Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Stabilization of Burgers' equation by constrained control ⋆

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Abstract: The work addresses constrained control of Burgers' equation by using distributed in space point actuation and measurements. An observer-based distributed-in-domain point control law is suggested to stabilize the system, where the controller employs averaged values of the observer. Constructive conditions are derived to ensure that the resulting closed-loop system is regionally exponentially stable. A numerical example demonstrates the efficiency of the results.

Keywords: Burgers' equation, point actuation, constrained control.

1. INTRODUCTION

Burgers' equation is a model for car clustering where the clustering occurs due to the difference of the inherent velocities of individual cars, which describes traffic problems (see e.g. Nagatani et al. (1998)). Burgers' equation describes also models in fluid mechanics, nonlinear acoustics and gas dynamics. Distributed / boundary stabilization of Burgers' equation has been extensively studied (see e.g.Balogh et al. (2000); Byrnes et al. (1998); Krstic (1999); Krstic et al. (2008); Liu et al. (2000); Ly et al. (1997) and the references therein). In Smyshlyaev et al. (2010), some further stability results were provided for boundary stabilization of Burgers' equation via backstepping method. In Krstic et al. (2008), observer design, trajectory generation and tracking problem of Burgers' equation was studied. In Meurer et al. (2011), PDE-based leader-enabled deployment problem was solved, where a modified viscous Burgers equation was used to model the motion of the mobile agent continuum.

A considerable amount of attention has been paid to constrained distributed control of PDEs (see e.g. El-Farra et al. (2003); Marx et al. (2015); Prieur et al. (2014); Slemrod (1989)). For practical application, the constraints on the control input should be taken into account in many cases. In El-Farra et al. (2003), the internal feedbacks with input constraints of quasi-linear heat equation were designed and the domains of attractions were found via the Galerkin method. In Marx et al. (2015); Prieur et al. (2014), global stabilization by distributed saturated control of 1-D Korteweg-de Vries and wave equations was studied. Global stabilization of linear or semilinear system in the Hilbert space by using constrained control was presented in Slemrod (1989). In Kang et al. (2017), Kang et al. (2018), regional boundary stabilization of coupled linear ODE-heat systemn and nonlinear Schrödinger equation under actuator saturation was presented respectively. The results in Kang et al. (2017, 2018) were based on the backstepping method Krstic et al. (2008) and on direct Lyapunov method for finding domains of attraction of the resulting target systems.

In Azouani et al. (2014); Pisano et al. (2017) point in-domain control of unstable diffusion equation under the collocated point state measurements was studied. In the absence of disturbances in the equation, a linear static output feedback may globally stabilize the system. However, in the presence of control input constraints, it is not clear if such a controller can achieve at least regional stability. Finding domain of attraction seems to be not possible here. It should be noticed that there are few works on regional analysis of distributed parameter system, which motivates our study (see e.g. El Jai et al. (1995)). In the present paper, for stabilization of Burgers' equation under point actuation and measurements, we suggest an observer-based control law that employs averaged values of the observer. This allows to regionally stabilize the system and to give an estimate on the domain of attraction via Lyapunov method. Moreover, sensors and actuators are not supposed to be collocated.

The work is organized as follows. The problem is formulated in next section. In Section 3, an observer-based point controller under point measurements is constructed. Sufficient LMI conditions are presented for the stability analysis of the closed-loop system. In Section 4, a constrained controller is introduced and an estimate on the domain of attraction is found. A numerical example illustrates the main results in Section 5. The conclusions are stated in Section 6.

⋆ This work was supported by Israel Science Foundation (Grant No 1128/14, 673/19), National Natural Science Foundation of China (grant No. 61803026), Beijing Science Foundation for the Excellent Youth Scholars (Grant No. 2018000020124G067), Outstanding Chinese and Foreign Youth Exchange Program of China Association of Science and Technology, Fundamental Research Funds for the Central Universities (Grant No. FRF- TP-18- 032A1), China Postdoctoral Science Foundation (Grant No. 2018M640065, 2019T120047).

Notations and preliminaries. Throughout the paper, $L^2(0,1)$ stands for the Hilbert space of square integrable scalar functions $v(x)$ on $(0,1)$ with the corresponding norm $\|v\|^2_{L^2(0,1)} = \int_0^1 |v(x)|^2 dx$. $H^1_0(0,1)$ is the closure in $H^1(0,1)$ of the set of smooth functions that are vanishing at $x=0$ and $x=1$. It is equipped with the norm $\|v\|^2_{H^1_0(0,1)} = \int_0^1 |v'(x)|^2 dx$.

Lemma 1. *(Wirtinger's inequality Fridman et al. (2018) or Krstic et al. (2008)): Let $v \in H^1_0(a,b)$. Then the following inequality holds:*

$$\int_a^b v^2(x)dx \leq \frac{(b-a)^2}{\pi^2}\int_a^b \left[\frac{dv}{dx}(x)\right]^2 dx.$$

Lemma 2. *(Agmon's inequality, see p. 20 in Krstic et al. (2008)) For any $v \in H^1(0,1)$, the following inequalities hold:*

$$\max_{x\in[0,1]} |v(x,t)|^2 \leq v^2(0) + 2\|v(t)\|_{L^2(0,1)}\|v_x(t)\|_{L^2(0,1)},$$
$$\max_{x\in[0,1]} |v(x,t)|^2 \leq v^2(1) + 2\|v(t)\|_{L^2(0,1)}\|v_x(t)\|_{L^2(0,1)}.$$

2. PROBLEM FORMULATION

Consider the following 1-D Burger's equation under the Dirichlet boundary conditions:

$$\begin{cases} z_t(x,t) = \gamma z_{xx}(x,t) - z(x,t)z_x(x,t) + \lambda z(x,t) \\ \qquad + \sum_{j=0}^{N-1} \delta(x-\bar{x}_j)u_j(t), \\ z(0,t) = z(1,t) = 0, \\ z(x,0) = z_0(x), \end{cases} \tag{1}$$

where $\gamma > 0$ is viscosity, $\lambda > 0$ denotes a constant coefficient, $z(x,t)$ is the state of Burger's equation, and $u_j(t)$ $(j = 0,1,\cdots,N-1)$ are the control inputs. If $\lambda < \gamma\pi^2$, the open-loop system (with $u_j(t) \equiv 0$) is exponentially stable. For $\lambda > \gamma\pi^2$, the open-loop system (with $u_j(t) \equiv 0$) may become unstable.

We assume that $\{\Omega_{u_j}\}_{j=0}^{N-1}$ is a partition of $[0,1]$. The intervals Ω_{u_j} are upper bounded by Δ_u:

$$0 < |\Omega_{u_j}| \leq \Delta_u,$$

where Δ_u is the maximum subdomain length $\max_j |\Omega_{u_j}|$. The control inputs $u_j(t)$ enter (1) through the Dirac delta function at some points $\bar{x}_j \in \Omega_{u_j}$.

Suppose that sensors are uncollocated with actuators providing point measurements of the state

$$y_k(t) = z(x_k,t), k = 0,...,M,$$

where $0 \leq x_0 < x_1 < \cdots < x_M \leq 1$.
Moreover,

$$x_{k+1} - x_k \leq \Delta_y, k = 0,...,M-1.$$

The aim is to introduce a constrained controller that regionally stabilizes the system which can be implemented by zero-order hold devices.

3. OBSERVER AND FEEDBACK FOR REGIONAL STABILIZATION OF SYSTEM

The following observer is constructed for system (1) under the point measurements:

$$\begin{cases} \hat{z}_t(x,t) = \gamma\hat{z}_{xx}(x,t) - \hat{z}(x,t)\hat{z}_x(x,t) + \lambda\hat{z}(x,t) \\ + \sum_{j=0}^{N-1} \delta(x-\bar{x}_j)u_j(t) - L\sum_{k=0}^{M-1}\mathcal{X}_k(x)[\hat{z}(x_k,t) - z(x_k,t)], \\ \hat{z}(0,t) = \hat{z}(1,t) = 0, \\ \hat{z}(x,0) = 0, \end{cases} \tag{2}$$

where $L > 0$ will be determined later.

As in Fridman et al. (2018), the characteristic functions are given by

$$\begin{cases} \mathcal{X}_k(x) = 0, \ x \notin \Gamma_k \triangleq [x_k, x_{k+1}) \\ \mathcal{X}_k(x) = 1, \ \textit{otherwise}, \end{cases} \quad k = 0,\cdots,M-1. \tag{3}$$

Let $e(x,t) = \hat{z}(x,t) - z(x,t)$. Then the estimation error satisfies

$$\begin{cases} e_t(x,t) = \gamma e_{xx}(x,t) - e(x,t)\hat{z}_x(x,t) - \hat{z}(x,t)e_x(x,t) \\ +e(x,t)e_x(x,t) + \lambda e(x,t) - L\sum_{k=0}^{M-1}\mathcal{X}_k(x)e(x_k,t), \\ e(0,t) = e(1,t) = 0, \\ e(x,0) = -z_0(x). \end{cases} \tag{4}$$

The following observer-based feedback controller is proposed for system (1):

$$u_j(t) = -K\int_{\Omega_{u_j}} \hat{z}(\xi,t)d\xi, \tag{5}$$

where $K > 0$ will be chosen later.
The closed-loop system (2), (4) corresponding to controller (5) becomes

$$\begin{cases} \hat{z}_t(x,t) = \gamma\hat{z}_{xx}(x,t) - \hat{z}(x,t)\hat{z}_x(x,t) + \lambda\hat{z}(x,t) \\ -L\sum_{k=0}^{M-1}\mathcal{X}_k(x)e(x_k,t) - \sum_{j=0}^{N-1}\delta(x-\bar{x}_j)K\int_{\Omega_{u_j}}\hat{z}(\xi,t)d\xi \\ \hat{z}(0,t) = \hat{z}(1,t) = 0, \\ e_t(x,t) = \gamma e_{xx}(x,t) - e(x,t)\hat{z}_x(x,t) - \hat{z}(x,t)e_x(x,t) \\ +e(x,t)e_x(x,t) + \lambda e(x,t) - L\sum_{k=0}^{M-1} b_k(x)e(x_k,t), \\ e(0,t) = e(1,t) = 0, \\ \hat{z}(x,0) = 0, \ e(x,0) = -z_0(x). \end{cases} \tag{6}$$

Now we study the well-posedness of (6). We investigate the coupled system (6) in the energy state space

$$\mathcal{H} = L^2(0,1) \times L^2(0,1)$$

with the norm $\|(f,g)\|^2_{\mathcal{H}} = \|f\|^2_{L^2(0,1)} + \|g\|^2_{L^2(0,1)}$.
Let

$$\mathcal{H}_1 = H^1_0(0,1) \times H^1_0(0,1)$$

be the Hilbert space with the norm:

$$\|(f,g)\|^2_{\mathcal{H}_1} = \|f'\|^2_{L^2(0,1)} + \|g'\|^2_{L^2(0,1)}.$$

Following Pisano et al. (2017) (see Definition 1 in Pisano et al. (2017)), we give the following definition of the solution to system (6):

Definition 1. *For any $T > 0$, a function $(\hat{z}(\cdot,t), e(\cdot,t)) \in C([0,T];\mathcal{H}) \cap L^2([0,T];\mathcal{H}_1)$, is said to be a weak solution of the boundary value problem (6) if for every $(\phi(\xi), \varphi(\xi)) \in \mathcal{H}_1$, the functions $\int_0^1 \hat{z}(\xi,t)\phi(\xi)d\xi$ and $\int_0^1 e(\xi,t)\varphi(\xi)d\xi$ are absolutely continuous on $[0,T]$ and relation*

$$\begin{cases} \frac{d}{dt}\int_0^1 \hat{z}(\xi,t)\phi(\xi)d\xi + \gamma\int_0^1 \hat{z}_\xi(\xi,t)\phi_\xi(\xi)d\xi \\ = -\int_0^1 \hat{z}(\xi,t)\hat{z}_\xi(\xi,t)\phi(\xi)d\xi + \lambda\int_0^1 \hat{z}(\xi,t)\phi(\xi)d\xi \\ -L\sum_{k=0}^{M-1} e(x_k,t)\int_{\Gamma_k}\phi(\xi)d\xi \\ -K\sum_{j=0}^{N-1}\phi(\bar{x}_j)\int_{\Omega_{u_j}}\hat{z}(\xi,t)d\xi, \\ \frac{d}{dt}\int_0^1 e(\xi,t)\varphi(\xi)d\xi + \gamma\int_0^1 e_\xi(\xi,t)\varphi_\xi(\xi)d\xi \\ = -\int_0^1 [e(\xi,t)\hat{z}_\xi(\xi,t) + \hat{z}(\xi,t)e_\xi(\xi,t)]\varphi(\xi)d\xi \\ +\int_0^1 e(\xi,t)e_\xi(\xi,t)\varphi(\xi)d\xi + \lambda\int_0^1 e(\xi,t)\varphi(\xi)d\xi \\ -L\sum_{k=0}^{M-1} e(x_k,t)\int_{\Gamma_k}\varphi(\xi)d\xi \end{cases} \tag{7}$$

holds for almost all $t \in [0,T]$.

The weak solution concept (7) is based on the integration-by-parts property

$$\int_0^1 \hat{z}_{\xi\xi}(\xi,t)\phi(\xi)d\xi = -\int_0^1 \hat{z}_\xi(\xi,t)\phi_\xi(\xi)d\xi,$$
$$\int_0^1 e_{\xi\xi}(\xi,t)\varphi(\xi)d\xi = -\int_0^1 e_\xi(\xi,t)\varphi_\xi(\xi)d\xi$$

of the Sobolev derivatives of $H_0^1(0,1)$- valued functions for any test function $(\phi(\xi),\varphi(\xi)) \in \mathcal{H}_1$. The well-posedness result can be obtained by Galerkin approximation method. More details can be found in Kang et al. (2020).

Theorem 1. *Consider the system (1) under the observer-based controller (5) , where $\hat{z}$ is governed by (2). Given positive scalars Δ_u, Δ_y, δ and tuning parameters $0 < \beta < 1$, $R > 0$, let there exist positive scalars K, L, $v_i (i = 1, 2)$ and nonnegative scalars $v_i (i = 3, 4)$ such that*

$$-\gamma + v_1 + v_3 + \frac{R}{2} \le 0, \tag{8}$$

$$-\gamma + v_2 + v_4 + \frac{R}{2} \le 0, \tag{9}$$

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & -\frac{K}{2} & -\frac{\beta L}{2} \\ * & -v_1\frac{\pi^2}{4\Delta_u^2} & 0 \\ * & * & \lambda_2 \end{bmatrix} \le 0, \tag{10}$$

$$\Lambda_2 = \begin{bmatrix} \lambda_3 & -\frac{L}{2} & -\frac{(1-\beta)L}{2} \\ * & -v_2\frac{\pi^2}{4\Delta_y^2} & -\frac{L}{2} \\ * & * & \lambda_4 \end{bmatrix} \le 0, \tag{11}$$

where

$$\begin{aligned} \lambda_1 &= \beta[-K - v_3\pi^2 + \lambda + \delta], \\ \lambda_2 &= \beta[-L - v_4\pi^2 + \lambda + \delta], \\ \lambda_3 &= (1-\beta)[-K - v_3\pi^2 + \lambda + \delta], \\ \lambda_4 &= (1-\beta)[-L - v_4\pi^2 + \lambda + \delta]. \end{aligned} \tag{12}$$

Then for any initial function $z_0 \in H^2(0,1) \cap H_0^1(0,1)$ satisfying $\|z_0\|_{L^2(0,1)} < R$, a unique solution of the closed-loop system exists and $z \in C([0,T]; H_0^1(0,1))$, $z_t \in L^\infty([0,T]; L^2(0,1)) \cap L^2([0,T]; H_0^1(0,1))$ for all $T > 0$. The solution of the closed-loop system satisfies

$$\|z(\cdot,t)\|_{L^2(0,1)} \le \sqrt{2}e^{-\delta t}\|z_0\|_{L^2(0,1)} \tag{13}$$

for all $t \ge 0$. Furthermore, if the strict LMIs (10), (11) are feasible for $\delta = 0$ and (8), (9) are satisfied, then the closed-loop system is exponentially stable with a small enough decay rate.

Proof. Consider the Lyapunov functional as follows:

$$V_0 = \frac{1}{2}\int_0^1 [\hat{z}^2(x,t) + e^2(x,t)]dx. \tag{14}$$

Denote $h_k(x,t) \triangleq \int_x^{x_k} e_\xi(\xi,t)d\xi$, $\omega_j(x,t) \triangleq \int_x^{\bar{x}_j} \hat{z}_\xi(\xi,t)d\xi$. For $t \ge 0$,

$$\begin{aligned} \dot{V}_0(t) &= -\gamma\int_0^1 [\hat{z}_x^2(x,t) + e_x^2(x,t)]dx \\ &+\lambda\int_0^1 [\hat{z}^2(x,t) + e^2(x,t)]dx \\ &-L\int_0^1 \hat{z}(x,t)e(x,t)dx - L\int_0^1 e^2(x,t)dx \\ &-L\sum_{k=0}^{M-1}\int_{\Gamma_k}[\hat{z}(x,t) + e(x,t)]h_k(x,t)dx \\ &-K\int_0^1 \hat{z}^2(x,t)dx - K\sum_{j=0}^{N-1}\int_{\Omega_{u_j}}\hat{z}(x,t)\omega_j(x,t)dx \\ &+\int_0^1 e(x,t)[-e(x,t)\hat{z}_x(x,t) - \hat{z}(x,t)e_x(x,t)]dx. \end{aligned} \tag{15}$$

Integrating by parts, using Young's, Agmon's and Cauchy-Schwartz's inequalities we obtain

$$\begin{aligned} &\int_0^1 e(x,t)[-e(x,t)\hat{z}_x(x,t) - \hat{z}(x,t)e_x(x,t)]dx \\ &\le \|e(\cdot,t)\|_{L^2(0,1)}\left[\frac{1}{2}\|e_x(\cdot,t)\|^2_{L^2(0,1)} + \frac{1}{2}\|\hat{z}_x(\cdot,t)\|^2_{L^2(0,1)}\right]. \end{aligned} \tag{16}$$

Wirtinger's inequality (Lemma 1) leads to

$$0 \le v_1\sum_{j=0}^{N-1}\left[\|\hat{z}_x(\cdot,t)\|^2_{L^2(\Omega_{u_j})} - \frac{\pi^2}{4\Delta_u^2}\|\omega_j^2(\cdot,t)\|^2_{L^2(\Omega_{u_j})}\right], \tag{17}$$

$$0 \le v_2\sum_{k=0}^{M-1}\left[\|e_x(\cdot,t)\|^2_{L^2(\Gamma_k)} - \frac{\pi^2}{4\Delta_y^2}\|h_k(\cdot,t)\|_{L^2(\Gamma_k)}\right], \tag{18}$$

$$0 \le v_3\left[\|\hat{z}_x(\cdot,t)\|^2_{L^2(0,1)} - \pi^2\|\hat{z}(\cdot,t)\|^2_{L^2(0,1)}\right], \tag{19}$$

$$0 \le v_4\left[\|e_x(\cdot,t)\|^2_{L^2(0,1)} - \pi^2\|e(\cdot,t)\|^2_{L^2(0,1)}\right], \tag{20}$$

where $v_i (i = 1, 2) > 0$ and $v_i (i = 3, 4) \ge 0$.
Substituting (16) into the right-hand side of (15), and adding (17)-(20) to (15), for any $\beta \in (0, 1)$

$$\begin{aligned}\dot{V}_0 + 2\delta V_0 \le& \sum_{j=0}^{N-1} \int_{\Omega_{u_j}} \sigma_1^\top \Lambda_1 \sigma_1 dx + \sum_{k=0}^{M-1} \int_{\Gamma_k} \sigma_2^\top \Lambda_2 \sigma_2 dx \\ &- \left(\gamma - v_1 - v_3 - \frac{1}{2}\|e\|_{L^2(0,1)}\right) \|\hat{z}_x\|^2_{L^2(0,1)} \\ &- \left(\gamma - v_2 - v_4 - \frac{1}{2}\|e\|_{L^2(0,1)}\right) \|e_x\|^2_{L^2(0,1)},\end{aligned} \tag{21}$$

where $\sigma_1 = \mathrm{col}\{\hat{z}, \omega_j, e\}$, $\sigma_2 = \mathrm{col}\{\hat{z}, h_k, e\}$, Λ_1 and Λ_2 are given by (10), (11) respectively.

We next prove that (13) is satisfied. Similar to Selivanov et al. (2017), we first assume that

$$\|e(\cdot,t)\|_{L^2(0,1)} < R, \ \forall t \in [0,\infty). \tag{22}$$

Then from (21), one gets

$$\dot{V}_0 + 2\delta V_0 \le 0 \tag{23}$$

if $\Lambda_1 \le 0$, $\Lambda_2 \le 0$, and (8), (9) hold.
implies

$$\|(\hat{z}(\cdot,t), e(\cdot,t))\|^2_{\mathcal{H}} \le e^{-2\delta t}\|(\hat{z}(\cdot,0), e(\cdot,0))\|^2_{\mathcal{H}}, \tag{24}$$

that together with Minkowski's inequality leads to

$$\|z(\cdot,t)\|_{L^2(0,1)} \le \sqrt{2}e^{-\delta t}\|z_0\|_{L^2(0,1)}.$$

Now we prove (22). Due to $V_0(t) = \frac{1}{2}\|(\hat{z}(\cdot,t), e(\cdot,t))\|^2_{\mathcal{H}}$, it is sufficient to show that

$$V_0(t) < \frac{R^2}{2}, \ \ \forall t \in [0,\infty). \tag{25}$$

Indeed, for $t = 0$, the inequality (25) holds. Let (25) be false for some $t_1 \in (0,\infty)$. Then $V_0(t_1) \ge \frac{R^2}{2} > V_0(0)$. Since V_0 is continuous in time, there must exist $t^* \in (0, t_1]$ such that

$$V_0(t) < \frac{R^2}{2} \ \ \forall t \in [0, t^*) \ \text{ and } \ V_0(t^*) = \frac{R^2}{2}. \tag{26}$$

The first relation of (26), together with the feasibility of $\Lambda_1 \le 0$, $\Lambda_2 \le 0$ and (8), (9), guarantees that $\dot{V}_0 + 2\delta V_0 \le 0$ on $[0, t^*)$. Therefore, $V_0(t^*) \le V_0(0) < \frac{R^2}{2}$. This contradicts the second relation of (26). Thus, (25) and consequently, (22), (23) are true, which implies (13) provided that $\|z_0\|_{L^2(0,1)} < R$.

Note that the feasibility of LMIs (10), (11) with $\delta = 0$ implies its feasibility with a small enough $\delta > 0$. Therefore, if LMIs (10), (11) hold for $\delta = 0$ and (8), (9) are satisfied, then the closed-loop system is exponentially stable with a small decay rate.

□

4. CONSTRAINED CONTROL

System (1) is considered with the point control law subject to the amplitude constraint:

$$|u_j(t)| \le \bar{u}, \ (j = 0, \cdots, N-1). \tag{27}$$

The following observer-based feedback controller is introduced:

$$u_j^{sat}(t) = \mathrm{sat}(u_j(t), \bar{u}), \ j = 0, \cdots, N-1, \tag{28}$$

where the saturation function is given by

$$\mathrm{sat}(u_j, \bar{u}) = \mathrm{sign}(u_j)\min(|u_j|, \bar{u}),$$

and $u_j(t)$ is given by (5).
Denoting the state trajectory of closed-loop system (2), (4) subject to (28) with the initial condition $(0, -z_0) \in \mathcal{H}$ by $(\hat{z}(x,t;0), e(x,t;-z_0))$, the domain of attraction of the closed-loop system is then the set

$$\mathcal{S} = \{(0, -z_0) \in \mathcal{H} : \lim_{t\to\infty} \|(\hat{z}(x,t;0), e(x,t;-z_0))\|_{\mathcal{H}} = 0\}.$$

An estimate $\mathcal{X}_R \subset \mathcal{S}$ will be obtained on the domain of attraction, where

$$\mathcal{X}_R = \{(0, -z_0) \in \mathcal{H} : \|(0, -z_0)\|_{\mathcal{H}} < R\},$$

and R is a scalar that will be maximized in the sequel.

Theorem 2. *Consider the system (1) under the observer-based constrained controller (28) governed by (2), (4). Given positive scalars K, L, Δ_u, Δ_y, δ, $\bar{u}$ and tuning parameters $0 < \beta < 1$, $R > 0$, let there exist scalars $v_i (i = 1,2) > 0$ and $v_i (i = 3,4) \ge 0$ such that (8)-(11) and*

$$\bar{u} \ge K(\Delta_u)^{\frac{1}{2}} R \tag{29}$$

hold. Then for any initial condition z_0 from the set

$$\mathcal{X}_R = \{z_0 \in L^2(0,1) : \|z_0\|_{L^2(0,1)} < R\}, \tag{30}$$

a unique solution of the closed-loop system exists. Moreover, the closed-loop system initialized with $z_0 \in \mathcal{X}_R$ is exponentially stable:

$$\|z(\cdot,t)\|_{L^2(0,1)} \le \sqrt{2}e^{-\delta t}\|z_0\|_{L^2(0,1)}, \ \forall t \ge 0. \tag{31}$$

Proof. From (5), the Cauchy-Schwartz inequality yields

$$|u_j(t)| = K\left|\int_{\Omega_{u_j}} \hat{z}(\xi,t)d\xi\right| \le K(\Delta_u)^{\frac{1}{2}}\|(\hat{z}, e)\|_{\mathcal{H}}. \tag{32}$$

Given $\bar{u} > 0$, we define the following set:

$$\mathcal{L}(K, \bar{u}) = \{(\hat{z}, e) \in \mathcal{H} : K(\Delta_u)^{\frac{1}{2}}\|(\hat{z}, e)\|_{\mathcal{H}} \le \bar{u}\}. \tag{33}$$

Then from (32) and the definition above, it follows that $(\hat{z}, e) \in \mathcal{L}(K, \bar{u}) \Longrightarrow |u_j(t)| \le \bar{u}, (j = 0, \cdots, N-1)$, and the saturation is avoided. Thus, the closed-loop system (2), (4) subject to (28) admits the linear representation (6).

From Theorem 1, it follows that if there exists $\delta > 0$ such that the strict LMIs (10), (11) are feasible and (8), (9) hold, then (24) is satisfied. Hence, the trajectories $(\hat{z}(x,t;0), e(x,t;-z_0))$ starting from initial function $(0, -z_0) \in \tilde{\mathcal{X}}_R \triangleq \{(0, -z_0) \in \mathcal{H} : \|(0, -z_0)\|_{\mathcal{H}} < R\}$ (i.e. $z_0 \in \mathcal{X}_R$) remain within

$$\mathcal{X} = \{(\hat{z}, e) \in \mathcal{H} : \|(\hat{z}, e)\|_{\mathcal{H}} < R\}.$$

The "ball" $\mathcal{X}$ is contained in $\mathcal{L}(K, \bar{u})$, if the following implication holds

$$\|(\hat{z}, e)\|_{\mathcal{H}} < R \Longrightarrow K(\Delta_u)^{\frac{1}{2}}\|(\hat{z}, e)\|_{\mathcal{H}} < \bar{u}$$

for all $(\hat{z}, e) \in \mathcal{H}$, i.e. if

$$K(\Delta_u)^{\frac{1}{2}}\|(\hat{z}, e)\|_{\mathcal{H}} \le R^{-1}\bar{u}\|(\hat{z}, e)\|_{\mathcal{H}}.$$

The latter inequality is guaranteed if (29) is satisfied. Therefore, the inequality (29) guarantees the saturation avoidance, and together with Theorem 1 imply that

$$\lim_{t\to\infty} \|(\hat{z}(x,t;0), e(x,t;-z_0))\|_{\mathcal{H}} = 0.$$

Hence, (31) holds. □

Remark 1. *It should be noticed that regional stabilization of Burgers' equation was established, where the nonlinear term zz_x allowed to find an estimate of domain of attraction. However, global stabilization result can not be achieved by using this method.*

5. EXAMPLE

Consider the system (1) with parameters $\gamma = 1$, $\lambda = 10.7$ under the point measurements. The open-loop system is unstable. For the observer-based constrained control law (28) governed by (2), (4) with $K = 21$ and $\bar{u} = 10.5$, by verifying LMI conditions of Theorem 2 with $\beta = 0.5$, $\Delta_u = 0.125$, $\Delta_y = 1/6$, $L = 15$, $\delta = 0.1$. We obtain that $\max R = 1$, and find that the closed-loop system (1), (2), (4), (28) preserves the exponential stability for $\|z_0\|_{L^2(0,1)} < 1$.

Under the observer-based constrained controller (28) governed by (2), (4), the state of the closed-loop system (1) is computed via a finite difference method. Take the same values of parameters and consider the initial condition $z_0(x) = 1.4\sin(\pi x)$, $0 \leq x \leq 1$. The steps of space and time are chosen as 0.025 and 0.002, respectively. Assume that there are 4 in-domain sensors transmitting point measurements at $x_0 = 0$, $x_1 = 1/6$, $x_2 = 1/3$, $x_3 = 1/2$, $x_4 = 2/3$, $x_5 = 5/6$, and $x_6 = 1$. Here $\Delta_y = 1/6$ (see Fig. 1). Simulation of solutions under the controller $u_j(t) = -21\int_{\Omega_{u_j}} \hat{z}(\xi, t)d\xi$ with $\Omega_{u_j} = \left[\frac{j}{8}, \frac{j+1}{8}\right)$ $(j = 0, \cdots, 7)$, and $\Delta_u = 0.125$, where the spatial domain is divided into eight sub-domains (see Fig. 1), shows that the closed-loop system is exponentially stable (see Fig. 2).

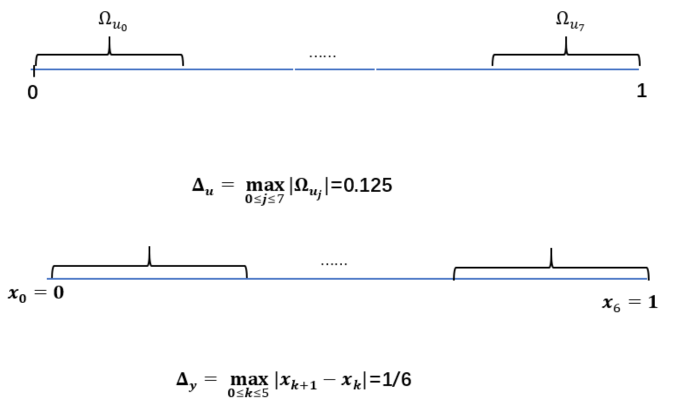
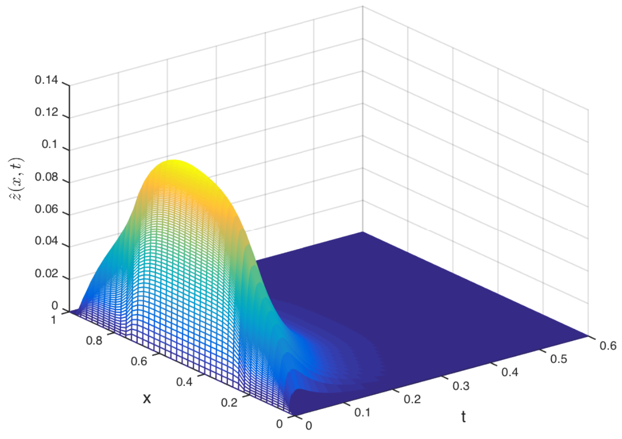
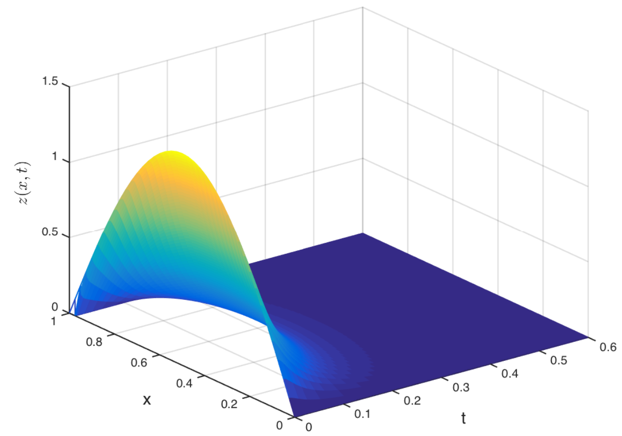


Fig. 1.



(a) Estimation $\hat{z}(x, t)$



(b) State $z(x, t)$

Fig. 2. Closed-loop system (with observer-based constrained controller) under point actuation and measurements

6. CONCLUSION

In this paper, an observer of Burgers' equation under the point measurements was constructed such that the exponential convergence of the observer is guaranteed. This allowed to achieve regional stabilization under the point in-domain constrained controller that employs the averaged values of the observer. An estimate on the domain of attraction was found by using LMIs. Our next step may be extension of the obtained results to the observer-based boundary control of coupled ODE-PDE system.

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