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Heterogeneous Multi-Population Evolutionary Dynamics with Migration Constraints ⋆

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Abstract: We present a novel distributed heterogeneous multi-population evolutionary dynamics approach, which can be used in diverse engineering applications as a distributed optimization-based algorithm. We also provide stability certificates of Nash equilibria under the proposed approach. Finally, over the end of this paper, an example illustrates the performance of the aforementioned multi-population evolutionary dynamics.

Keywords: Heterogeneous populations, multi-population evolutionary dynamics, distributed learning

1. INTRODUCTION

The evolutionary game dynamics have become a powerful tool in the modeling of strategic interactions as in Hofbauer and Sigmund [1988], Barreiro-Gomez and Tembine [2018a]. Recently, this approach has been implemented in many engineering applications as in Quijano et al. [2017]. Moreover, in Tian et al. [2019], authors propose dynamical attacking strategies in the simulation of reputation management scheme evaluation applying evolutionary games. In Kawano et al. [2019], authors study the evolutionary dynamics of two different types of communities in an evolving environment, and in Stella and Bauso [2019], a behavior from honeybee swarms is generalized to duopolistic competition and opinion dynamics in the context of evolutionary dynamics.

Evolutionary dynamics represent the evolution of a population composed of a large number of agents. In this regard, the evolutionary dynamics are a non-atomic and anonymous approach, i.e., the decisions of an individual agent have a negligible influence over the whole population. Likewise, this approach implies that all the agents are homogeneous. Therefore, population dynamics assume that the switching rates, for a given pairwise interaction, have the same structure for all the decision makers within the population. In contrast, this paper suggests to take into account that there must be different types of revision protocols within a same mass of agents. For convenience, we consider a society that comprises the collection of all the heterogenous agents in a strategic interaction. Furthermore, there are different classes of masses that are forming diverse populations. Due to the fact that each population evolves differently, i.e., its evolution is given by different evolutionary dynamics, then the proposed approach becomes a multi-population case. Moreover, we allow different populations to interact to each other within the same society. Indeed, even though the evolution of two different populations are described by the same evolutionary dynamics, they might have different structures regarding migration constraints. Hence, strategies play different roles. For instance, there are strategies that allow the interaction to other populations known as migration nodes, but there are also some strategies that only allow an interaction within the same population.

The contribution of this paper is a novel distributed heterogeneous multi-population evolutionary dynamics approach, in which the evolution of different populations composing a unique society is described by means of a coupled and distributed system. We present stability certificates, showing that the multi-population approach can still be used for the same purposes as the single-population counterpart. To this end, we present an off-line economic dispatch problem.

Notation: Let $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$ denote the set of real, positive real, and non-negative real numbers, respectively. The

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decision variables of population p are mostly denoted with an associated super-index, e.g., $x^p \in \mathbb{R}^{n_p}$. Moreover, $\mathbb{1}_n$ ($\mathbf{0}_n$) denotes the vector with n unitary (null) entries, i.e., $\mathbb{1}_n = [1 \quad \dots \quad 1]^\top$ $\left(\mathbf{0}_n = [0 \quad \dots \quad 0]^\top\right)$, and $\mathbb{I}_n$ is the $n \times n$ identity matrix. Boundary of the set Δ is denoted by $\partial\Delta$. Finally, consider the operator $[\cdot]_+ = \max(0, \cdot)$.

2. HETEROGENEOUS MULTI-POPULATION EVOLUTIONARY DYNAMICS

Let us consider a society composed of a set of $q \in \mathbb{N}_{>0}$ *heterogeneous* populations $\mathcal{P} = \{1, \dots, q\}$ (e.g., Figure 1). In addition, the society comprises a large and finite number of rational agents or decision makers. Let $\pi^{\mathrm{s}} \in \mathbb{R}_{>0}$ denote the mass of the society. Each population $p \in \mathcal{P}$ has an associated mass $\pi^p \leq \pi^{\mathrm{s}}$, which is composed of a large and finite number of agents. These agents make the decision to select a certain strategy within the population they belong to, or can migrate to a different population within the society pursuing to enhance their payoff. Let $\mathcal{S}^p = \{1, \dots, n^p\}$ be the set of available strategies in the population $p \in \mathcal{P}$, and the scalar x_i^p be the portion of agents selecting the strategy $i \in \mathcal{S}^p$ in the population $p \in \mathcal{P}$. The vector $x^p = [x_1^p \quad \dots \quad x_{n^p}^p]^\top \in \mathbb{R}^{n^p}$ denote the p-population state, or the strategic distribution in the p^{th} population. Each population has a set of feasible population states given by $\Delta_+^p(\pi^p) = \left\{x^p \in \mathbb{R}_{\geq 0}^{n^p} : \sum_{i \in \mathcal{S}^p} x_i^p = \pi^p\right\}$, $\forall p \in \mathcal{P}$, and the tangent space of the simplex $\Delta_+^p(\pi^p)$ is given by $\mathrm{T}\Delta_+^p = \left\{z^p \in \mathbb{R}^{n^p} : \sum_{i \in \mathcal{S}^p} z_i^p = 0\right\}$, for all $p \in \mathcal{P}$. It follows that there exists a relationship between the population states, the population mass and the society mass, i.e., $\sum_{p \in \mathcal{P}} \sum_{i \in \mathcal{S}^p} x_i^p = \sum_{p \in \mathcal{P}} \pi^p = \pi^{\mathrm{s}}$. Let $\pi = [\pi^1 \quad \dots \quad \pi^q]^\top \in \mathbb{R}_{\geq 0}^q$ denote the society state or the strategic distribution along the entire society, i.e., the allowed mass within each population is decided throughout the society. Then, consider the following simplex set

$$\Delta^{\mathrm{s}} = \left\{\pi \in \mathbb{R}_{>0}^q : \sum_{p \in \mathcal{P}} \pi^p = \pi^{\mathrm{s}}\right\}. \tag{1}$$

As previously mentioned, agents make decisions in order to maximize their payoffs, which are determined by a fitness function, i.e., let $f_i^p : \Delta_+^p(\pi^p) \to \mathbb{R}$ be the function corresponding to the portion of agents x_i^p selecting the strategy $i \in \mathcal{S}^p$, for all $p \in \mathcal{P}$. Hence, let $f^p : \Delta_+^p(\pi^p) \to \mathbb{R}^{n^p}$ denote the population fitness function, i.e., $f^p(x^p) = [f_i^p(x^p) \quad \dots \quad f_{n^p}^p(x^p)]^\top$.

At each population, there is a portion of agents in charge of coordinating with other populations. In other words, there are strategies working as a gate for both emigration and immigration. Let $\mathcal{M}^p \subset \mathcal{S}^p$ be the set corresponding to the migration strategies in the population $p \in \mathcal{P}$. Hence, Let $g^p = f_i^p(x^p) : \Delta_+^p(\pi^p) \to \mathbb{R}$ be the fitness function corresponding to the migration strategy $i \in \mathcal{M}^p$, for all $p \in \mathcal{P}$. Let $\mathcal{M} = \{1, \dots, m\}$ be the set of $m \geq q$ migration nodes. Notice that, there are not isolated populations (it is possible that agents immigrate and/or emigrate to/from each population).

Assumption 1. For simplicity, we consider there is a unique strategy at each population for the migration to other populations, i.e., the set of migration strategies $\mathcal{M}^p$ is a singleton set, for all $p \in \mathcal{P}$. □

In addition, there are two different types of migration: (1) switching among strategies within the same population, and (2) migration among populations. Indeed, there are constraints for all the possible migrations in the entire society represented by undirected and connected graphs as explained next.

Let $\mathcal{G}^p = (\mathcal{S}^p, \mathcal{E}^p, A^p)$ be the connected graph representing the possible migration among strategies within the same population $p \in \mathcal{P}$, where $\mathcal{E}^p \subseteq \{(i,j) : i,j \in \mathcal{S}^p\}$ represents the possible interaction among the portion of agents selecting the strategies, i.e., if $(i,j) \in \mathcal{E}^p$, then the portion of agents x_i^p and x_j^p can interact to each other and migration between the strategies i and j can occur. Moreover, $A^p \in \{0,1\}^{n^p \times n^p}$ is the adjacency matrix of the graph $\mathcal{G}^p$. On the other hand, let $\mathcal{G} = (\mathcal{M}, \mathcal{E}, A)$ be the undirected graph representing the possible migration among populations, where $\mathcal{E} \subseteq \{(p,r) : p,r \in \mathcal{M}\}$ represents the possible interaction among populations. It follows that considering Assumption 1, if $(p,r) \in \mathcal{E}$ then agents in the population $p \in \mathcal{P}$ can migrate to population $r \in \mathcal{P}$, or equivalently, portion of agents x_i^p and x_j^r can interact, where $i \in \mathcal{M}^p$ and $j \in \mathcal{M}^r$. Similarly, the matrix $A \in \{0,1\}^{m \times m}$ is the adjacency matrix of the graph $\mathcal{G}$.

This work is developed under the framework of games with monotone fitness functions and/or full-potential games as defined below.

Definition 1. The game $f^p : \Delta_+^p(\pi^p) \to \mathbb{R}^{n^p}$ for $p \in \mathcal{P}$ is monotone decreasing if $(x^p - y^p)^\top (f^p(x^p) - f^p(y^p)) \leq 0$, for all $x^p, y^p \in \Delta_+^p(\pi^p)$. Alternatively, if $f^p(x^p)$ is continuously differentiable and monotone decreasing, a sufficient condition is $Df^p(x^p) \preceq 0$. □

Definition 2. If there exists a continuously differentiable potential function $V^p : \mathbb{R}_{\geq 0}^{n^p} \to \mathbb{R}$, such that $f^p(x^p) = \nabla V^p(x^p)$, for all $x^p \in \mathbb{R}_{\geq 0}^n$, then $f^p(x^p)$ is a full-potential game. □

Additionally, we consider two different solution concepts, i.e., the Nash equilibria in its traditional form (Definition 3), and over graphs depending on the social interaction structure, or equivalently, considering migration constraints (Definition 4).

Definition 3. The population state $x^{p*} \in \Delta_+^p(\pi^p)$ is a Nash equilibrium if each used strategy entails the maximum benefit for the proportion who is choosing it as in Sandholm [2010], i.e., the set $\mathrm{NE}(f^p(x^p)) = \{x^p \in \Delta_+^p(\pi^p) : x_i^p > 0 \implies f_i^p(x^p) \geq f_j^p(x^p),\ \forall\ i,j \in \mathcal{S}^p\}$, for all $p \in \mathcal{P}$, corresponds to the Nash equilibria. □

Definition 4. Let $A^p \in \{0,1\}^{n^p \times n^p}$ be the adjacency matrix of the graph $\mathcal{G}^p$ representing the migration constraints in the p^{th} population. The population state $x^{p*} \in \Delta_+^p(\pi^p)$ is a Nash equilibrium if each used strategy entails the maximum benefit in the neighborhood for the proportion who is choosing it. The set $\mathrm{NEG}(f^p(x^p), A^p) = \{x^p \in \Delta_+^p(\pi^p) : x_i^p > 0 \implies f_i^p(x^p) \geq f_j^p(x^p),\ \forall\ i \in \mathcal{S}^p, j \in \mathcal{N}_i^p\}$, for all $p \in \mathcal{P}$, corresponds to the Nash equilibria on graphs. □

There exists an equivalence between $\mathrm{NE}(f^p(x^p))$ and $\mathrm{NEG}(f^p(x^p), A^p)$ depending on the graph $\mathcal{G}$ as stated below in Lemma 1.

Lemma 1. If the possible interaction in a population is given by an undirected connected graph $\mathcal{G}^p$, then the set of

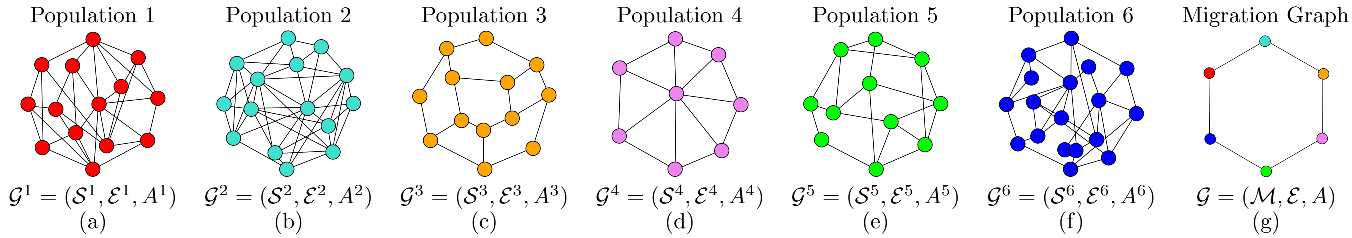

Fig. 1. Population and migration graphs.

equilibria $\text{NE}(f^p(x^p)) = \text{NEG}(f^p(x^p), A^p)$ (see [Barreiro-Gomez and Tembine 2018b, Lemma 1]). □

Proof. Let $\text{supp}(x^p) = \{i \in \mathcal{S}^p : x_i^p > 0\}$. Since $x^{p*} \in \Delta^p$, then $\text{supp}(x^{p*}) \neq \emptyset$, i.e., there exists an $i \in \mathcal{S}^p$ for which $x_i^{p*} > 0$. Now, let $f_i^p(x^{p*}) = \max_{j \in \mathcal{N}_i^p} f_j^p(x^{p*})$. If $x_i^{p*} = \pi^p$, then $x_j^{p*} = 0$, for all $j \in \mathcal{S}^p \backslash \{i\}$, and it is concluded that $x^{p*} \in \text{NE}(f^p(x^p))$. If $x_i^{p*} < \pi^p$ then, there is another strategy $j \in \mathcal{S}^p$ such that $x_j^{p*} > 0$. Moreover, let $f_j^p(x^{p*}) = \max_{a \in \mathcal{S}^p} f_a^p(x^{p*})$. Notice that, since the graph $\mathcal{G}^p$ is connected, then there exists a path on $\mathcal{G}^p$ connecting i to j, i.e., a path $\tilde{\mathcal{E}}^p{}_{i \to j} \subseteq \mathcal{E}^p$. It follows that

$$f_i^p(x^{p*}) = \max_{a \in \mathcal{N}_i} f_a^p(x^{p*}) = \max_{a \in \mathcal{N}_i^p} \left(\max_{k \in \mathcal{N}_a^p} f_k^p(x^{p*}) \right),$$

$$\vdots$$

$$= \max_{a \in \mathcal{N}_i^p} \left(\max_{k \in \mathcal{N}_a^p} \cdots \left(\max_{\ell \in \mathcal{N}_r^p} f_\ell(x^{p*}) \right) \right), \text{(covering } \tilde{\mathcal{E}}_{i \to j}^p),$$

then $f_i^p(x^{p*}) = \max_{a \in \mathcal{S}^p} f_a^p(x^{p*})$, for which it is concluded that $x^{p*} \in \text{NE}(f^p(x^p))$. ■

The interactions are determined by a revision protocol (Sandholm [2010]). Consider the function $\varrho^p : \Delta_+^p(\pi^p) \times \mathbb{R}^{n^p} \times \{0,1\}^{n^p \times n^p} \to \mathbb{R}_{\geq 0}^{n^p \times n^p}$ known as revision protocol, which describes how agents make decisions in the population $p \in \mathcal{P}$. The revision protocol takes a population state, a population fitness function and the adjacency matrix representing the migration restrictions in the population, and returns a non-negative matrix. Moreover, $\varrho_{ij}^p(x^p, f^p(x^p), A^p)$ is known as the switching rate from the i^{th} to j^{th} strategy in the p^{th} population. Agents selecting the strategy $i \in \mathcal{S}^p$ have incentives to migrate to the strategy $j \in \mathcal{S}^p$ only if $\varrho_{ij}^p(x^p, f^p(x^p), A^p) > 0$.

Definition 5. A revision protocol $\varrho_{ij}^p(x^p, f^p(x^p), A^p)$ is pair-wise if it has the following form:

$$\varrho_{ij}^p(x^p, f^p(x^p), A^p) = a_{ij}\phi(x^p)[f_j^p(x^p) - f_i^p(x^p)]_+,$$

with $\phi(x^p) > 0$, for all $x^p \in \Delta^p(\pi^p) \backslash \partial \Delta^p(\pi^p)$ and $\phi(x^p) \geq 0$, for all $x^p \in \partial \Delta^p(\pi^p)$. □

The evolutionary game dynamics, describing the evolution process for each population, emerge from the combination of the population game f^p and the switching rates $\varrho_{ij}^p(x^p, f^p, A^p)$. Consider a novel distributed version of the mean dynamics considering birth and death for each population (reproduction rate $\delta_i^p(x^p)$), i.e.,

$$\begin{aligned} \dot{x}_i^p &= D^p(x^p), \\ &= \sum_{j \in \mathcal{S}^p} x_j^p \varrho_{ji}^p(x^p, f^p(x^p), A^p) \qquad (2) \\ &\quad - x_i^p \sum_{j \in \mathcal{S}^p} \varrho_{ij}^p(x^p, f^p(x^p), A^p) + \delta_i^p(x^p), \ \forall i \in \mathcal{S}^p, p \in \mathcal{P}, \end{aligned}$$

where $D^p : \mathbb{R}^{n^p} \to \mathbb{R}^{n^p}$, and $\delta_i^p(x^p) = 0$ if $x^p \in \Delta_+^p(\pi^p)$. We take the following reproduction rate

$$\delta_i^p(x^p) = \alpha^p \left(\pi^p - \sum_{j \in \mathcal{S}^p} x_j^p \right), \ \forall i \in \mathcal{S}^p, p \in \mathcal{P}, \qquad (3)$$

where $\alpha^p \in \mathbb{R}_{>0}$. Each population might have different revision protocol, e.g., first population evolving with the distributed replicator $\varrho_{ij}^1(x^1, f^1(x^1), A^1) = x_j^1 a_{ij}^1 [f_j^1(x^1) - f_i^1(x^1)]_+$, second with the distributed Smith $\varrho_{ij}^2(x^2, f^2(x^2), A^2) = a_{ij}^2 [f_j^2(x^2) - f_i^2(x^2)]_+$, third with the distributed projection $\varrho_{ij}^3(x^3, f^3(x^3), A^3) = a_{ij}^3 [f_j^3(x^3) - f_i^3(x^3)]_+ / x_i^3$, among others.

Proposition 1. *$\Delta^p(\pi^p) = \left\{x^p \in \mathbb{R}^{n^p} : \sum_{i \in \mathcal{S}^p} x_i^p = \pi^p\right\}$, for all $p \in \mathcal{P}$, is invariant under the density-dependent mean dynamics in (2). Moreover, if the component-wise inequality $D^p(x^p) \geq 0$ holds for all $x^p \in \partial \Delta_+^p(\pi^p)$ and $p \in \mathcal{P}$. Then, the simplex set $\Delta_+^p(\pi^p)$ is invariant under the density-dependent mean dynamics in (2).* □

Proof. First, we compute $\sum_{i \in \mathcal{S}^p} \dot{x}_i^p$ to verify the invariance of the simplex $\Delta^p(\pi^p)$, i.e.,

$$\begin{aligned} \sum_{i \in \mathcal{S}^p} \dot{x}_i^p &= \sum_{i \in \mathcal{S}^p} \sum_{j \in \mathcal{S}^p} x_j^p \varrho_{ji}^p(x^p, f^p(x^p), A^p) \\ &\quad - \sum_{i \in \mathcal{S}^p} \sum_{j \in \mathcal{S}^p} x_i^p \varrho_{ij}^p(x^p, f^p(x^p), A^p) + \sum_{i \in \mathcal{S}^p} \delta_i^p(x^p), \\ &= 0, \end{aligned}$$

for all $p \in \mathcal{P}$, for which $\Delta^p(\pi^p)$ is invariant under (2). Moreover, if $D^p(x^p) \geq 0$ holds for all $x^p \in \partial \Delta_+^p(\pi^p)$ and $p \in \mathcal{P}$. Then, when $x_i^p = 0$, $\dot{x}_i^p \geq 0$ showing the invariance of $\Delta_+^p(\pi^p)$. ■

Proposition 2. *Let $D^p(x^p) \geq 0$ holds for all $x^p \in \partial \Delta_+^p(\pi^p)$ and $p \in \mathcal{P}$. The population simplex $\Delta_+^p(\pi^p)$ is locally asymptotically stable under the density-dependent mean dynamics in (2) with the reproduction rate (3).* □

Proof. Consider the following Lyapunov function (Poveda and Quijano [2015]): $L_1(x^p) = \frac{1}{2}\left(\sum_{i \in \mathcal{S}^p} x_i^p - \pi^p\right)^2$, where $L_1(x^p) > 0$ for all $x^p \notin \Delta_+^p(\pi^p)$, and $L_1(x^p) = 0$ for all $x^p \in \Delta_+^p(\pi^p)$. Therefore,

$$\begin{aligned} \dot{L}_1(x^p) &= \left(\sum_{i \in \mathcal{S}^p} x_i^p - \pi^p \right) \sum_{i \in \mathcal{S}^p} \dot{x}_i^p, \\ &= -\alpha^p n^p \left(\sum_{i \in \mathcal{S}^p} x_i^p - \pi^p \right)^2 \leq 0. \end{aligned}$$

Since $D^p(x^p) \geq 0$ holds for all $x^p \in \partial\Delta_+^p(\pi^p)$ and $p \in \mathcal{P}$, then $\dot{L}_1(x^p) = 0$ holds only if $x^p \in \Delta_+^p(\pi^p)$. Then $\Delta_+^p(\pi^p)$ is asymptotically stable completing the proof. ■

Corollary 1. Let conditions stated in Proposition 2 hold. Then, the social state $\pi = [\pi^p]_{p\in\mathcal{P}}$ asymptotically converges to the social simplex Δ^s. □

Proof. It immediately follows from the definition of the social simplex Δ^s in (1), i.e., if $x^p \in \Delta_+^p(\pi^p)$, for all $p \in \mathcal{P}$ (Proposition 2), then it implies $\pi \in \Delta^s$. ■

Theorem 1. Let $f^p(x^p)$ be a full-potential game with strictly concave potential function $V^p(x^p)$, and the revision $\varrho_{ij}^p(x^p, f^p(x^p), A^p)$ be a pair-wise comparison protocol (see Definition 5). Then, the Nash equilibrium over graphs $x^{p*} \in \Delta^p(\pi^p)\backslash\partial\Delta^p(\pi^p)$ is locally asymptotically stable under the dynamics in (2). □

Proof. The stability analysis under the distributed dynamics in (2) is developed by using the potential function $V^p(x^p)$ to define a Lyapunov function, i.e., consider the following Lyapunov function candidate: $L_2^p(x^p) = V^p(x^{p*}) - V^p(x^p)$, where $L_2^p(x^p) > 0$, for all $x^p \neq x^{p*}$, and $L_2^p(x^{p*}) = 0$. Hence,

$$\dot{L}_2^p(x^p) = -[\nabla V^p(x^p)]^\top \dot{x}^p = -\sum_{j\in\mathcal{S}^p} \dot{x}_j^p f_j^p(x^p),$$

since it is assumed that $x^p \in \Delta_+^p(\pi^p)$, then $\delta_i^p(x^p) = 0$. It follows that

$$\dot{L}_2^p(x^p) = -\sum_{j\in\mathcal{S}^p}\sum_{i\in\mathcal{S}^p} x_i^p \varrho_{ij}^p(x^p, f^p(x^p), A^p) f_j^p(x^p) + \sum_{j\in\mathcal{S}^p}\sum_{i\in\mathcal{S}^p} x_j^p \varrho_{ji}^p(x^p, f^p(x^p), A^p) f_j^p(x^p),$$

since the revision protocol is of the form $\varrho_{ij}^p(x^p, f^p(x^p), A^p) = a_{ij}\phi(x^p)[f_j^p(x^p) - f_i^p(x^p)]_+$, with $\phi(x^p) > 0$, for all $x^p \in \Delta^p(\pi^p)\backslash\partial\Delta^p(\pi^p)$, and due to the fact $A^p = A^{p\top}$

$$\dot{L}_2^p(x^p) = -\sum_{j\in\mathcal{S}^p}\sum_{i\in\mathcal{S}^p} a_{ij}\phi(x^p) x_i^p [f_j^p(x^p) - f_i^p(x^p)]_+^2 \leq 0.$$

Notice that the term in the latter inequality is equivalent to have $\dot{x}^{p\top} f^p(x^p) \geq 0$, which is the positive correlation property in Sandholm [2010]. Hence, notice that $\dot{L}_2^p(x^p) = 0$ if and only if $x^p \in \mathrm{NEG}(f^p(x^p), A^p)$ (assuming $\mathcal{G}^p$ is connected, then $\dot{L}_2^p(x^p) = 0$ if and only if $x^p \in \mathrm{NE}(f^p(x^p))$). Therefore, applying the LaSalle-invariance principle, it is concluded that the equilibrium point x^{p*} is locally asymptotically stable. ■

3. ESTIMATION AND MIGRATION DYNAMICS

3.1 Distributed Estimations About Collective Information

Each population dynamics in (2) are not distributed since the reproduction rate depends on the population mass deviation, which is computed by using both the total desired population mass π^p and the knowledge about the current population mass $\sum_{i\in\mathcal{S}^p} x_i^p$. In order to make the evolutionary dynamics in (2) non-centralized, two different distributed projection-based estimators are proposed. Therefore, consider auxiliary variables for each population $w^p \in \Delta^p(0)$ with a possible interaction given by the population graph $\mathcal{G}^p$. Also, let $h_i^p : \Delta^p(0) \to \mathbb{R}$ be an auxiliary fitness function. Hence, the distributed projection dynamics introduced in Barreiro-Gomez et al. [2017] are given by

$$\dot{w}_i^p = \sum_{j\in\mathcal{S}^p} a_{ij}^p\big(h_j^p(w_j^p) - h_i^p(w_i^p)\big),\ \forall i \in \mathcal{S}^p, p \in \mathcal{P}. \quad (4)$$

In addition, let $h_i^p(w_i^p) = -w_i^p - c_i^p$, with $i \in \mathcal{S}^p$, where $c_i^p \in \mathbb{R}_{\geq 0}$ is a constant related to the information to propagate, for all $i \in \mathcal{S}^p$, $p \in \mathcal{P}$.

Corollary 2. Let $\mathcal{G}^p$ be connected. Then, the set $\Delta^p(0)$ is invariant under the dynamics in (4), and the Nash equilibrium over graphs $w^{p*} \in \Delta^p(0)$, such that $h^p(w^{p*}) \in \mathrm{span}\{\mathbb{1}_{n^p}\}$, is locally asymptotically stable under the dynamics in (4). □

Proof. Notice that (4) is equivalent to (2) considering $w^p = x^p$, $h^p(w^p) = f^p(x^p)$, and the pairwise comparison revision protocol $\varrho_{ij}^p(x^p, f^p(x^p), A^p) = a_{ij}^p[f_j^p(x^p) - f_i^p(x^p)]_+/x_i^p$. Hence, both claims immediately follow from Proposition 1, and Theorem 1 taking into account that $\mathcal{G}^p$ is connected, and the fact $h^p(w^p) = f^p(x^p)$ represents a full-potential game as in Definition 2, and also a monotone fitness function as in Definition 1. ■

Proposition 3. The equilibrium point in (4) is given by $w_i^{p*} = \sum_{j\in\mathcal{S}^p} c_j^p/n^p - c_i^p$, for all $i \in \mathcal{S}^p$ and $p \in \mathcal{P}$. □

Proof. The equilibrium fitness is given by $h^p(w^{p*}) \in \mathrm{span}\{\mathbb{1}_{n^p}\}$. Therefore,

$$h_i^p(w_i^{p*}) = -w_i^{p*} - c_i^p,\ \forall i \in \mathcal{S}^p,$$
$$= \frac{1}{n^p}\sum_{j\in\mathcal{S}^p} h_j^p(y_j^{p*}) = -\frac{1}{n^p}\sum_{j\in\mathcal{S}^p} c_j^p,$$

since $w^p \in \Delta^p(0)$, showing that $w_i^{p*} = \frac{1}{n^p}\sum_{j\in\mathcal{S}^p} c_j^p - c_i^p$, for all $i \in \mathcal{S}^p$, which is the desired result. ■

Using the result in Proposition 3, the dynamics in (4) can be performed as a distributed estimator about collective information as discussed in the folowing Sections.

Estimating Mean State Within Populations: In the dynamics (2), information about the mean population state is required. These dynamics are centralized within the population since information from all the portion of agents is required at each strategy. Let $z^p \in \mathbb{R}^{n^p}$, for all $p \in \mathcal{P}$. Each element z_i^p is associated to the estimation of the mean state $\sum_{j\in\mathcal{S}^p} x_j^p/n^p$ for the corresponding strategy $i \in \mathcal{S}^p$ in the population $p \in \mathcal{P}$. The estimation is performed by using following dynamics:

$$\dot{z}_i^p = \sum_{j\in\mathcal{S}^p} a_{ij}^p\Big(\Big(z_i^p + x_i^p\Big) - \Big(z_j^p + x_j^p\Big)\Big), \quad (5)$$

for all $i \in \mathcal{S}^p$, $p \in \mathcal{P}$.

Corollary 3. Let x^p be fixed and $z^p \in \Delta^p(0)$, for all $p \in \mathcal{P}$. Then, $z_i^{p*} + x_i^p \to \sum_{j\in\mathcal{S}^p} \frac{x_j^p}{n^p}$, for all $i \in \mathcal{S}^p$, $p \in \mathcal{P}$. □

Proof. Notice that dynamics in (5) are equivalent to (4) considering $z^p = w^p$, and $h_i^p(z_i^p) = -z_i^p - x_i^p$, for all $i \in \mathcal{S}^p$, $p \in \mathcal{P}$. Then, the result immediately follows from Proposition 3 where it is shown that $z_i^{p*} = \sum_{j\in\mathcal{S}^p} x_j^p/n^p - x_i^p$, for all $i \in \mathcal{S}^p$, then, $z_i^{p*} + x_i^p \to \sum_{j\in\mathcal{S}^p} x_j^p/n^p$, for all $i \in \mathcal{S}^p$, $p \in \mathcal{P}$ completing the proof. ■

Estimating Mass Within Populations: Dynamics in (2) also require centralized knowledge about the desired population mass π^p, for all $p \in \mathcal{P}$. Moreover, notice that this information is only known by the corresponding migration node $\mathcal{M}^p$ (only one according to Assumption 1). Hence, information π^p should be propagated from $\mathcal{M}^p$ to all other nodes $\mathcal{S}^p \backslash \mathcal{M}$ within the same population, and in a distributed fashion. To this end, let $y^p \in \mathbb{R}^{n^p}$, for all $p \in \mathcal{P}$, and consider the distributed estimator given by

$$\dot{y}_i^p = \sum_{j \in \mathcal{S}^p} a_{ij}^p \left(\left(y_i^p + r_i^p \right) - \left(y_j^p + r_j^p \right) \right), \tag{6a}$$

$$r_i^p = \begin{cases} \pi^p & \text{if } i \in \mathcal{M}^p \\ 0 & \text{otherwise} \end{cases}, \tag{6b}$$

for all $i \in \mathcal{S}^p$, $p \in \mathcal{P}$.

Corollary 4. Let π^p be fixed and $y^p \in \Delta^p(0)$, for all $p \in \mathcal{P}$. Then, $y_i^{p*} \to \frac{\pi^p}{n^p}$, for all $i \in \mathcal{S}^p \backslash \mathcal{M}^p$, and $p \in \mathcal{P}$. □

Proof. Notice that dynamics in (6a) are equivalent to (4) considering $y^p = w^p$, and $h_i^p(y_i^p) = -y_i^p - r_i^p$, for all $i \in \mathcal{S}^p$, $p \in \mathcal{P}$. Then, according to Proposition 3, it is known that, $y_i^{p*} = \sum_{j \in \mathcal{S}^p} r_j^p / n^p - r_i^p$, for all $i \in \mathcal{S}^p$. Hence, the result immediately follows since $y_i^{p*} + r_i^p \to \pi^p / n^p$ and considering that $r_i^p = 0$, for all $i \in \mathcal{S}^p \backslash \mathcal{M}^p$. ■

3.2 Distributed Migration Dynamics Among Populations

Consider there is a strategic interaction among the heterogeneous populations throughout the migration strategies. To this end, let the migration nodes $\mathcal{M}$ perform as leaders of each population by using information $f_i^p(x^{p*})$ where $i \in \mathcal{M}^p$, for all $p \in \mathcal{P}$, i.e., $i \in \mathcal{M}^p$ makes decisions based on the population fitness, which is inferred from its own fitness at equilibrium.

In this regard, the strategic interaction among populations is represented by the graph $\mathcal{G}$ shown in Figure 1(g). The population strategic interaction can be seen as a game with q strategies from the set $\mathcal{M}$ (for simplicity $\mathcal{P}$, according to Assumption 1) whose respective fitness functions can be seen as $g^p(\pi^p)$, for all $p \in \mathcal{P}$. Notice that the function g^p is monotone decreasing with respect to π^p assuming that $g^p(\pi^p) = f_i^p(x^{p*})$ where x^{p*} is the population equilibrium for a given population mass π^p. The population mass dynamics are

$$\dot{\pi}^p = \sum_{q \in \mathcal{M}} a_{pq}(g^p(\pi^p) - g^q(\pi^q)), \ \forall p \in \mathcal{P}, \tag{7}$$

which corresponds to the distributed projection dynamics of the same form as in (4).

Proposition 4. The simplex set Δ^s in (1) is invariant under dynamics in (7), and the Nash equilibrium over graphs $\pi^{p*} \in \Delta^s$, such that $g^p(\pi^{p*}) \in \text{span}\{\mathbb{1}_m\}$, is locally asymptotically stable under the dynamics in (7) with region of attraction Δ^s. □

Proof. Notice that (7) is equivalent to (2) considering $\pi = x^p$, $g(\pi) = f^p(x^p)$, being $g(\pi) = [g^1(\pi^1) \quad \dots \quad g^m(\pi^m)]^\top$, and the pairwise comparison revision protocol

$$\varrho_{ij}^p(x^p, f^p(x^p), A^p) = a_{ij}^p[f_j^p(x^p) - f_i^p(x^p)]_+ / x_i^p.$$

Hence, the invariance claims immediately follow from Proposition 1. However, notice that even though $g^p(\pi^p)$ is monotone decreasing with respect to π^p for all $p \in \mathcal{P}$, Theorem 1 cannot be applied since the game is not full potential. Therefore, we use a different Lyapunov function as in [Barreiro-Gomez et al. 2017, Theorem 4], i.e., $L_3(\pi) = \sum_{p \in \mathcal{M}} \sum_{q \in \mathcal{M}} \frac{a_{pq}}{2}[g^q(\pi^q) - g^p(\pi^p)]_+^2$, it follows that

$$\begin{aligned} \frac{\partial}{\partial \pi^\ell} L_3(\pi) &= \sum_{p \in \mathcal{M}} \sum_{q \in \mathcal{M}} \Bigg(a_{pq}[g^q(\pi^q) - g^p(\pi^p)]_+ \\ &\quad - a_{pq}[g^p(\pi^p) - g^q(\pi^q)]_+ \Bigg) \frac{\partial}{\partial \pi^\ell} g^q(\pi^q), \\ &= \sum_{q \in \mathcal{M}} \dot{\pi}^q \frac{\partial}{\partial \pi^\ell} g^q(\pi^q). \end{aligned} \tag{8}$$

Then, we compute $\dot{L}_3(\pi)$ by using expression in (8), i.e, $\dot{L}_3(\pi) = [\nabla L_3(\pi)]^\top \dot{\pi}$, or equivalently $\dot{L}_3(\pi) = \dot{\pi}^\top Dg(\pi) \dot{\pi}$. It follows that $\dot{\pi}^\top Dg(\pi) \dot{\pi} \leq 0$ since $Dg(\pi) \preceq 0$ according to Definition 1. Therefore, it is concluded that $\dot{L}_3(\pi) \leq 0$, and equality $\dot{L}_3(\pi) = 0$ holds only when $\pi \in \text{NE}(g)$ completing the proof. ■

4. OVERALL DISTRIBUTED HETEROGENEOUS MULTI-POPULATION DYNAMICS

Thanks to the previously presented mathematical analysis, it is possible to establish the following distributed heterogeneous multi-population game dynamics approach:

$$\begin{aligned} \beta^p \dot{x}_i^p &= \sum_{j \in \mathcal{S}^p} x_j^p \varrho_{ij}^p(x^p, f^p(x^p), A^p) \\ &\quad - x_i^p \sum_{j \in \mathcal{S}^p} \varrho_{ji}^p(x^p, f^p(x^p), A^p) + \alpha^p \left(y_i^p - z_i^p - x_i^p \right), \end{aligned} \tag{9a}$$

$$\varepsilon^p \dot{y}_i^p = \sum_{j \in \mathcal{S}^p} a_{ij}^p \left(\left(y_i^p + r_i^p \right) - \left(y_j^p + r_j^p \right) \right), \tag{9b}$$

$$\varepsilon^p \dot{z}_i^p = \sum_{j \in \mathcal{S}^p} a_{ij}^p \left(\left(z_i^p + x_i^p \right) - \left(z_j^p + x_j^p \right) \right), \tag{9c}$$

$$\dot{\pi}^p = \sum_{q \in \mathcal{M}} a_{pq}(g^p(\pi^p) - g^q(\pi^q)), \tag{9d}$$

$$r_i^p = \begin{cases} \pi^p & \text{if } i \in \mathcal{M}^p, \\ 0 & \text{otherwise.} \end{cases} \tag{9e}$$

for all $i \in \mathcal{S}^p$, $p \in \mathcal{P}$. Initial conditions are $x^p(0)$, $y^p(0)$, $z^p(0) \in \mathbb{R}^{n^p}$, for all $p \in \mathcal{P}$; and $\pi(0) \in \mathbb{R}_{\geq 0}^q$.

Remark 1. Notice that the system in (9) is totally distributed (all the sums have the corresponding entry from the adjacency matrix, i.e., (9) can be written in term of the set of neighbors for each node in the graphs $\mathcal{G}^p$, for all $p \in \mathcal{P}$, and $\mathcal{G}$), and that the dynamics of the portion of agents (9a) is an initialization-free algorithm. □

The algorithm in (9) has three different time scales, i.e., the parameters ε^p and β^p are selected such that the distributed estimations about collective information in (9b) and (9c) are the fastest dynamics, and dynamics in (9d) are the slowest. This implies that dynamics in (9a) become the dynamics in (2) since $y_i^{p*} - z_i^{p*} - x_i^p = \left(\pi^p - \sum_{j \in \mathcal{S}^p} x_j^p \right) / n^p$ according to Corollaries 3 and 4. It follows that at each population, a Nash equilibrium

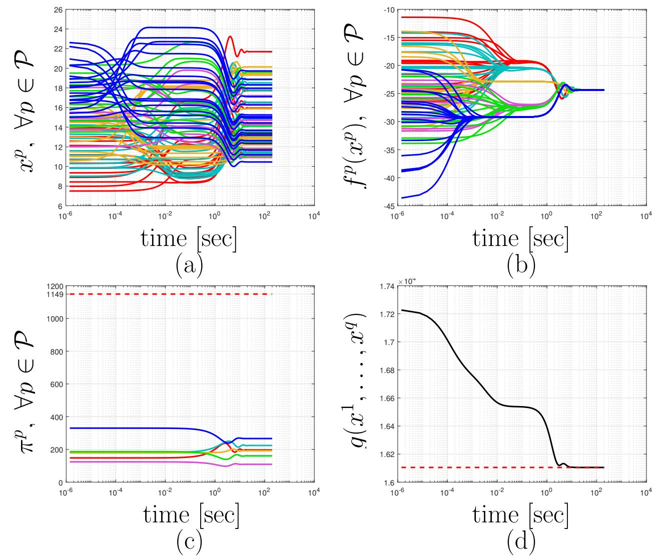

Fig. 2. Distributed heterogeneous multi-population evolutionary dynamics performing as a distributed economic dispatch solver. Evolution of: (a) population states, (b) population fitness functions, (c) population mass (society mass is shown in dashed line), and (d) cost function (optimal cost is shown in dashed line).

is achieved for the corresponding mass π^p according to Proposition 2 and Theorem 1. Finally, depending on the comparison of the fitness functions for different migration nodes, dynamics in (9d) adjust the population masses such that all the fitness functions get the same value for the whole population.

5. DISTRIBUTED OPTIMIZATION: ECONOMIC DISPATCH PROBLEM

Let us consider a distributed power system that provides electricity to $q = 6$ towns, i.e., $\mathcal{P} = \{1, \ldots, 6\}$. Each town has installed a set of distributed electric generators (DGs). The number of DGs of town $1, \ldots, 6$ is 14, 8, 15, 11, 13 and 18, respectively. Therefore, the power system has a total of 79 DGs. Let $x_i^p \in \mathbb{R}_{\geq 0}$ be the power generated by the generator $i \in \mathcal{S}^p$ in the town $p \in \mathcal{P}$. The distributed power system has to supply an electric demand given by $\pi^s \in \mathbb{R}$, i.e., it is necessary to guarantee that $\sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{S}^p} x_i^p = \pi^s$. The objective is to minimize the cost of generating the demanded power, considering that each DG has an associated cost function $g_i^p : \mathbb{R} \to \mathbb{R}$ given by $g_i^p = a_i^p (x_i^p)^2 + b_i^p x_i^p$, where $a_i^p \in \mathbb{R}_{>0}$ and $b_i^p \in \mathbb{R}$ are cost function's parameters. Thus, the problem can be formulated as follows:

$$\min_{x^1,\ldots,x^q} g(x^1, \ldots, x^q) = \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{S}^p} g_i^p(x_i^p),$$
$$\text{s.t.} \quad \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{S}^p} x_i^p = \pi^s, \quad \mathbf{0}_{n^p} \leq x^p,\ \forall p \in \mathcal{P},$$

where $g : \mathbb{R}^{\sum_{p\in\mathcal{P}} n^p} \to \mathbb{R}$ is convex. For simulation purposes, cost function's parameters are randomly selected from the intervals $a_i^p \in [1, 2]$ and $b_i^p \in [-1, 1]$. Furthermore, we assume that the demand is $\pi^s = 1149$ p.d.u.. Therefore, a full-potential game for each population emerges with potential function $V_p = \sum_{i\in\mathcal{S}^p} g_i^p(x_i^p)$ and fitness functions given by $f_i^p(x_i^p) = \frac{\partial V_p}{\partial x_i^p} = -2x_i^p c_i^p - b_i^p$, for all $i \in \mathcal{S}^p$, and $p \in \mathcal{P}$. Results of the distributed algorithm in (9) applied for solving the economic dispatch problem are shown in Figure 2. Notice that the proposed algorithm asymptotically reaches the optimal solution while constraints are satisfied all time.

6. CONCLUSIONS

We have proposed novel distributed heterogeneous multi-population evolutionary dynamics including stability certificates. The presented approach discussed in this paper considers a society composed of multiple populations, which evolve by using different revision protocols, i.e., populations evolve according to different distributed evolutionary dynamics. We motivate the approach as an alternative to consider heterogeneous types or classes of agents in a strategic interaction. Furthermore, the proposed multi-population approach allows migration among strategies within the same population, but also migration among different populations. Besides, diverse population communication structures or migration constraints are taken into account. Finally, populations can be isolated, preserving the classical population dynamics counterpart.

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