

Stability Analysis for Nonlinear Weakly Hard Real-Time Control Systems

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Abstract: This paper considers the stability analysis for nonlinear sampled-data systems with failures in the feedback loop. The failures are caused by shared resources, and modeled by a weakly hard real-time (WHRT) dropout description. The WHRT dropout description restricts the considered dropout sequences with a non-probabilistic, window based constraint, that originates from schedulability analysis. The proposed approach is based on the emulation of a controller for the nonlinear sampled-data system from a continuous-time feedback. The emulation technique is extended and combined with non-monotonic Lyapunov functions and a graph description for the WHRT constraints to guarantee asymptotic stability. The effectiveness of the proposed approach is illustrated with a numerical example from literature.

Keywords: Networked embedded control systems, control over networks, control under communication constraints

1. INTRODUCTION

In many modern control applications as e.g. in networked or embedded control systems, it may be unavoidable that the control system runs in open-loop from time to time due to failures in the feedback channel. Such failures can be caused e.g. by an unreliable communication system connecting sensors, controllers and actuators (Kauer et al. (2014)), or since the control input is calculated on a micro-processor that runs many different tasks in parallel, such that the on-time calculation of the control input cannot always be guaranteed (Bernat et al. (2001); Árzén et al. (2005)). Nevertheless, it is important to guarantee stability of the control system despite the unreliable feedback channel.

For random loss processes, this problem has been thoroughly studied, cf. Schenato et al. (2007) for a survey. However, drawbacks of a random dropout description are that stability can only be guaranteed in a mean-square sense and that the random dropout description does often not yield a precise characterization of the loss process, cf. Blind and Allgöwer (2015): If there are at most 10% of failures, then this might e.g. mean that there is one failure within 10 tries or that there are 100 consecutive failures after 900 consecutive successes.

An alternative approach to describe the loss process is to determine an upper bound on the successive number of failures in the feedback loop. This maximum number of dropouts approach was studied e.g. in Xiong and Lam (2007); Kauer et al. (2014) for linear systems. Moreover, results that guarantee stability, if the time between two arriving control inputs is bounded, as e.g. the emulation technique for nonlinear systems (cf. Carnevale et al.

(2007); Nesic et al. (2009)), can be used to guarantee stability despite a bounded number of successive failures in the feedback loop. This works by choosing the sampling period smaller or equal than the maximum admissible sampling period (MASP) for the respective approach divided by the maximum number of successive failures plus 1.

However, to avoid conservatism, it is advantageous to use preferably much information on the loss process for the controller design instead of only employing knowledge about the maximum number of successive failures in the feedback loop. A conceptual framework for a description of the loss process is given by weakly hard real-time (WHRT) constraints (Bernat et al. (2001)). WHRT constraints provide guarantees for time windows of fixed size and include a large class of scheduling constraints as e.g. (m, k) -firmness from Hamdaoui and Ramanathan (1995).

In van Horssen et al. (2016), the problem of controlling a linear system with an (m, k) -firmness dropout description has been investigated. For linear systems with general WHRT constraints as dropout description, a sufficient stability condition was proposed in Blind and Allgöwer (2015) and controller design methods have been presented in Linsenmayer and Allgöwer (2017); Linsenmayer et al. (2020). Conditions for observability and controllability for linear systems with a similar dropout model have been presented in Jungers et al. (2018). Whereas stability analysis and controller design methods for linear systems with failures in the feedback loop subject to WHRT constraints are thus well established, there are no comparable results for nonlinear systems available in literature.

In this paper, we propose an approach for the stability analysis of nonlinear sampled-data systems with failures in the feedback loop that are described by a weakly hard real-time dropout description. The proposed approach is based

* Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - 285825138; 390740016.

on an extension of the emulation approach from Nesic et al. (2009). A controller that is stabilizing for continuous-time feedback is emulated and an upper bound on a Lyapunov function candidate for the nonlinear sampled-data system is derived. This bound is combined with the concepts of non-monotonic Lyapunov functions (cf. Michel et al. (2015)) and WHRT graphs (cf. Linsensmayer et al. (2020)) to obtain a sufficient condition for asymptotic stability. The proposed approach can be used for a wide class of nonlinear systems and is less conservative than considering only the maximum number of dropouts. We illustrate the benefits of the proposed approach with a numerical example from literature that relates the proposed approach to the maximum number of dropouts approach.

The remainder of this paper is structured as follows. First, we specify in Section 2 the considered setup and recap some results from literature. Then we discuss in Section 3 the controller emulation technique from Nesic et al. (2009) and modify it to our setup. In Section 4, we present sufficient conditions for asymptotic stability of the WHRT control system. Section 5 contains an example to illustrate the proposed approach. A conclusion is given in Section 6. Some proofs are omitted due to spacial limitations. They can be found in the preprint Hertneck et al. (2020).

Notation The positive, respectively nonnegative, real numbers are denoted by $\mathbb{R}_{>0}$, respectively $\mathbb{R}_{\geq 0} = \mathbb{R}_{>0} \cup \{0\}$. The positive natural numbers are denoted by \mathbb{N} , and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a class \mathcal{K} function if it is strictly increasing and $\alpha(0) = 0$. It is a class \mathcal{K}_∞ function if it is of class \mathcal{K} and it is unbounded. The notation t^- is used as $t^- := \lim_{s < t, s \rightarrow t} s$. A continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \setminus \{0\}$.

2. SETUP

In this section, we specify the setup considered in this paper, introduce the notation of WHRT constraints and non-monotonic Lyapunov functions and formalize the control objective of the paper.

2.1 Control System

We consider a nonlinear, time-invariant plant

$$\dot{x} = f_p(x, u) \quad (1)$$

with a continuously differentiable vector valued function $f_p : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x}$ satisfying $f_p(0, 0) = 0$, the system state $x(t) \in \mathbb{R}^{n_x}$ with initial condition $x(0) = x_0$ and the input $u(t) \in \mathbb{R}^{n_u}$ that is applied by the actuator.

The system is sampled periodically with a fixed sampling period h to be specified later. Thus, at each discrete time instant satisfying $t = kh$ for some $k \in \mathbb{N}_0$, a new sample of the system state is sent to the controller. When the controller receives a sample of the state, then a control input \hat{u} is generated as $\hat{u}(kh) = \kappa(x(kh))$ with the nonlinear feedback law $\kappa : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$. We assume that κ asymptotically stabilizes the continuous-time system, according to the following assumption.

Assumption 1. There is a locally Lipschitz, positive definite function $V : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ satisfying for all $x \in \mathbb{R}^{n_x} \setminus \{0\}$

$$\underline{\alpha}_V(\|x\|) \leq V(x) \leq \bar{\alpha}_V(\|x\|) \quad (2)$$

for $\underline{\alpha}_V, \bar{\alpha}_V \in \mathcal{K}_\infty$ and

$$\langle \nabla V(x), f_p(x, \kappa(x)) \rangle < 0. \quad (3)$$

From time to time, transmissions of the sampled state to the controller, the calculation of the control input, or its transmission to the actuator, fail due to higher priority tasks on the microprocessor or due to imperfections of the communication system. The time instants, when a new input is received at the actuator are given by the infinite sequence $(\tau_z)_{z \in \mathbb{N}_0}$ and define a discrete set

$$\mathcal{T} := \{\tau_0, \tau_1, \tau_2, \dots\}. \quad (4)$$

For simplicity, we assume subsequently that the control law for the initial state is received successfully at the actuator at the first sampling time and have thus $\tau_0 = 0$. Moreover, we assume that there is no transmission delay, i.e., for each $z \in \mathbb{N}_0$, $\tau_z = kh$ for some $k \in \mathbb{N}_0$. With \mathcal{T} , we can furthermore specify an infinite binary sequence $\nu := (\nu_k)_{k \in \mathbb{N}_0}$ that describes, if a new input is received at sampling instants. We set $\nu_k = 1$ if $kh \in \mathcal{T}$, i.e., if a new input is received at sampling instant k , and $\nu_k = 0$ if $kh \notin \mathcal{T}$, i.e., if no new input arrives. Between times from \mathcal{T} , the actuator holds the last received feedback, i.e.,

$$u(t) = \hat{u}(\tau_z) = \kappa(x(\tau_z)), \tau_z \leq t < \tau_{z+1} \quad (5)$$

for all $z \in \mathbb{N}_0$. We define the sampling error as

$$e(t) = x(\tau_z) - x(t), \tau_z \leq t < \tau_{z+1} \quad (6)$$

for all $z \in \mathbb{N}_0$. We can thus model the closed-loop system composed of (1) with input defined by (5) and a dropout sequence ν as the discontinuous dynamical system (DDS)

$$\left. \begin{aligned} \dot{x}(t) &= f(x(t), e(t)) := f_p(x, \kappa(x + e)) \\ \dot{e}(t) &= g(x(t), e(t)) := -f_p(x, \kappa(x + e)) \end{aligned} \right\}, \quad \tau_z \leq t < \tau_{z+1},$$

$$\left. \begin{aligned} x(t) &= x(t^-) \\ e(t) &= 0 \end{aligned} \right\}, \quad t = \tau_{z+1}, \quad \forall z \in \mathbb{N}_0 \quad (7)$$

with $x(0) = x_0$ and $e(0) = 0$. A trajectory that satisfies (7) for all $t \geq 0$ exists, if a unique solution to

$$\begin{aligned} \dot{\tilde{x}}(t) &= f(\tilde{x}(t), \tilde{e}(t)) \\ \dot{\tilde{e}}(t) &= g(\tilde{x}(t), \tilde{e}(t)) \end{aligned} \quad (8)$$

exists for $\tau_z \leq t \leq \tau_{z+1}$ for all $z \in \mathbb{N}_0$ and arbitrary $\tilde{x}(\tau_z)$ for $\tilde{e}(\tau_z) = 0$. We assume from now on that such solutions exist and justify later why this assumption is reasonable.

2.2 Weakly Hard Real-Time Constraints

As in the recent works Blind and Allgöwer (2015); Linsensmayer and Allgöwer (2017); Linsensmayer et al. (2020), we use WHRT constraints to model the loss process, i.e., to specify the sequence ν from the previous subsection. Thus, we shall also use the following definitions that are taken from Bernat et al. (2001) and Blind and Allgöwer (2015).

Definition 2. (cf. (Blind and Allgöwer, 2015, Def. 1)) A constraint η is a function that maps an infinite binary sequence ν to the Boolean values **true** and **false**, i.e., $\eta : \{0, 1\}^\infty \rightarrow \{\mathbf{true}, \mathbf{false}\}$. We say that a sequence ν satisfies a constraint η , denoted by $\nu \vdash \eta$, when $\eta(\nu) = \mathbf{true}$.

Definition 3. (cf. (Bernat et al., 2001, Def. 3)) A transmission sequence ν “meets any n in m deadlines” ($m \geq 1, 1 \leq n \leq m$) and it is denoted by $\nu \vdash \binom{n}{m}$ if, in any window of m consecutive transmissions, there are at least n transmissions in any order, that are successful.

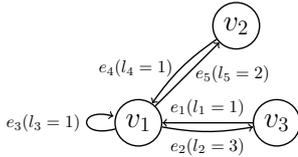


Fig. 1. WHRT graph for $\langle 2 \rangle_5$.

Definition 4. (cf. (Bernat et al., 2001, Def. 4)) A transmission sequence ν “meets row n in m deadlines” ($m \geq 1, 1 \leq n \leq m$) and it is denoted by $\nu \vdash \langle n \rangle_m$ if, in any window of m consecutive transmissions, there are at least n consecutive transmissions, that are successful.

Definition 5. (cf. (Bernat et al., 2001, Def. 6)) A transmission sequence ν “misses less than row n in m deadlines” ($m \geq 1, 1 \leq n \leq m$) and it is denoted by $\nu \vdash \overline{\langle n \rangle}_m$ if, in any window of m consecutive transmissions, it is never the case that n consecutive transmissions are unsuccessful.

Definition 6. (cf. (Bernat et al., 2001, Def. 10)): Given two constraints, η and η' , we say that η' is harder than η (η is easier than η'), denoted by $\eta' \preceq \eta$ if all sequences that satisfy η' also satisfy η .

Definition 7. The maximum number of consecutive lost transmissions, that may be contained in a sequence ν , such that ν can still satisfy η , is denoted by $w(\eta)$.

Henceforth, we use the graph representation for WHRT constraints, that has been proposed in Linselmayer and Allgöwer (2017) and extended in Linselmayer et al. (2020), to capture all possible transmission sequences that satisfy a constraint η . A labeled directed graph is described by a tuple $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \dots, v_{n_{\mathcal{V}}}\}$ is the set of nodes and $\mathcal{E} = \{e_1, \dots, e_{n_{\mathcal{E}}}\}$ is the set of edges. If there exists an edge from node v_{i_p} to node v_{j_p} with weight l_p , then $e_p = (i_p, j_p, l_p)$ is contained in \mathcal{E} . A walk P for a graph \mathcal{G} is a sequence of edges of \mathcal{G} and can be described by a (possibly infinite) index sequence $\mathbb{I}(P) = (\mathbb{I}_1(P), \mathbb{I}_2(P), \dots)$ of edge indices, where $\mathbb{I}_k(P) \in \mathbb{N}$ for all $k \in \mathbb{N}$. Thus, the sequence of edges of P is described by $(e_{\mathbb{I}_1(P)}, e_{\mathbb{I}_2(P)}, \dots)$. If the number of elements in P is finite, then we denote this number by n_P . Note, that an edge may be contained multiple times in a walk. The cost of a walk is the sum of the edge weights of the walk.

In order to describe a constraint η by a graph \mathcal{G}_η , we use the definition of WHRT graphs from Linselmayer et al. (2020), that can be given for our setup as follows.

Definition 8. (cf. Linselmayer et al. (2020), Def. 9) A labeled directed graph \mathcal{G}_η is a WHRT graph for a given weakly hard real-time constraint η , if for all sequences $\nu \vdash \eta$, there exists an infinite walk P for \mathcal{G}_η , such that

$$\nu = (1 \quad \underbrace{0 \dots 0}_{l_{\mathbb{I}_1(P)}-1 \text{ times}} \quad 1 \quad \underbrace{0 \dots 0}_{l_{\mathbb{I}_2(P)}-1 \text{ times}} \quad 1 \dots). \quad (9)$$

Thus, the edge labels of a WHRT graph \mathcal{G}_η do represent the number of sampling periods between two successfully received inputs, and we can generate all dropout sequences that satisfy a constraint η by concatenating edges of \mathcal{G}_η in a similar fashion as in (9). For a constraint η of type $\langle n \rangle_m$, a WHRT graph \mathcal{G}_η , can be constructed automatically¹

¹ Matlab code can be found at <https://www.ist.uni-stuttgart.de/institute/team/pdf/SL/WhrtGraphMatlab.zip>.

using Algorithm 1 from Linselmayer et al. (2020). For the constraint types $\langle n \rangle_m$ and $\overline{\langle n \rangle}_m$, construction algorithms can easily be derived based on the same main ideas as in Algorithm 1 from Linselmayer et al. (2020). The WHRT graph for $\langle 2 \rangle_5$ is given as example in Figure 1. For this graph, an exemplary walk P_e can be described by the index sequence $\mathbb{I}(P_e) = (2, 1, 3, 5, 4, 2)$ that generates the sequence $\nu = (1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1)$. We will later use furthermore the following definition.

Definition 9. We denote by $\mathcal{S}(\mathcal{G}, c_{\text{walk}})$ the set of all walks of a graph \mathcal{G} with cost larger or equal than $c_{\text{walk}} \in \mathbb{N}$, that are such that removing the last edge from the walk makes the cost of the walk smaller than c_{walk} .

Each walk from $\mathcal{S}(\mathcal{G}_\eta, c_{\text{walk}})$ can be used to create a (finite) binary transmission sequence. Note, that we can generate each infinite sequence $\nu \vdash \eta$ that starts with a successful transmission by appending suitable walks from $\mathcal{S}(\mathcal{G}_\eta, c_{\text{walk}})$ to an infinite walk and concatenating the edge labels of that walk according to (9).

2.3 Non-Monotonic Lyapunov Functions

Next, we present a sufficient stability condition for the DDS model (7), that can be derived similarly as Theorem 6.4.6 from Michel et al. (2015). This condition is based on a non-monotonic Lyapunov function that may increase for some times, as long as one can still guarantee an average decrease, and will thus turn out to be useful in our setup with a WHRT dropout description for the feedback loop. It can be formulated as follows.

Proposition 10. Observe the DDS given by (7). Let $\xi = [x^\top, e^\top]^\top$. Assume that there is an unbounded discrete subset $\tilde{\mathcal{T}} \subseteq \mathcal{T}$, described by an infinite sequence $(\tilde{\tau}_z)_{z \in \mathbb{N}_0}$ that satisfies

$$0 < \underline{h} \leq \tilde{\tau}_{z+1} - \tilde{\tau}_z \leq \bar{h}, \quad \forall z \in \mathbb{N}_0 \quad (10)$$

and $\tilde{\tau}_0 = 0$. Moreover, assume there is a continuous positive definite function $V_n : \mathbb{R}^{2n_x} \rightarrow \mathbb{R}$, satisfying

$$\alpha_1(\|\xi\|) \leq V_n(\xi) \leq \alpha_2(\|\xi\|) \quad (11)$$

such that for all $z \in \mathbb{N}_0$,

$$V_n(\xi(\tilde{\tau}_z + r)) \leq \alpha_3(V_n(\xi(\tilde{\tau}_z))), \quad 0 \leq r < \tilde{\tau}_{z+1} - \tilde{\tau}_z \quad (12)$$

and

$$\frac{1}{\tilde{\tau}_{z+1} - \tilde{\tau}_z} [V_n(\xi(\tilde{\tau}_{z+1})) - V_n(\xi(\tilde{\tau}_z))] \leq -\alpha_4(V_n(\xi(\tilde{\tau}_z))) \quad (13)$$

hold with class \mathcal{K}_∞ functions α_1, α_2 and class \mathcal{K} functions α_3, α_4 . Then the equilibrium $\xi = 0$ is globally asymptotically stable for (7).

The proof of Proposition 10 can be found in the preprint Hertneck et al. (2020). The main difference of Proposition 10 in comparison to Theorem 6.4.6 from Michel et al. (2015) is that additional jumps of the DDS (7) at times from $\mathcal{T} \setminus \tilde{\mathcal{T}}$ may occur.

2.4 Problem Statement

For a given WHRT constraint η and a given sampling period h , the goal of this paper is to derive a sufficient stability condition for control systems with unreliable

feedback loops, described by the DDS (7) (or respectively (1) and (5)) with a dropout sequence $\nu \vdash \eta$, that exploits the knowledge about ν that is described by η .

3. EMULATION OF THE CONTROLLER

In this section, we adapt the emulation technique from Nescic et al. (2009) that can be used to obtain stability guarantees for the sampled-data system (7) with emulated continuous-time controller based on a monotonic Lyapunov function, to the setup with non-monotonic Lyapunov functions considered in this paper. More precisely, we modify the explicit bound on the MASP from Nescic et al. (2009) such that we are able to guarantee a decrease of $V(x)$, or at least to bound the amount of increase of $V(x)$ by a known reference, depending on the time between two arriving inputs. First, we state a basic assumption.

Assumption 11. There exist a locally Lipschitz function $W : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$, $L, \gamma \in \mathbb{R}_{>0}$, $\epsilon \in \mathbb{R}$ and $\underline{\alpha}_W, \bar{\alpha}_W \in \mathcal{K}_\infty$, such that for all $e \in \mathbb{R}^{n_x}$ and $x \in \mathbb{R}^{n_x}$,

$$\underline{\alpha}_W(\|e\|) \leq W(e) \leq \bar{\alpha}_W(\|e\|) \quad (14)$$

and

$$\left\langle \frac{\partial W^2(e)}{\partial e}, g(x, e) \right\rangle \leq 2W(e)(LW(e) + H(x)) \quad (15)$$

hold. Moreover, for $V(x)$ from Assumption 1,

$$\langle \nabla V(x), f(x, e) \rangle \leq -\epsilon V(x) - H^2(x) + \gamma^2 W^2(e) \quad (16)$$

holds.

This assumption is comparable to Assumption 1 in Nescic et al. (2009). There are two noteworthy differences.

We use $-\epsilon V(x)$ instead of $-\rho(x)$ in (16) and thus require exponential stabilizability of the continuous time system for $\epsilon > 0$, which is also often done in literature, see e.g. Assumption 3 in Postoyan et al. (2014) and Assumption 8.1 in Heijmans (2019). A possible relaxation in comparison to those assumptions from literature is that ϵ in Assumption 11 not necessarily needs to be positive, which will play an important role in our main result. Note that Assumption 11 can hold for a DDS (7) simultaneously with different combinations of ϵ , γ and L . In general, a smaller value for ϵ will allow us to choose γ smaller and vice versa. We will require in our main result at least one parameter set with $\epsilon > 0$, but also exploit different parameter sets for arbitrary ϵ when possible.

The second difference is that we have replaced condition (9) from Nescic et al. (2009) by (15). In fact, if $W^2(e)$ is differentiable, which is, e.g., the case for $W(e) = \|e\|$, then (15) is essentially the same as condition (9) of Assumption 1 in Nescic et al. (2009).

The technique from Nescic et al. (2009) to determine a bound on the the MASP for systems with emulated controller, that satisfy Assumption 11 for $\epsilon > 0$, γ and L , is to choose the maximum time between two arriving inputs smaller than the bound T_{\max} , where

$$T_{\max}(\gamma, \Lambda) := \begin{cases} \frac{1}{\Lambda r} \arctan(r) & \gamma > \Lambda \\ \frac{1}{\Lambda} & \gamma = \Lambda \\ \frac{1}{\Lambda r} \operatorname{arctanh}(r) & \gamma < \Lambda \end{cases} \quad (17)$$

with

$$r := \sqrt{\left| \left(\frac{\gamma}{\Lambda} \right)^2 - 1 \right|} \quad (18)$$

for $\Lambda = L$. Then we can explicitly state a Lyapunov function that is monotonically decreasing. Note that the value of $T_{\max}(\gamma, \Lambda)$ increases as γ or Λ decrease.

Our goal is to combine the emulation approach with non-monotonic Lyapunov functions. Doing so, we can even tolerate an increase of the non-monotonic Lyapunov function, but need to upper-bound the amount of its increase or decrease at any time. To find such a bound, we study now the effect of arbitrary ϵ and Λ on the time evolution of $V(x(t))$ and $W(e(t))$, if the time between two arriving inputs is bounded by $T_{\max}(\gamma, \Lambda)$ for γ, ϵ and L that satisfy Assumption 11. For this setup, we can state the following proposition.

Proposition 12. Let Assumption 11 hold for some γ, ϵ, L . Let $0 < \tau_{z+1} - \tau_z < T_{\max}(\gamma, \Lambda)$ for some $z \in \mathbb{N}_0$ and $\Lambda > 0$. Then, a unique solution to (8) exists for $\tau_z \leq t \leq \tau_{z+1}$. Furthermore, for (7), it holds that

$$\begin{aligned} & V(x(\tau_{z+1})) \\ & \leq \exp(\max\{-\epsilon, 2(L - \Lambda)\}(\tau_{z+1} - \tau_z)) V(x(\tau_z)). \end{aligned} \quad (19)$$

Moreover,

$$V(x(t)) \leq k_1 V(x(\tau_z)) \quad (20)$$

and

$$\|e(t)\| \leq \alpha_{w2}(V(x(\tau_z))) \quad (21)$$

hold for $\tau_z \leq t < \tau_{z+1}$, $k_1 \in \mathbb{R}_{>0}$ and $\alpha_{w2} \in \mathcal{K}$.

The proof for Proposition 12 can be found in the preprint Hertneck et al. (2020). Proposition 12 delivers a bound on the increase or decrease of $V(x)$ between two arrival times of inputs that depends on $\max\{-\epsilon, 2(L - \Lambda)\}$, if the time span between the two arrival times is upper bounded by $T_{\max}(\gamma, \Lambda)$. For $\Lambda > L$ and $\epsilon > 0$, we can guarantee a certain amount of decrease for $V(x)$ between both arrival times at the cost of a smaller admissible time span between the arrival times. On the other hand, if $\Lambda < L$ or $\epsilon < 0$, which allows a larger time span between two arriving inputs, then $V(x)$ may increase between the arrival times, but the amount of increase remains bounded. We will combine this with WHRT graphs and non-monotonic Lyapunov functions to derive a sufficient condition for asymptotic stability of the origin of the DDS (7) in the next section.

Remark 13. Proposition 12 justifies also our assumption, that a unique solution to (8) exists for $\tau_z \leq t \leq \tau_{z+1}$ for all $z \in \mathbb{N}_0$, arbitrary $x(\tau_z)$ and $e(\tau_z) = 0$, if $\tau_{z+1} - \tau_z$ is small enough, which will be ensured by the choice of h in the next section.

4. STABILITY RESULTS

Now, we present sufficient stability conditions for the DDS (7) despite failures in the feedback loop.

4.1 Maximum Number of Dropouts Approach

A straightforward approach to guarantee stability despite the loss process, that does only take into account the worst case number of sampling periods between two arriving inputs, i.e., $w(\eta) + 1$, is the maximum number of dropouts

approach. Here, the sampling period h has to satisfy $h < \frac{T_{\max}(\gamma, L)}{w(\eta)+1}$, with γ and L chosen such, that Assumption 11 holds for some $\epsilon > 0$. Then, asymptotic stability can be guaranteed with the standard emulation technique. For a DDS (7), in general, there are possible multiple choices for $\epsilon > 0, \gamma$ and L , such that Assumption 11 holds. Thus, for the maximum number of dropouts approach, it is beneficial to seek for the combination that maximizes $T_{\max}(\gamma, L)$ and satisfies Assumption 11 with $\epsilon > 0$.

4.2 A Stability Condition Based on WHRT Constraints

It can be easily verified that $\langle \binom{n}{m} \rangle \preceq \overline{\langle \binom{m-n+1}{*} \rangle}$ and $\binom{n}{m} \preceq \overline{\langle \binom{m-n+1}{*} \rangle}$ for $*$ larger than $m - n + 1$. Observe, that the maximum number of dropouts approach is in principle based on constraints of the type $\langle \binom{w(\eta)+1}{*} \rangle$, where $*$ represents an arbitrary window size larger than $w(\eta)$, instead of the actual constraint type. In other words, if the dropout sequence satisfies constraints of the types $\langle \binom{n}{m} \rangle$ or $\binom{n}{m}$, then a weaker constraint is taken into account by the maximum number of dropouts approach to verify stability, and not the full knowledge on the original constraint is used. We propose in this subsection a stability condition, which in contrast exploits additional knowledge on the dropout sequence for constraints of the types $\langle \binom{n}{m} \rangle$ and $\binom{n}{m}$.

This sufficient condition is based on the WHRT graph \mathcal{G}_η , which we can compute with tools from Linsenmayer et al. (2020) for a WHRT constraint η . For given $c_{\text{walk}} > 0$, we can moreover determine $\mathcal{S}(\mathcal{G}_\eta, c_{\text{walk}})$ for \mathcal{G}_η with a simple recursion. Then, since a new input is received at the first sampling time, we can generate any sequence $\nu \vdash \eta$ by appending walks from $\mathcal{S}(\mathcal{G}_\eta, c_{\text{walk}})$. The next step is to compute an upper bound on $V(x)$ for all transmission sequences that can be generated by the walks from $\mathcal{S}(\mathcal{G}_\eta, c_{\text{walk}})$. If this bound is decreasing for each such sequence, then we can show asymptotic stability of the origin of the DDS (7) with Proposition 10 for \tilde{T} chosen such, that it contains always the starting times and the end times of each appended walk from $\mathcal{S}(\mathcal{G}_\eta, c_{\text{walk}})$. As a result, we obtain the following sufficient stability condition.

Theorem 14. Consider the closed-loop system (7) with a sampling period $h \in \mathbb{R}_{>0}$ and a dropout sequence $\nu \vdash \eta$ for a given WHRT constraint η . Let \mathcal{G}_η be a WHRT graph for η and let $c_{\text{walk}} \in \mathbb{N}$. Let Assumption 1 hold. Assume moreover, there are parameters $(\gamma_i, L_i, \Lambda_i, \epsilon_i)$ for each $i \in \{1, \dots, w(\eta) + 1\}$, such that Assumption 11 holds for $(\gamma_i, L_i, \epsilon_i)$, and

$$ih < T_{\max}(\gamma_i, \Lambda_i) \quad (22)$$

holds for all $i \in \{1, \dots, w(\eta) + 1\}$. Assume for each walk $P \in \mathcal{S}(\mathcal{G}_\eta, c_{\text{walk}})$ with index sequence $\mathbb{I}(P)$, and sequence of edge weights of the walk $(l_{\mathbb{I}_1(P)}, \dots, l_{\mathbb{I}_{n_P}(P)})$, that

$$\sum_{\theta=1}^{n_P} h l_{\mathbb{I}_\theta(P)} \max \left\{ -\epsilon_{l_{\mathbb{I}_\theta(P)}}, 2(L_{l_{\mathbb{I}_\theta(P)}} - \Lambda_{l_{\mathbb{I}_\theta(P)}}) \right\} < 0 \quad (23)$$

holds. Then the origin of (7) is asymptotically stable for all sequences $\nu \vdash \eta'$ with $\eta' \preceq \eta$.

The proof of Theorem 14 can be found in the preprint Hertneck et al. (2020).

Remark 15. For constraints of type $\overline{\langle \binom{n}{m} \rangle}$, Theorem 14 and the maximum number of dropouts approach are equivalent. For constraints of the types $\langle \binom{n}{m} \rangle$ and $\binom{n}{m}$, Theorem 14 can exploit the additional information to guarantee stability for significantly larger sampling periods than the maximum number of dropouts approach.

The parameter c_{walk} can be chosen arbitrarily. However, in most cases it is advantageous to chose it equal to the window length of the constraint η . For given parameters $\gamma_i, L_i, \epsilon_i$ and Λ_i , for all $i \in \{1, \dots, w(\eta) + 1\}$, condition (23) can be automatically verified. For that, the set $\mathcal{S}(\mathcal{G}_\eta, c_{\text{walk}})$ can be determined by a recursive search through \mathcal{G}_η . Then, the left hand side of (23) can be computed for any walk from $\mathcal{S}(\mathcal{G}_\eta, c_{\text{walk}})$ in order to to verify (23). Thus to be able to show stability of a DDS (7) for a preferably large sampling period h , it remains to find the parameters $\gamma_i, L_i, \epsilon_i$ and Λ_i for all $i \in \{1, \dots, w(\eta)\}$, such that the conditions of Theorem 14 hold. For a given h , the least conservative parameters are given for each $i \in \{1, \dots, w(\eta)\}$ by the solution of

$$\begin{aligned} (\gamma_i, L_i, \epsilon_i, \Lambda_i) &= \arg \inf_{\gamma_i, L_i, \epsilon_i, \Lambda_i} \{ \max \{ -\epsilon_i, 2(L_i - \Lambda_i) \} \} \\ &\text{s.t. Ass. 11 holds} \\ T_{\max}(\gamma_i, \Lambda_i) &> hi. \end{aligned} \quad (24)$$

Solving this optimization problem is in general not an easy task, since the solution has to satisfy Assumption 11.

Fortunately, it suffices to find a feasible parameter set for the constraints of (24), instead of a solution to (24), in order to use Theorem 14. If (23) holds for that suboptimal parameter set, then we can conclude asymptotic stability for the DDS (7). We will demonstrate in Section 5 for an example system from literature how a (possibly suboptimal) but feasible set of parameters can be found with SOSTOOLS (Papachristodoulou et al. (2013)), such that Theorem 14 yields a significantly less conservative stability condition than the maximum number of dropouts approach. (or equivalently than using Theorem 14 with a constraint of the type $\overline{\langle \binom{m-n+1}{*} \rangle}$).

Remark 16. The recent results from (Heijmans, 2019, Chapter 8), can be interpreted as an alternative approach to deal with additional knowledge about dropout sequences. Therein, the worst case MASP has been significantly prolonged for systems with distributed sensors in comparison to the MASP from Carnevale et al. (2007) as long as the average sampling time is small enough. However, the results from (Heijmans, 2019, Chapter 8) are only beneficial for systems with distributed sensors that are not sampled simultaneously and do not improve the MASP for the sampled-data setup that is considered in this paper in comparison to the maximum number of dropouts approach with $T_{\max}(\gamma, L)$ according to Nesic et al. (2009).

5. EXAMPLE

Next, we apply Theorem 14 to an example from literature and demonstrate that it yields a less conservative stability condition than using the maximum number of dropouts approach, when the dropout sequence satisfies a WHRT constraint. We consider the unstable example system

$$\dot{x} = d_2 x^2 - x^3 + u, \quad (25)$$

Table 1. Feasible parameters for (24).

i	ϵ_i	γ_i	Λ_i	L_i	$T_{\max}(\gamma_i, \Lambda_i)$
1	1.5	5.77	2.75	2	0.211 s
2	0.5	2.38	2.25	2	0.428 s
3	-2	2.00	1	2	0.605 s
4	-4	2.00	0.001	2	0.787 s

that was also used in Nesic et al. (2009), and the constraint $\eta = \binom{17}{20}$ with $w(\eta) = 3$. The Graph \mathcal{G}_η can be constructed using Algorithm 1 from Linsenmayer et al. (2020). We choose $d_2 = 1$, $\kappa(x) = -2x$, $V(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 + 2x^2$ and $W(e) = \|e\|$ and obtain thus $f(x, e) = -2x + x^2 - x^3 + 2e$ and $g(x, e) = -f(x, e)$. We compute

$$\left\langle \frac{\partial}{\partial e} W^2(e), g(x, e) \right\rangle = 2\|e\| \operatorname{sign}(e) (g(x, e))$$

$$\leq 2W(e) (2\|e\| + \|2x - x^2 + x^3\|)$$

and observe thus that $L = 2$ and $H(x) = \|2x - x^2 + x^3\|$. Feasible value pairs for ϵ and γ , for which (16) holds, can be found with SOSTOOLS (Papachristodoulou et al. (2013)). The largest value for $T_{\max}(\gamma, L)$ for $\epsilon > 0$ is achieved for $\gamma = 2$. Thus, for the maximum number of dropouts approach, the best achievable bound on the MASP is $h < \frac{T_{\max}(2,2)}{4} = 0.125$ s. Further parameter sets, for which Assumption 11 holds are given in Table 1. For $h \leq 0.195$ s, these parameters are feasible for the constraints of (24) and are thus suitable to use Theorem 14, even though they are not the (optimal) solution to (24).

With the WHRT graph \mathcal{G}_η , (23) can be verified automatically for the parameters from Table 1 and all walks from $\mathcal{S}(\mathcal{G}_\eta, c_{\text{walk}})$. We observe, that (23) holds for $c_{\text{walk}} = 20$. Thus, we can use Theorem 14 to show asymptotic stability for the example system and $h = 0.195$ s, i.e., for a sampling period that is 1.56 times larger than the maximum sampling period for which stability can be guaranteed with the maximum number of dropouts approach.

6. CONCLUSION

In this paper, we have discussed the stability analysis of nonlinear sampled-data systems that run in open-loop from time to time due to failures in the feedback channel. WHRT constraints were used as precise dropout description. The overall approach is based on the emulation technique from Nesic et al. (2009), and is in general less conservative than considering only the maximum number of successive dropouts. The efficiency of the proposed approach has been illustrated for an example system from literature.

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