static) friction and sign (the multi-valued signum function) is defined by

$$\mathrm{sign}(v) = \begin{cases} 1 & \text{if } v > 0, \\ -1 & \text{if } v < 0, \\ [-1,1] & \text{if } v = 0. \end{cases} \tag{4}$$

It is well known that in this case, for any initial state $x(0) = [z(0)\ \dot{z}(0)]$ and any continuous function u, (1) has a unique solution. If u and d are sufficiently small, then this solution stops in finite time (at a point that may depend on u and on $x(0)$), see for instance Amann and Diaz (2003), Diaz and Millot (2003). Thus, the system has a continuum of equilibrium points, none of which is locally asymptotically stable. If we replace $\mathcal{N}(v)$ with the shifted version $\mathcal{N}(v - v_0)$ (which is still monotone, and represents Coulomb friction with respect to a moving platform having velocity $v_0 \neq 0$) then the system has a globally asymptotically stable equilibrium point.

The second example (below) will illustrate that the addition of the damping term $\mathcal{N}$ in (1) does not necessarily improve the stability properties of the system. We should not necessarily think of $\mathcal{N}$ as a clever addition to the system, meant to improve it, but as something present that needs to be modelled.

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Example 1.2. Consider the mechanical system consisting of two masses, two springs and a damper. The equations of the system are

$$m_1\ddot{x}_1(t) + k_1x_1(t) + k_2(x_1(t) - x_2(t)) = 0,$$
$$m_2\ddot{x}_2(t) - k_2(x_1(t) - x_2(t)) + d\dot{x}_2(t) = u(t),$$
$$y(t) = \dot{x}_2(t).$$

Assuming that all the constants are positive, it is easy to check that this system is exponentially stable. If, in addition, there is also static friction between the second mass and the fixed supporting surface, then the second equation changes to

$$m_2\ddot{x}_2(t) - k_2(x_1(t) - x_2(t)) + d\dot{x}_2(t) + \beta\,\mathrm{sign}(\dot{x}_2(t)) = u(t),$$

where the multivalued function sign is as in (4). This system can be put into the framework (1), by defining $E = \mathbb{R}^2$, $U = \mathbb{R}$, $C_0 = [0\ 1]$, $B_0 = C_0^\top$,

$$\left\langle \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle_E = m_1 z_1 x_1 + m_2 z_2 x_2,$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{d}{m_2} \end{bmatrix}, \qquad A_0 = \begin{bmatrix} \dfrac{k_1 + k_2}{m_1} & -\dfrac{k_2}{m_1} \\ -\dfrac{k_2}{m_2} & \dfrac{k_2}{m_2} \end{bmatrix},$$

$$\mathcal{N}\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \beta\,\mathrm{sign}(z_2) \end{bmatrix}.$$

The addition of static friction does not improve the stability of the system, quite the contrary. Assuming that $u = 0$, a typical state trajectory is such that $\dot{x}_2(t)$ becomes zero in finite time, after which $x_1(t)$ oscillates with constant amplitude.

This paper is about infinite-dimensional versions of (1)–(2), and our main interest is the well-posedness of the system, considering input signals $u \in L^2_{\mathrm{loc}}([0,\infty); U)$ (U is the input and output space, and is a Hilbert space), initial states $x(0) \in X$ (X is the state space, also a Hilbert space) and expecting final states $x(\tau) \in X$ and output signals in $y \in L^2_{\mathrm{loc}}([0,\infty); U)$. Well-posedness means that on any time interval $[0, \tau]$, the mapping from $x(0)$ and the u (restricted to $[0, \tau]$) to $x(\tau)$ and to y (restricted to $[0, \tau]$) is continuous, see Section 2.

There is a very large literature on systems described by partial differential equations that are linear except for a nonlinear damping term, which can act in the interior or on the boundary of the domain. We mention, as a representative sample only (in alphabetical order) the papers Alabau-Boussouira and Ammari (2011) (abstract second order in time equations), Berrahmoune (2002) (beam equations), Conrad and Rao (1993) (wave equations), Eller, Lagnese and Nicaise (2002) (Maxwell's equations), Lasiecka (1989) (wave and plate equations), Lasiecka and Tataru (1993) (wave equations), Zuazua (1990) (wave equations). As far as we are aware, most of the papers on this topic treat the well-posedness of the associated Cauchy problem, i.e., the existence of a non-linear strongly continuous semigroup on the state space, and its various asymptotic stability properties (decay rates). It seems that no attention has been devoted to systems with input and output signals that are described by equations containing a nonlinear damping term. Our aim in this paper is to fill this gap in an abstract and fairly general framework. Our main tools are the theory of maximal monotone operators, the generation theory of nonlinear contraction semigroups, the theory of scattering passive linear systems, and the Lax-Phillips semigroup associated to a well-posed system, for which we introduce here a nonlinear version.

Example 1.3. Consider the vibrations in a fixed vertical plane of a vertical beam clamped at the bottom, with a rigid body with a large mass M mounted on the top. Such a system could represent, for instance, a wind turbine tower with the nacelle and the turbine together playing the role of the rigid body. If we adopt the homogeneous Euler-Bernoulli model for the beam, then this is the famous SCOLE system, introduced by Littman and Markus (1988a,b) (they had in mind an antenna on a flexible mast). Suppose that a perturbation force f acting horizontally on the rigid body in the fixed vertical plane causes the beam to vibrate. In the case of the wind turbine, this force would represent the wind acting on the turbine and the nacelle. The well-posedness and other properties of this system were analyzed in Zhao and Weiss (2010), and many more relevant references are listed there.

We try to dampen the vibrations of the system described above by placing a trolley of mass m in contact with the rigid body, such that there is friction between them. This friction has a component of viscous friction with constant D and a component of static friction with amplitude f_0. The idea is to absorb the vibration energy via these frictions. Such dampers or more sophisticated versions, called tuned mass dampers, are often used to dampen the vibrations of very tall buildings, see, for instance, Hrovat, Barak and Rabins (1983), Varadarajan and Nagarajaiah (2004) and the references therein.

Assuming that the beam is uniform, with height l, the model (the SCOLE system coupled with the trolley, defined for $(x, t) \in ((0, l) \times [0, \infty))$ is

$$\begin{cases} \rho w_{tt}(x,t) + EIw_{xxxx}(x,t) = 0, \\ w(0,t) = 0, \qquad w_x(0,t) = 0, \\ Mw_{tt}(l,t) - EIw_{xxx}(l,t) = f(t) - D[w_t(l,t) - \xi_t(t)] \\ \qquad -f_0\mathrm{sign}[w_t(l,t) - \xi_t(t)] - k[w(l,t) - \xi(t)], \\ Jw_{xtt}(l,t) + EIw_{xx}(l,t) = 0, \\ m\xi_{tt}(t) = D[w_t(l,t) - \xi_t(t)] \\ \qquad +f_0\mathrm{sign}[w_t(l,t) - \xi_t(t)] + k[w(l,t) - \xi(t)], \end{cases} \tag{5}$$

where the subscripts t and x denote derivatives with respect to the time t and the position x, respectively. We have denoted by w the transverse displacement of the beam, and by ξ the horizontal displacement of the trolley with respect to an equilibrium position. The positive constants EI, ρ and J are the flexural rigidity of the beam, the mass density of the beam and the moment of inertia of the rigid body. We have assumed that the trolley is connected to the rigid body by a spring with constant k, whose role is to prevent the trolley from drifting away too far from its equilibrium position. (In practical applications there is a limited range for the displacement of the trolley.) The linear version of this system (corresponding to $f_0 = 0$) but with a non-uniform beam, has been investigated in Section 5 of Zhao and Weiss (2017).

The signal f is the force input acting on the rigid body. $-EIw_{xxxx}(x,t)\mathrm{d}x$ is the total lateral force acting on a slice of the beam of length $\mathrm{d}x$, located at the position x and the time t. $EIw_{xxx}(l,t)$ and -$EIw_{xx}(l,t)$ are the force and the torque acting on the rigid body from the beam at the time t. We define the input and output signals of the SCOLE model, u and y, as follows:

$$u = f\,, \qquad y = w_t(l,\cdot)\,. \tag{6}$$

The natural state of this nonlinear coupled system has two parts, z and q:

$$z(t) = \begin{bmatrix} \sqrt{\rho}\, w_t(\cdot,t) \\ \sqrt{k}(\xi(t) - w(l,t)) \\ \sqrt{J}\, w_{xt}(l,t) \end{bmatrix}, \quad q(t) = \begin{bmatrix} \sqrt{EI}\, w_{xx}(\cdot,t) \\ \sqrt{M}\, w_t(l,t) \\ \sqrt{m}\, \xi_t(t) \end{bmatrix}.$$

The state space where (z,q) evolves is

$$X = L^2[0,l] \times \mathbb{R}^2 \times L^2[0,l] \times \mathbb{R}^2\,,$$

with the natural norm. The (physical) energy of the system is $E(t) = \frac{1}{2}\|z(t)\|^2 + \frac{1}{2}\|q(t)\|^2$. This system is *impedance passive*, which means that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leqslant \langle u(t), y(t) \rangle.$$

The linear version of this system, which corresponds to $f_0 = 0$, can be expressed as an impedance passive system corresponding to the *Maxwell class of systems*, as in equation (1.8) in Staffans and Weiss (2012) (with slightly changed notation): For $t \geqslant 0$,

$$\begin{aligned} \begin{bmatrix} \dot z(t) \\ \dot q(t) \end{bmatrix} &= \begin{bmatrix} 0 & -L \\ L^* & G \end{bmatrix} \begin{bmatrix} z(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K_0^* \end{bmatrix} u(t), \\ y(t) &= [0 \quad K_0] \begin{bmatrix} z(t) \\ q(t) \end{bmatrix}. \end{aligned} \tag{7}$$

In the cited reference, the state space is $X = H \oplus E$, and E_0 is a dense subspace of E with continuous embedding. Denoting by E_0' the dual of E_0 with respect to the pivot space E, the framework from Staffans and Weiss (2012) requires the following conditions:

$$L \in \mathcal{L}(E_0, H)\,, \qquad K \in \mathcal{L}(E_0, U)\,, \qquad G \in \mathcal{L}(E_0, E_0'),$$

$\mathrm{Re}\, G \leqslant 0$ and the operator $\left[\begin{smallmatrix} L \\ K_0 \end{smallmatrix}\right]$ has to be closed as an unbounded operator from E to $H \oplus U$. All these requirements are true for the linear version of our coupled system (5), if we put $H = E = L^2[0,l] \times \mathbb{R}^2$, $E_0 = \mathcal{H}_l^1(0,1) \times \mathbb{R}^2$ and $U = \mathbb{R}$, where

$$\mathcal{H}_l^1(0,l) = \{\varphi \in \mathcal{H}^1(0,l) \;\mid\; \varphi(0) = 0\}.$$

The operators L and K_0 from (7) are defined by

$$L = \begin{bmatrix} \sqrt{\frac{EI}{\rho}} \frac{\mathrm{d}^2}{\mathrm{d}x^2} & 0 & 0 \\ 0 & \sqrt{\frac{k}{M}} & -\sqrt{\frac{k}{m}} \\ \sqrt{\frac{EI}{J}} \delta_l^* & 0 & 0 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 0 & \frac{1}{\sqrt{M}} & 0 \end{bmatrix},$$

where δ_l^* is the operator of point evaluation at l. The operator G from (7) is defined by

$$G = \begin{bmatrix} 0 & \sqrt{\frac{EI}{M}} \delta_l' & 0 \\ \sqrt{\frac{EI}{M}} \delta_l^* \frac{\mathrm{d}}{\mathrm{d}x} & -\frac{D}{M} & \frac{D}{\sqrt{Mm}} \\ 0 & \frac{D}{\sqrt{Mm}} & -\frac{D}{m} \end{bmatrix}$$

and then one can verify by computations that (7) is equivalent to the linear version of (5).

With the state space X this system is not well-posed, not even in the linear case, i.e., when $f_0 = 0$. In the linear case it becomes well-posed if we use instead the state space $\tilde X = \tilde H \oplus \tilde E$, where $\tilde H = \tilde E = \mathcal{H}_l^1(0,l) \times \mathbb{R}^2$.

Let us introduce new input and output signals via the scattering transformation

$$u_{\mathrm{sca}} = \frac{1}{\sqrt{2}}(u + y), \qquad y_{\mathrm{sca}} = \frac{1}{\sqrt{2}}(u - y).$$

With these signals, our system becomes scattering passive, which means that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leqslant \|u_{\mathrm{sca}}(t)\|^2 - \|y_{\mathrm{sca}}(t)\|^2\,.$$

Moreover, this nonlinear system is well-posed, as follows from the theory presented here later. Following the reasoning in (Staffans and Weiss, 2012, Sect. 5), in the linear case ($f_0 = 0$) the equations of this scattering passive system are exactly (7). In the nonlinear case, G should be replaced with $G - \mathcal{N}$. The nonlinear monotone set-valued function $\mathcal{N}$ is given as:

$$\mathcal{N}(q) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{f_0}{\sqrt{M}} & 0 \\ 0 & 0 & -\frac{f_0}{\sqrt{m}} \end{bmatrix} \mathrm{sign}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{M}} & -\frac{1}{\sqrt{m}} \\ 0 & \frac{1}{\sqrt{M}} & -\frac{1}{\sqrt{m}} \end{bmatrix} q(t) \right),$$

where $\mathrm{sign}(\cdot)$ is as defined in (4) and it is applied here component-wise.

2. THE MAXWELL CLASS OF SCATTERING PASSIVE SYSTEMS

A *well–posed linear system* is a linear time-invariant system such that on any finite time interval, the operator from the initial state and the input function to the final state and the output function is bounded. To express this more clearly, and also to clarify our notation, let us denote by U the input space, by X the state space and by Y the output space of a well-posed linear system Σ. U, X and Y are Hilbert spaces (to prepare for nonlinear extensions, we work with real Hilbert spaces) and the input and output functions are $u \in L^2_{\mathrm{loc}}([0,\infty); U)$ and $y \in L^2_{\mathrm{loc}}([0,\infty); Y)$. For any $u \in L^2_{\mathrm{loc}}([0,\infty); U)$ and any $\tau \geqslant 0$, we denote by $\mathbf{P}_\tau u$ its truncation to the interval $[0,\tau]$. Then the well-posed system Σ consists of the family of bounded operators $\Sigma = (\Sigma_\tau)_{\tau \geqslant 0}$ such that

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix} = \Sigma_\tau \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}. \tag{8}$$

Here $x : [0,\infty) \to X$ is the state trajectory of Σ corresponding to the initial state $x(0)$ and the input function u, and y is the corresponding output function. The boundedness property mentioned at the beginning of this section means that the operators Σ_τ are bounded. Denoting $c_\tau = \|\Sigma_\tau\|$, this can be written as

$$\|x(\tau)\|^2 + \int_0^\tau \|y(t)\|^2\, \mathrm{d}t \leqslant c_\tau^2 \left(\|x(0)\|^2 + \int_0^\tau \|u(t)\|^2\, \mathrm{d}t \right). \tag{9}$$

The operators Σ_τ are partitioned in a natural way (corresponding to the two product spaces) as follows:

$$\Sigma_\tau = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix}. \tag{10}$$

The four families of operators appearing on the right-hand side above must satisfy four functional equations expressing the causality and the time-invariance of Σ (these functional equations are parts of the definition of a well-posed system). In particular, the family $(\mathbb{T}_\tau)_{\tau \geqslant 0}$ is a strongly continuous operator semigroup on X and its generator A is called the *semigroup generator* of Σ. In the sequel, we assume that the reader is familiar with the basics of the theory of well-posed linear systems, as can be found for instance in Curtain and Weiss (1989), Salamon (1987), Staffans (2004), Staffans and Weiss (2002), Tucsnak and Weiss (2014), Weiss (1994a).

Every well-posed system with input space U, state space X and output space Y has a *system operator* $S : \mathcal{D}(S) \to X \times Y$, where $\mathcal{D}(S)$ is a dense subspace of $X \times U$ and S closed. Hence, the space $\mathcal{D}(S)$ may be regarded as a Hilbert space with the graph norm of S. How to find S from the operators Σ_τ is a long story for which we refer to the just mentioned references. In the particular case of a finite dimensional system described by the equations $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$, where A, B, C, D are matrices of suitable dimensions, we have $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. In general, the operator S gives a local description of the state trajectories and output functions of Σ, as explained in the following proposition, that is a part of Theorem 3.1 in Staffans and Weiss (2002).

Proposition 2.1. Assume that $u \in \mathcal{H}^1_{\mathrm{loc}}([0,\infty); U)$ and $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(S)$. The state trajectory x and the output function y of Σ are defined as in (8). Then

$$x \in C^1([0,\infty); X), \qquad \begin{bmatrix} x \\ u \end{bmatrix} \in C([0,\infty); \mathcal{D}(S)),$$

$$y \in \mathcal{H}^1_{\mathrm{loc}}([0,\infty); Y),$$

and for every $t \geqslant 0$ we have that

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{11}$$

From the above it can be shown (by density and continuous extension) that Σ is completely determined by its system operator S. For example, the semigroup generator of Σ is the "left upper" corner of S:

$$Ax_0 = [I \ 0] S \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \qquad \forall\, x_0 \in \mathcal{D}(A). \tag{12}$$

The system operator S can be split according to the product structure of its range space $X \times Y$, as follows:

$$S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}. \tag{13}$$

The operators $A\&B$ and $C\&D$ have the same domain $\mathcal{D}(S)$, $A\&B$ is closed but $C\&D$ usually is not.

Scattering passive systems are a subclass of the well-posed linear systems. A well-posed linear system Σ is called "scattering passive" if the following *balance inequality* holds:

$$\|x(\tau)\|^2 + \int_0^\tau \|y(t)\|^2 \mathrm{d}t \leqslant \|x(0)\|^2 + \int_0^\tau \|u(t)\|^2 \mathrm{d}t, \tag{14}$$

which is equivalent to $\|\Sigma_\tau\| \leqslant 1$ (for all $\tau \geqslant 0$).

In Staffans and Weiss (2012), Weiss and Staffans (2013), a class of scattering passive linear systems with a special structure has been introduced. This class appears in models of various systems from mathematical physics, such as wave, plate and Maxwell equations and it contains the class of systems "from thin air" introduced in Tucsnak and Weiss (2003), Weiss and Tucsnak (2003). Following our recent survey paper Tucsnak and Weiss (2014), we call this the "Maxwell class" of systems. This paper deals with a nonlinear extension of this class of systems. Thus it is necessary to briefly recall the definition of this class and the main facts about it, which we do in this section.

If Σ is a system in the *Maxwell class*, then its state space X can be decomposed as $X = H \oplus E$, where H and E are Hilbert spaces. The Hilbert space U is both the input space and the output space of Σ. We identify H, E and U with their duals H', E' and U'. The Hilbert space E_0 is a dense subspace of E and the embedding $E_0 \hookrightarrow E$ is continuous. We denote by E_0' the dual of E_0 with respect to the pivot space E, so that

$$E_0 \subset E \subset E_0', \tag{15}$$

densely and with continuous embeddings. We decompose the state of Σ as follows:

$$x_0 = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, \qquad z_0 \in H, \quad w_0 \in E.$$

The following theorem is extracted from the main results of Staffans and Weiss (2012).

Theorem 2.2. Let H, E, U and E_0 be as in the previous paragraph, and let the operators $L \in \mathcal{L}(E_0, H)$, $K \in \mathcal{L}(E_0, U)$ and $G \in \mathcal{L}(E_0, E_0')$ be such that

$$\begin{bmatrix} L \\ K \end{bmatrix} : E \to H \oplus U \text{ (with domain } E_0) \text{ is closed,} \tag{16}$$

$$\mathrm{Re}\, \langle Gw_0, w_0 \rangle_{E_0', E_0} \leqslant 0 \qquad \forall\, w_0 \in E_0. \tag{17}$$

Define the operator S by

$$S = \begin{bmatrix} 0 & -L & 0 \\ L^* & (G - \frac{1}{2}K^*K) & K^* \\ 0 & -K & I \end{bmatrix}, \tag{18}$$

$$\mathcal{D}(S) = \left\{ \begin{bmatrix} z_0 \\ w_0 \\ u_0 \end{bmatrix} \in \begin{matrix} H \\ \times \\ E_0 \\ \times \\ U \end{matrix} \;\middle|\; \begin{matrix} L^* z_0 + \left(G - \frac{1}{2}K^*K\right) w_0 \\ +K^* u_0 \in E \end{matrix} \right\}.$$

Then S is the system operator of a scattering passive system Σ. Moreover, the following claims hold:

1. If the input function u and the initial state $\left[\begin{smallmatrix} z(0) \\ w(0) \end{smallmatrix}\right]$ of S satisfy

$$u \in \mathcal{H}^1_{\mathrm{loc}}(0,\infty; U), \qquad \begin{bmatrix} z(0) \\ w(0) \\ u(0) \end{bmatrix} \in \mathcal{D}(S), \tag{19}$$

then the corresponding state trajectory $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ and output function y of S satisfy

$$\begin{bmatrix} z \\ w \end{bmatrix} \in C^1([0,\infty); H \oplus E), \qquad \begin{bmatrix} z \\ w \\ u \end{bmatrix} \in C([0,\infty); \mathcal{D}(S)),$$

$$y \in \mathcal{H}^1_{\mathrm{loc}}(0,\infty; Y), \tag{20}$$

and

$$\begin{bmatrix} \dot{z}(t) \\ \dot{w}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} z(t) \\ w(t) \\ u(t) \end{bmatrix} \qquad \forall\, t \geqslant 0. \tag{21}$$

2. The semigroup generator A of S is the restriction of the operator

$$\overline{A} = \begin{bmatrix} 0 & -L \\ L^* & (G - \frac{1}{2}K^*K) \end{bmatrix} \tag{22}$$

(defined on $H \times E_0$, with values in $H \times E_0'$) to the domain

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in \begin{matrix} H \\ \times \\ E_0 \end{matrix} \;\middle|\; L^* z_0 + \left(G - \tfrac{1}{2}K^*K\right) w_0 \in E \right\}.$$

3. We denote by X_1 the space $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(I - A)z\|$ and by X_{-1} the completion of X with respect to the norm $\|z\|_{-1} = \|(I - A)^{-1} z\|$. We have

$$X_1 \subset H \times E_0 \subset X \subset H \times E_0' \subset X_{-1},$$

densely and with continuous embeddings. A has a unique extension to an operator $A \in \mathcal{L}(X, X_{-1})$, whose restriction to $H \times E_0$ is $\overline{A}$ from (22).

4. If the functions $u, x = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ and y are as in (19)–(21), then they satisfy the following *power balance equation* for every $t \geqslant 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2 + 2\mathrm{Re}\,\langle Gw(t), w(t)\rangle\,.$$

We remark that the additional Claim 1 in the theorem is a consequence of the main statement together with Proposition 2.1, while Claim 2 follows from the main statement together with (12).

The aim of this paper is to generalize this class of systems by allowing the appearance of a monotone nonlinear operator $\mathcal{N}$ subtracted from G in (18). Thus, the operator $\overline{A}$ is replaced with the nonlinear operator

$$\overline{A} = \begin{bmatrix} 0 & -L \\ L^* & G - \mathcal{N} - \frac{1}{2}K^*K \end{bmatrix}, \tag{23}$$

where $\mathcal{N} : E \to E$ is a set-valued mapping that is *monotone*, meaning that for any $u_1, u_2 \in E$,

$$v_1 \in \mathcal{N}(u_1),\ v_2 \in \mathcal{N}(u_2) \ \Rightarrow\ \langle v_1 - v_2, u_1 - u_2\rangle \geqslant 0\,.$$

Moreover, we assume that $\mathcal{N}$ is *maximal monotone*, which means that it does not have a proper monotone extension (in the sense of inclusion of the graphs).

3. SOME BACKGROUND ABOUT THE LAX-PHILLIPS SEMIGROUP

Starting from an arbitrary well-posed linear system Σ, it is possible to define a strongly continuous semigroup which resembles those encountered in the scattering theory of Lax and Phillips (1967, 1973), and which contains all the information about Σ. We recall the basics about this semigroup, following Staffans and Weiss (2002) (related material is also in Staffans (2004), Chen and Weiss (2005)).

Like in the previous section, we assume that Σ is a well-posed linear system with component operator families as in (10), and we continue to use also the notation U, X and Y. For any $\tau \geqslant 0$, we denote by $\mathbf{S}_\tau$ the (unilateral) right shift operator on $\mathcal{U} = L^2([0,\infty); U)$ and also on $\mathcal{Y} = L^2((-\infty, 0]; Y)$, so that the adjoint $\mathbf{S}_\tau^*$ is the left shift by τ on the same spaces. We also introduce the bilateral right shift $\mathcal{S}_t$ acting on $L^2((-\infty,\infty); Y)$ (where $t \in \mathbb{R}$). We regard $\mathcal{Y}$ as a subspace of $L^2((-\infty,\infty); Y)$, by extending functions in $\mathcal{Y}$ to be zero for $t > 0$.

Proposition 3.1. For all $t \geqslant 0$ we define on $\mathcal{Y} \times X \times \mathcal{U}$ the operator $\mathfrak{T}_t$ by

$$\mathfrak{T}_t = \begin{bmatrix} \mathcal{S}_{-t} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathbf{S}_t^* \end{bmatrix} \begin{bmatrix} I & \Psi_t & \mathbb{F}_t \\ 0 & \mathbb{T}_t & \Phi_t \\ 0 & 0 & I \end{bmatrix}.$$

Then $\mathfrak{T} = (\mathfrak{T}_t)_{t\geqslant 0}$ is a strongly continuous semigroup.

If we take $y_0 \in \mathcal{Y}$, $x_0 \in X$ and $u_0 \in \mathcal{U}$ to represent the input function of Σ, its initial state and its past output (for negative time), then at any time $t \geqslant 0$, the first component of $\mathfrak{T}_t \left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right]$ represents the past output up to t, the second component represents the present state $x(t)$ and the third component represents the future input that will reach the system after t.

The operator semigroup $\mathfrak{T}$ introduced in the last proposition is called the *Lax-Phillips semigroup* of Σ. Much useful information on how to translate scattering theory into the language of systems theory is found in Helton (1976). The generator of $\mathfrak{T}$ can be characterized as follows:

Proposition 3.2. Let $\mathfrak{T}$ be the Lax–Phillips semigroup of the well-posed system Σ, with system operator S. We denote the generator of $\mathfrak{T}$ by $\mathfrak{A}$, and we use the notation $A\&B$ and $C\&D$ from (13).

(i) The domain of $\mathfrak{A}$, $\mathcal{D}(\mathfrak{A})$ consists of all the vectors $\left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{H}^1((-\infty, 0]; Y) \times X \times \mathcal{H}^1([0,\infty); U)$ which satisfy $\left[\begin{smallmatrix} x_0 \\ u_0(0) \end{smallmatrix}\right] \in \mathcal{D}(S)$ and $y_0(0) = C\&D \left[\begin{smallmatrix} x_0 \\ u_0(0) \end{smallmatrix}\right]$, and on $\mathcal{D}(\mathfrak{A})$, $\mathfrak{A}$ is given by

$$\mathfrak{A}\begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} y_0' \\ A\&B \left[\begin{smallmatrix} x_0 \\ u_0(0) \end{smallmatrix}\right] \\ u_0' \end{bmatrix}. \tag{24}$$

(ii) The following two conditions are equivalent:

(a) $\left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(\mathfrak{A})$ and $\left[\begin{smallmatrix} y \\ x \\ u \end{smallmatrix}\right] = \mathfrak{A} \left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right]$,

(b) $y_0 \in \mathcal{H}^1((-\infty, 0]; Y)$, $x_0 \in X$, $u_0 \in \mathcal{H}^1([0,\infty); U)$, $\left[\begin{smallmatrix} x_0 \\ u_0(0) \end{smallmatrix}\right] \in \mathcal{D}(S)$ and

$$\begin{bmatrix} x \\ y_0(0) \end{bmatrix} = S \begin{bmatrix} x_0 \\ u_0(0) \end{bmatrix}, \qquad \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} y_0' \\ u_0' \end{bmatrix}. \tag{25}$$

This proposition has been extracted (as a particular case) from Theorem 6.3 in Staffans and Weiss (2002).

Proposition 3.3. We use the notation of the previous two propositions. The following conditions are equivalent:

(i) Σ is scattering passive.
(ii) For every $\tau > 0$, the operator Σ_τ from (10) is a contraction (from $X \times L^2([0,\tau]; U)$ to $X \times L^2([0,\tau]; Y)$).
(iii) The Lax-Phillips semigroup induced by Σ is contractive (equivalently, $\|\mathfrak{T}_\tau\| = 1$ for all $t \geqslant 0$).

This proposition has been extracted from Proposition 7.2 in Staffans and Weiss (2002). The fact that $\|\mathfrak{T}_\tau\| \leqslant 1$ is equivalent to $\|\mathfrak{T}_\tau\| = 1$ follows from the structure of $\mathfrak{T}_\tau$: it contains blocks that are left shifts, hence $\|\mathfrak{T}_\tau\|$ cannot be less than 1.

4. THE MAIN RESULT

First we have to clarify what we mean by a (possibly nonlinear) time-invariant well-posed system. Our definition is far from being the most general or "the best" in any sense. Indeed, we assume that the input, state and output spaces are Hilbert spaces and the input and output functions are of class L^2_{loc}, because this is the framework that we are used to from the linear time-invariant case, but entirely different frameworks are conceivable.

Giving an axiomatic definition of a well-posed system in the spirit of Weiss (1994a) is possible but cumbersome. We prefer to define a well-posed system via its (non-linear version of the) Lax-Phillips semigroup.

For the basics about strongly continuous semigroups of nonlinear operators we refer to Brézis (1974), Crandall and Pazy (1969), Kato (1970), Showalter (1991). We recall here just a bare minimum of facts: Let Z be a Hilbert space. An operator $\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \to Z$ is called (maximal) dissipative if $-\mathfrak{A}$ is (maximal) monotone. For every maximal dissipative operator $\mathfrak{A}$ and for every $x \in \mathcal{D}(\mathfrak{A})$, the set $\mathfrak{A}x$ is convex and closed. If $\mathfrak{A}$ is maximal dissipative and densely defined, then we call it *m-dissipative*.

A strongly continuous semigroup of nonlinear operators $\mathfrak{T}$ acting on Z is defined exactly as in the linear case, without requiring that the operators are linear. If $\mathfrak{T}$ is such a semigroup, then its generator is defined by:

$$\mathfrak{A}^0 z = \lim_{t\to 0,\, t>0} \frac{1}{t}\left[\mathfrak{T}_t z - z\right],$$

$$\mathcal{D}(\mathfrak{A}^0) = \{z \in Z \mid \text{the above limit exists}\}.$$

A semigroup as above is called *contractive* if

$$\|\mathfrak{T}_t z_1 - \mathfrak{T}_t z_2\| \leqslant \|z_1 - z_2\| \qquad \forall\, z_1, z_2 \in Z,\, t \geqslant 0.$$

For any strongly continuous contractive operator semigroup on a Hilbert space Z, its generator $\mathfrak{A}^0$ is a dissipative operator whose domain $\mathcal{D}(\mathfrak{A}^0)$ is dense in Z. This operator has a unique (possibly set-valued) m-dissipative extension $\mathfrak{A}$, and $\mathcal{D}(\mathfrak{A}) = \mathcal{D}(\mathfrak{A}^0)$. For every $x \in \mathcal{D}(\mathfrak{A})$, $\mathfrak{A}^0 x$ is the unique element of smallest norm in the set $\mathfrak{A}x$. The set $\mathcal{D}(\mathfrak{A})$ is invariant under $\mathfrak{T}_t$ and for any $x_0 \in \mathcal{D}(\mathfrak{A})$, the trajectory $x(t) = \mathfrak{T}_t x_0$ is Lipschitz continuous and right differentiable. An important fact, often referred to as the Crandall-Pazy theorem, is the following: If $\mathfrak{A}$ is m-dissipative, then $\mathfrak{A}^0$ (which assign to every $x \in \mathcal{D}(\mathfrak{A})$ the element of smallest norm in $\mathfrak{A}x$) is the generator of a strongly continuous contractive semigroup on Z.

As in Sect. 3, U, X and Y will denote Hilbert spaces and $\mathcal{U} = L^2([0,\infty); U)$, $\mathcal{Y} = L^2((-\infty, 0]; Y)$.

Definition 4.1. A *time invariant well-posed system* with input space U, state space X and output space Y consists of two families of (possibly nonlinear) continuous operators

$$\Sigma^{\mathrm{st}} = (\Sigma^{\mathrm{st}}_t)_{t\geqslant 0}, \quad \Sigma^{\mathrm{out}} = (\Sigma^{\mathrm{out}}_t)_{t\geqslant 0},$$

where $\Sigma^{\mathrm{st}}_t : X \times \mathcal{U} \to X$ and $\Sigma^{\mathrm{out}}_t : X \times \mathcal{U} \to L^2([0,\infty); Y)$ such that the following is a strongly continuous semigroup of (possibly nonlinear) operators $\mathfrak{T}^{\mathcal{N}} = (\mathfrak{T}^{\mathcal{N}}_t)_{t\geqslant 0}$ acting on $\mathcal{Y} \times X \times \mathcal{U}$: for every $t \geqslant 0$,

$$\mathfrak{T}^{\mathcal{N}}_t = \begin{bmatrix} \mathcal{S}_{-t} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathbf{S}^*_t \end{bmatrix} \begin{bmatrix} I & \Sigma^{\mathrm{out}}_t \\ 0 & \Sigma^{\mathrm{st}}_t \\ 0 & 0 \quad I \end{bmatrix}. \tag{26}$$

To understand the meaning of this definition, one should compare it to Proposition 3.1. It is clear that in the linear case, we have

$$\Sigma^{\mathrm{st}}_t = [\mathbb{T}_t \;\; \Phi_t], \qquad \Sigma^{\mathrm{out}}_t = [\Psi_t \;\; \mathbb{F}_t],$$

but in the nonlinear case, in general we cannot split these operators in a similar way as above. Note that the semigroup property for the family of operators $\mathfrak{T}^{\mathcal{N}}$ implies functional equations for the families of operators Σ^{st} and Σ^{out} which, in the linear case, reduce to the functional equations in the definition of a well-posed linear system, as given for instance in Weiss (1994a). For example, we must have for all $t, \tau > 0$, $x_0 \in X$ and $u, v \in \mathcal{U}$,

$$\Sigma^{\mathrm{st}}_{t+\tau} \begin{bmatrix} x_0 \\ u \underset{\tau}{\diamondsuit} v \end{bmatrix} = \Sigma^{\mathrm{st}}_t \begin{bmatrix} \Sigma^{\mathrm{st}}_\tau \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \\ v \end{bmatrix}.$$

Here we have used the notation for the concatenation of two functions $u, v \in \mathcal{U}$, as defined in Weiss (1994a) and many other papers. Naturally, we call $\mathfrak{T}^{\mathcal{N}}$ the *Lax-Phillips semigroup* of the system $\Sigma^{\mathcal{N}} = (\Sigma^{\mathrm{st}}, \Sigma^{\mathrm{out}})$.

Scattering passive (possibly nonlinear) systems are defined exactly as in the linear case. A time-invariant well-posed system is scattering passive if and only if its Lax-Phillips semigroup $\mathfrak{T}^{\mathcal{N}}$ is contractive. Scattering passive systems come with the big advantage that they can be described locally in time, via the generator of $\mathfrak{T}^{\mathcal{N}}$.

The generator of the Lax-Phillips semigroup for the (possibly nonlinear) scattering passive well-posed system $\Sigma^{\mathcal{N}} = (\Sigma^{\mathrm{st}}, \Sigma^{\mathrm{out}})$ has the following structure: There exist two nonlinear operators $[A\&B]^{\mathcal{N}}$ (possibly multi-valued in X) and $[C\&D]^{\mathcal{N}}$ (with single values in Y), both defined on $\mathcal{D}(S^{\mathcal{N}})$, which is a dense subset of $X \times U$, such that

$$\mathfrak{A}^{\mathcal{N}} = \begin{bmatrix} y_0' \\ [A\&B]^{\mathcal{N}} \begin{bmatrix} x_0 \\ u_0(0) \end{bmatrix} \\ u_0' \end{bmatrix}, \tag{27}$$

$$\mathcal{D}(\mathfrak{A}^{\mathcal{N}}) \subset \left\{ \begin{array}{c} \mathcal{H}^1((-\infty,0);Y) \\ \times \\ X \\ \times \\ \mathcal{H}^1((0,\infty);U) \end{array} \;\middle|\; \begin{array}{c} \begin{bmatrix} x_0 \\ u_0(0) \end{bmatrix} \in \mathcal{D}(S^{\mathcal{N}}) \\ y_0(0) = [C\&D]^{\mathcal{N}} \begin{bmatrix} x_0 \\ u_0(0) \end{bmatrix} \end{array} \right\}$$

If

$$\begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix} \in \mathcal{D}(\mathfrak{A}^{\mathcal{N}}), \qquad \begin{bmatrix} y_t \\ x_t \\ u_t \end{bmatrix} = \mathfrak{T}^{\mathcal{N}}_t \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix},$$

then $[y_t \;\, x_t \;\, u_t]^\top \in \mathcal{D}(\mathfrak{A}^{\mathcal{N}})$ for all $t \geqslant 0$ and it is right differentiable. A detailed explanation will be provided in the journal version of this paper (Singh, Weiss and Tucsnak (2020)).

The *system operator* of $\Sigma^{\mathcal{N}}$ is now defined by

$$S^{\mathcal{N}} = \begin{bmatrix} [A\&B]^{\mathcal{N}} \\ [C\&D]^{\mathcal{N}} \end{bmatrix}.$$

Clearly the system operator determines the system, via the generator $\mathfrak{A}^{\mathcal{N}}$ of its Lax-Phillips semigroup $\mathfrak{T}^{\mathcal{N}}$.

Theorem 4.2. We use the notation and the assumptions of Theorem 2.2, so that Σ is a scattering passive system from the Maxwell class. Let $\mathcal{N}$ be a (set-valued) maximal monotone operator defined on all of E. Consider the nonlinear operator $\Sigma^{\mathcal{N}}$ that is obtained from Σ by replacing G with $G - \mathcal{N}$. Thus,

$$S^{\mathcal{N}} = \begin{bmatrix} 0 & -L & 0 \\ L^* & (G - \mathcal{N} - \frac{1}{2}K^*K) & K^* \\ 0 & -K & I \end{bmatrix}, \tag{28}$$

with $\mathcal{D}(S_{\mathcal{N}}) = \mathcal{D}(S)$ (defined after (18)).

Then $S^{\mathcal{N}}$ is the system operator of a scattering passive time invariant well-posed system.

Sketch of the proof. According to (24) and (27), the generator $\mathfrak{A}^{\mathcal{N}}$ of Lax-Phillips semigroup $\mathfrak{T}$ (if it exists at all) must have the following structure:

$$\mathfrak{A}^{\mathcal{N}} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\xi} & 0 & 0 & 0 \\ 0 & 0 & -L & 0 \\ 0 & L^* & (G - \mathcal{N} - \frac{1}{2}K^*K) & K^* \\ 0 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}\xi} \end{bmatrix}, \tag{29}$$

with the same domain $\mathcal{D}(\mathfrak{A})$ as in the linear case, described in equation (25). In order to prove that $\mathfrak{A}^{\mathcal{N}}$ is m-dissipative, we split it as follows:

$$\mathfrak{A}^{\mathcal{N}} = \underbrace{\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\xi} & 0 & 0 & 0 \\ 0 & 0 & -L & 0 \\ 0 & L^* & G - \frac{1}{2}K^*K & K^* \\ 0 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}\xi} \end{bmatrix}}_{\mathfrak{A}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{N} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{-\tilde{\mathcal{N}}}.$$

The first term on the right hand-side, $\mathfrak{A}$ is m-dissipative since it is the generator of the Lax-Phillips semigroup of the linear scattering passive system Σ described in Theorem 2.2. The second term $-\tilde{\mathcal{N}}$ is m-dissipative because $\mathcal{N}$ is maximally monotone, by assumption. $\mathcal{N}$ is defined on the entire Hilbert space E, hence $\mathcal{D}(\tilde{\mathcal{N}}) = \mathcal{Y} \times X \times \mathcal{U}$. Therefore, $\mathcal{D}(\mathfrak{A}) \cap \left(\mathrm{int}\mathcal{D}(\tilde{\mathcal{N}})\right) = \mathcal{D}(\mathfrak{A})$, which is dense. According to Theorem 1 of Rockafellar (1970), it follows that $\mathfrak{A}^{\mathcal{N}}$ is m-dissipative (and densely defined) on $\mathcal{Y} \times X \times \mathcal{U}$. According to the main result of Crandall and Pazy (1969) (see also Theorem 5 in Brézis (1974)), $\mathfrak{A}^{\mathcal{N}}$ generates a contraction semigroup $\mathfrak{T}^{\mathcal{N}}$ on $\mathcal{Y} \times X \times \mathcal{U}$.

Take $[y_0\ x_0\ u_0]^\top \in \mathcal{D}(\mathfrak{A}^{\mathcal{N}})$. Then the trajectory

$$\begin{bmatrix} y(t) \\ x(t) \\ u(t) \end{bmatrix} = \mathfrak{T}_t^{\mathcal{N}} \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix},$$

is a classical solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} y(t) \\ x(t) \\ u(t) \end{bmatrix} = \mathfrak{A}^{\mathcal{N}} \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix}.$$

In particular the first component satisfies $\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \frac{\mathrm{d}}{\mathrm{d}\xi}y(\xi)$ in the space $\mathcal{Y}$. Hence $y(t) = \mathcal{S}_{-t}y_0$, therefore the first component evolves according to left shift semigroup on $\mathcal{Y}$. Also, the last component satisfies the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \frac{\mathrm{d}}{\mathrm{d}\xi}u(\xi)$ in the space $\mathcal{U}$. It follows that $u(t) = \mathbf{S}_t^* u_0$, i.e., the last component evolves according to the left shift semigroup on $\mathcal{U}$. Thus, we have verified that the third line of $\mathfrak{T}_t^{\mathcal{N}}$ has the structure as required in (26).

The remaining details will be given in the journal version of this paper, Singh, Weiss and Tucsnak (2020). □

5. CONCLUSION

In this paper we have introduced the generalized Maxwell class of systems obtained from linear time invariant system in the Maxwell class, with a nonlinear damping term $\mathcal{N}$. We have shown that such systems are well-posed and scattering passive, provided that the nonlinear term $\mathcal{N}$ is maximally monotone and is defined over the entire state space.

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