Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Finite-time Sliding Mode Control Under Dynamic Event-triggered Scheme ⋆

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Abstract: In this work, we investigate the finite-time boundedness (FTB) problem for a class of continuous-time uncertain systems via sliding mode control (SMC) method. A dynamic event-triggered scheme is introduced to determine whether the measurement output will be transmitted. And then, a state observer is designed by means of the transmitted output information, based on which a sliding surface in the estimation space is construct. It is shown that the corresponding SMC law can drive the system trajectories onto the specified sliding surface in a finite (possibly short) time. Meanwhile, a partitioning strategy is introduced to analyze the FTB over both reaching phase and the sliding motion, respectively. Finally, a numerical simulation is given.

Keywords: Sliding mode control, finite-time boundedness, event-triggered control, external disturbance, partitioning strategy.

1. INTRODUCTION

Compared with traditional control systems, cyber-physical systems includes a mass of shared communication information, which has received hot research concerns on how to save the communication resources. Very recently, the event-triggered strategy (ETS), instead of the time-triggered strategy, has been proposed to reduce the transmission frequency. The key feature of ETS is that the information is transmitted only when some condition is satisfied (Peng and Li (2018)). Based on the above design philosophy, some researches have been reported about various of ETS transmission rules such as static ETS (Tabuada (2007)), dynamic ETS (Dolk et al. (2017)), and adaptive ETS (Gu et al. (2018)). Among these, it is worthy of mentioning that the dynamic ETS include an internal dynamic variable which effectively increase the minimum inter-event time (Girard (2015)), and has been widely applied for both continuous-time and discrete-time systems.

Sliding mode control (SMC) is a well known robust control method for its distinguished ability to withdraw parameter uncertainties and external disturbances. The basic idea of SMC method is to force the system state trajectories to the pre-designed sliding mode surface and remain thereafter (Polyakov and Fridman (2014); Li and Niu (2019)). Up to date, some interesting results has been obtained on the event-triggered SMC method, see (Kumari et al. (2019); Wu et al. (2017)) and the references therein. In (Zhao et al. (2019)), the event-triggered SMC problems was studied for the switched systems subject to Denial-of-Service attacks, where the sliding motion and reachability were analyzed under the per-designed ETS condition. Besides, the SMC problem under dynamic ETS was investigated for discrete-time systems in (Song and Niu (2019)). However, it still remains open for continuous-time systems. Several conditions, especially, the analysis of Zeno phenomenon bring some difficulties.

On the other hand, finite-time boundness (FTB) has a wide practical significance (ElBsat and Yaz (2013)), especially in the field of spacecraft. FTB concerns the transient systems performance, which means the norm of system state not escaping from the specified threshold during a given time interval if the initial condition is known (Amato et al. (2001)). Up to date, several excellent works have been done on the finite-time SMC problem. For example, Song, Niu and Zou proposed a partitioning strategy, which divides a whole finite-time interval into the reaching part and the sliding motion part (Song et al. (2017)). Later, this work was extended to some complex system such as switched systems (Zhao and Niu (2019)) and Markov jump systems (Cao et al. (2019)). However it should be pointed out that the dynamic ETS has not been investigated for the finite-time SMC problem, which motivates our work.

In this paper, we investigate the finite-time SMC problem for the continuous-time systems under ETS, in which

⋆ This work was supported in part by the NNSF of China (61803255, 61673174) and Natural Science Foundation of Shanghai under Grant 18ZR1416700.

the measured output signals will be transmitted to the controller according to a dynamic event-trigger rule. An SMC law is designed such that the state trajectories can be driven to the sliding surface within the interval $[0, T^*]$ with $T < T^*$ a given finite time T. Moreover, by utilizing the partitioning strategy, the FTB for both the reachability phase in $[0, T^*]$ and the sliding phase in $[T^*, T]$ are analyzed respectively. Thus, the FTB over whole finite interval $[0, T]$ can be achieved. The main contributions are summarized below:

1) This is the first attempt to investigate the finite-time SMC problem under ETS, especially for the dynamic ETS.
2) The continuous dynamic ETS for SMC problem has been considered, and the minimum inter-event time has been given to avoid Zeno phenomena.
3) Both the FTB over reaching phase within $[0, T^*]$ and over sliding motion within $[T^*, T]$ have been analysed.

Notation: $\mathbb{R}^n$ is the n dimensional Euclidean space, while $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices. I stands for the identity matrix of appropriate dimensions. diag$\{\cdots\}$ denotes a block-diagonal matrix with the elements of the argument on its main diagonal. $|\cdot|$ and $\|\cdot\|$ refer to the 1-norm and Euclidean vector norm or its induced matrix norm. The symbol He(M) refers to represent $M + M^T$. If not specified stated, matrices are assumed to have compatible dimensions.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following uncertain systems:

$$\begin{aligned} \dot{x}(t) &= \big(A + \Delta A(t)\big)x(t) + Bu(t) + D\omega(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^p$ is the measurement output. A, B, C, D, E and H are the known constant matrices, and the matrix B is of full column rank. The parameter uncertainty $\Delta A(t)$ satisfies the norm-bounded assumption i.e., $\Delta A(t) = EF(t)H$, where $F(t)$ is an unknown time-varying matrix satisfying $F^T(t)F(t) \le I$. $\omega(t) \in \mathbb{R}^q$ is the external disturbance with known peak value $\|\omega(t)\| \le \bar{\omega}$.

In this work, only the measurement output $y(t)$ is assumed to be available. In order to save the network resources, the dynamic event-triggered mechanism is utilized between the sensor/controller network. The event-triggered instant t_k is determined by the following rule:

$$\begin{aligned} t_{k+1} = \inf\big\{& t \in \mathbb{R} | t > t_k, \\ & \frac{1}{\theta}\eta(t) + \alpha y^T(t)My(t) - \delta^T(t)N\delta(t) \le 0\big\} \end{aligned} \tag{2}$$

where the event-triggering error $\delta(t) \triangleq y(t_k) - y(t)$ is the difference between the current measured output and the transmitted output signal at the last triggering instant. M and N are known matrices, and the internal dynamical variable $\eta(t)$ satisfies

$$\dot{\eta}(t) = -c_0\eta(t) + \alpha y^T(t)My(t) - \delta^T(t)N\delta(t) \tag{3}$$

with $c_0 > 0$ and $\alpha \in (0, 1)$ given constants, and the initial value of $\eta(0) = \eta_0 > 0$.

Next, utilizing the transmitted signal $y(t_k)$ at triggering instant t_k, the following state observer is constructed:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\big(y(t_k) - C\hat{x}(t)\big) \tag{4}$$

for $t \in [t_k, t_{k+1})$, where $\hat{x}(t)$ is the state estimate and the matrix L will be designed later. Denote the estimation error $e(t) \triangleq x(t) - \hat{x}(t)$, and we can obtain the estimation error dynamics from (1) and (4):

$$\begin{aligned} \dot{e}(t) =& \Delta A(t)\hat{x}(t) + \big(A + \Delta A(t) - LC\big)e(t) \\ & - L\delta(t) + D\omega(t) \end{aligned} \tag{5}$$

3. MAIN RESULTS

Choose the following integral sliding function:

$$s(t) = G\hat{x}(t) - G\int_0^t (A + BK)\hat{x}(r)\mathrm{d}r \tag{6}$$

where the matrix $G \in \mathbb{R}^{m\times n}$ should be designed to ensure the non-singularity of matrix GB. It can be easily shown that the non-singularity of GB can be ensured by choosing $G = B^TX$ with the matrix $X > 0$.

According to the SMC theory, when the state trajectories are driven to the sliding surface, i.e., $s(t) = \dot{s}(t) = 0$, we can obtain the equivalent control law:

$$u_{eq}(t) = K\hat{x}(t) - (GB)^{-1}GL\big(y(t_k) - C\hat{x}(t)\big) \tag{7}$$

Then, we design the sliding mode controller as follows:

$$u(t) = u_{eq}(t) - \tilde{\rho}(t) \tag{8}$$

where $\tilde{\rho}(t) = \rho\mathrm{sgn}(s(t))$ is the switching control term, and the parameter ρ will be designed later in Theorem 1.

Denote $\xi(t) \triangleq [\hat{x}^T(t), e^T(t)]^T$. Then, substituting (8) into (4), yields the following augmented closed-loop system:

$$\dot{\xi}(t) = \mathcal{M}\xi(t) + \mathcal{N}\delta(t) + \mathcal{X}\omega(t) + \mathcal{Y}\tilde{\rho}(t) \tag{9}$$

with

$$\mathcal{M} \triangleq \begin{bmatrix} A + BK & \tilde{G}LC \\ \Delta A & A + \Delta A - LC \end{bmatrix},\ \mathcal{N} \triangleq \begin{bmatrix} \tilde{G}L \\ -L \end{bmatrix},$$
$$\mathcal{X} \triangleq \begin{bmatrix} 0 \\ D \end{bmatrix},\ \mathcal{Y} \triangleq \begin{bmatrix} -B \\ 0 \end{bmatrix},\ \tilde{G} \triangleq I - B(GB)^{-1}G.$$

Definition 1. Give a time interval $[t_1, t_2]$, two positive scalars c_1, c_2 ($c_1 < c_2$), and a weighted matrix $R > 0$. The argument system (9) with $u(t) = 0$ is said to be FTB with respect to $(c_1, c_2, [t_1, t_2], R, \bar{\omega})$ if

$$\xi^T(t_1)R\xi(t_1) \le c_1 \ \Rightarrow \xi^T(t)R\xi(t) \le c_2,\ \forall t \in [t_1, t_2].$$

In the sequel, we shall analyze the reachability of the sliding surface $s(t) = 0$ within the interval $[0, T^*]$ with $T^* < T$ and the FTB over the whole interval $[0, T]$.

3.1 The reachability with $T^* < T$

Theorem 1. The closed-loop system (9) can be driven to the specified sliding surface $s(t) = 0$ in a finite time T^* with $T^* < T$ for any given $T > 0$, if the adjustable parameter ρ satisfies:

$$\rho \ge \frac{\lambda_{\max}(GB)^{-1}}{T - \beta}\|G\hat{x}(0)\|. \tag{10}$$

with a small constant $0 < \beta < T$.

Proof. Choose the Lyapunov function as

$$U(t) \triangleq \frac{1}{2}s^T(t)(GB)^{-1}s(t) \tag{11}$$

From (6) we have

$$\dot{U}(t) = s^T(t)(GB)^{-1}\dot{s}(t) = -\rho s^T(t)\text{sgn}(s(t)) \\ = -\rho|s(t)| \le -\rho\|s(t)\| \quad (12)$$

Due to $U(t) \le \frac{1}{2}\lambda_{\max}(GB)^{-1}\|s(t)\|^2$, thus

$$\|s(t)\| \ge \sqrt{\frac{2U(t)}{\lambda_{\max}(GB)^{-1}}} \quad (13)$$

Substituting (13) into (12), then integrating from 0 to T^*, one has

$$\int_0^{T^*} \frac{1}{\sqrt{U(t)}}\text{d}U(t) \le \int_0^{T^*} -\rho\sqrt{\frac{2}{\lambda_{\max}(GB)^{-1}}}\text{d}t \quad (14)$$

Due to $U(T^*) = 0$, we have

$$T^* \le \frac{\lambda_{\max}(GB)^{-1}}{\rho}\|G\hat{x}(0)\| \quad (15)$$

Moreover, it is obtained from (10) and (15) that $T^* \le T - \beta < T$. So, the trajectories of the closed-loop system (9) can be driven to the specified sliding surface $s(t) = 0$ within the interval $[0, T^*]$ with $T* < T$ the given finite time T. ■

3.2 FTB over reaching phase within $[0, T^*]$

Now, we shall analyze the FTB of the closed-loop system over the reaching phase within $[0, T^*]$.

Theorem 2. Given parameters $(c_1, c^*, [0, T^*], R, \bar{\omega})$ and $\gamma > 0$, if there exist the positive constant c^* and matrices $P > 0$, K, and L satisfying:

$$\Gamma \triangleq \begin{bmatrix} \Gamma^{(1,1)} & \tilde{P}\mathcal{N} & \tilde{P}\mathcal{X} & \tilde{P}\mathcal{Y} & 0 \\ * & -\hat{\kappa}_1 N & 0 & 0 & 0 \\ * & * & -\gamma I & 0 & 0 \\ * & * & * & -\gamma I & 0 \\ * & * & * & * & \varpi_1 \end{bmatrix} < 0 \quad (16)$$

$$\bar{\lambda}c_1 + \eta_0 + \gamma\bar{\omega}^2 T^* + \gamma m\rho^2 T^* < e^{-\gamma T^*}\underline{\lambda}c^* \quad (17)$$

where

$$\Gamma^{(1,1)} \triangleq \mathcal{M}^T\tilde{P} + \tilde{P}\mathcal{M} + \hat{\kappa}_1\alpha\tilde{C}^T M\tilde{C} - \gamma\tilde{P}$$
$$\tilde{P} \triangleq \text{diag}\{P, P\},\ \tilde{C} \triangleq \begin{bmatrix} C, & C \end{bmatrix},$$
$$\varpi_1 \triangleq \frac{\kappa_1}{\theta} - \gamma - c_0,\ \hat{\kappa}_1 \triangleq 1 + \kappa_1,$$
$$\underline{\lambda} \triangleq \lambda_{\min}\big(R^{-\frac{1}{2}}\tilde{P}R^{-\frac{1}{2}}\big),\ \overline{\lambda} \triangleq \lambda_{\max}\big(R^{-\frac{1}{2}}\tilde{P}R^{-\frac{1}{2}}\big),$$

the closed-loop system (9) is FTB with respect to $(c_1, c^*, [0, T^*], R, \bar{\omega})$.

Proof. Note that the dynamic event-triggered condition (2) implies for any $t \in [t_k, t_{k+1})$

$$\frac{1}{\theta}\eta(t) + \alpha y^T(t)My(t) - \delta^T(t)N\delta(t) > 0 \quad (18)$$

From the dynamic equation (3), we have

$$\dot{\eta}(t) > -c_0\eta(t) - \frac{1}{\theta}\eta(t) = -(c_0 + \frac{1}{\theta})\eta(t) \quad (19)$$

Then, integrating from 0 to t, yields

$$\eta(t) > \eta_0 e^{-(c_0 + \frac{1}{\theta})t} \quad (20)$$

For $\eta_0 > 0$, one obtains $\eta(t) > 0$ for $t \in [0, \infty)$.

Choose the Lyapunov functional candidate:

$$V_1(t) \triangleq \xi^T(t)\tilde{P}\xi(t) + \eta(t) \quad (21)$$

One obtains:

$$\begin{aligned}\dot{V}_1(t) =& \xi^T(t)(\mathcal{M}^T\tilde{P} + \tilde{P}\mathcal{M} + \alpha\tilde{C}^T M\tilde{C})\xi(t) \\ &+ \xi^T(t)\tilde{P}\mathcal{N}\delta(t) + \delta^T(t)\mathcal{N}^T\tilde{P}\xi(t) \\ &+ \xi^T(t)\tilde{P}\mathcal{X}\omega(t) + \omega^T(t)\mathcal{X}^T\tilde{P}\xi(t) \\ &+ \xi^T(t)\tilde{P}\mathcal{Y}\tilde{\rho}(t) + \tilde{\rho}^T(t)\mathcal{Y}^T\tilde{P}\xi(t) \\ &- c_0\eta(t) - \delta^T(t)N\delta(t) \end{aligned} \quad (22)$$

Define the following auxiliary function:

$$J_1 \triangleq \dot{V}_1(t) - \gamma V_1(t) - \gamma\omega^T(t)\omega(t) - \gamma\tilde{\rho}^T(t)\tilde{\rho}(t) \quad (23)$$

If the dynamic event-triggering condition (2) is violated, for any scalar $\kappa_1 > 0$, we can obtain from (22):

$$\begin{aligned} J_1 \le& J_1 + \kappa_1\Big(\frac{1}{\theta}\eta(t) + \alpha y^T(t)My(t) - \delta^T(t)N\delta(t)\Big) \\ \le& \xi^T(t)(\mathcal{M}^T\tilde{P} + \tilde{P}\mathcal{M} + (1+\kappa_1)\alpha\tilde{C}^T M\tilde{C} - \gamma\tilde{P})\xi(t) \\ &+ \xi^T(t)\tilde{P}\mathcal{N}\delta(t) + \delta^T(t)\mathcal{N}^T\tilde{P}\xi(t) + \xi^T(t)\tilde{P}\mathcal{X}\omega(t) \\ &+ \omega^T(t)\mathcal{X}^T\tilde{P}\xi(t) + \xi^T(t)\tilde{P}\mathcal{Y}\tilde{\rho}(t) + \tilde{\rho}^T(t)\mathcal{Y}^T\tilde{P}\xi(t) \\ &+ (\frac{\kappa_1}{\theta} - \gamma - c_0)\eta(t) - (1+\kappa_1)\delta^T(t)N\delta(t) \\ &- \gamma\omega^T(t)\omega(t) - \gamma\tilde{\rho}^T(t)\tilde{\rho}(t) \triangleq \mu_1^T(t)\Gamma\mu_1(t) < 0 \end{aligned} \quad (24)$$

with $\mu_1(t) \triangleq [\xi^T(t), \delta^T(t), \omega^T(t), \tilde{\rho}^T(t), \sqrt{\eta(t)}]^T$. Thus, if the condition (16) holds, we have $J_1 < 0$ from (24) by Schur complement.

Then, we obtain

$$\dot{V}_1(t) < \gamma V_1(t) + \gamma\omega^T(t)\omega(t) + \gamma\tilde{\rho}^T(t)\tilde{\rho}(t) \quad (25)$$

Multiplying $e^{-\gamma t}$ for both sides of expression (25) and integrating from 0 to t with $t \in [0, T^*]$, one has

$$\begin{aligned} &e^{-\gamma t}V_1(t) \\ <& V_1(0) + \gamma\int_0^t e^{-\gamma v}\omega^T(v)\omega(v)\text{d}v + \gamma\int_0^t e^{-\gamma v}\tilde{\rho}^T(v)\tilde{\rho}(v)\text{d}v \\ <& V_1(0) + \gamma\bar{\omega}^2T^* + \gamma m\rho^2 T^* \end{aligned} \quad (26)$$

Note the fact that

$$\underline{\lambda}\xi^T(t)R\xi(t) \le \xi^T(t)\tilde{P}\xi(t) \le \overline{\lambda}\xi^T(t)R\xi(t) \quad (27)$$

we can achieve

$$V_1(0) = \xi^T(0)\tilde{P}\xi(0) + \eta(0) \le \bar{\lambda}c_1 + \eta_0 \quad (28)$$

Thus, from (26)-(28) we have

$$\xi^T(t)R\xi(t) < \frac{e^{\gamma t}(\bar{\lambda}c_1 + \eta_0 + \gamma\bar{\omega}^2T^* + \gamma m\rho^2T^*)}{\underline{\lambda}} \quad (29)$$

And the condition (17) implies $\xi^T(t)R\xi(t) < c^*$ for $t \in [0, T^*]$. According to Definition 1, the FTB over the interval $[0, T^*]$ is ensured. ■

3.3 FTB over sliding motion within $[T^*, T]$

Next, we shall analyze the FTB of sliding motion during the interval $[T^*, T]$, where the state trajectories maintain in the sliding surface $s(t) = 0$. Substituting the equivalent control law (7) into (4), we yields the following sliding mode dynamics:

$$\dot{\xi}(t) = \mathcal{M}\xi(t) + \mathcal{N}\delta(t) + \mathcal{X}\omega(t) \quad (30)$$

Theorem 3. Consider the closed-loop system (30) within the interval $[T^*, T]$. If there exist the positive constant

c^* and matrices $P > 0$, K and L, satisfying the following conditions:

$$\Lambda \triangleq \begin{bmatrix} \Lambda^{(1,1)} & \tilde{P}\mathcal{N} & \tilde{P}\mathcal{X} & 0 \\ * & -\hat{\kappa}_2 N & 0 & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & \varpi_2 \end{bmatrix} < 0 \tag{31}$$

$$c_1 < c^* < c_2 \tag{32}$$

$$e^{-\gamma T^*}\bar{\lambda}c^* + \eta_0 e^{-(\gamma+c_0+\frac{1}{\theta})T^*} + \gamma\bar{\omega}^2\tilde{T} < e^{-\gamma T}\underline{\lambda}c_2 \tag{33}$$

with

$$\Lambda^{(1,1)} \triangleq \mathcal{M}^T\tilde{P} + \tilde{P}\mathcal{M} + \hat{\kappa}_2\alpha\tilde{C}^T M\tilde{C} - \gamma\tilde{P},$$
$$\varpi_2 \triangleq \frac{\kappa_2}{\theta} - \gamma - c_0,\ \hat{\kappa}_2 \triangleq 1 + \kappa_2,\ \tilde{T} \triangleq T - T^*$$

and then the closed-loop system (30) is FTB with respect to $(c^*, c_2, [T^*, T], R, \bar{\omega})$.

Proof. Choose the Lyapunov function as

$$V_2(t) \triangleq \xi^T(t)\tilde{P}\xi(t) + \eta(t) \tag{34}$$

Then, define an auxiliary function:

$$J_2 \triangleq \dot{V}_2(t) - \gamma V_2(t) - \gamma\omega^T(t)\omega(t) \tag{35}$$

one has

$$\begin{aligned} J_2 \le\, & J_2 + \kappa_2\Big(\frac{1}{\theta}\eta(t) + \alpha y^T(t)My(t) - \delta^T(t)N\delta(t)\Big) \\ \le\, & \xi^T(t)(\mathcal{M}^T\tilde{P} + \tilde{P}\mathcal{M} + (1+\kappa_2)\alpha\tilde{C}^T M\tilde{C} - \gamma\tilde{P})\xi(t) \\ & + \xi^T(t)\tilde{P}\mathcal{N}\delta(t) + \delta^T(t)\mathcal{N}^T\tilde{P}\xi(t) + \xi^T(t)\tilde{P}\mathcal{X}\omega(t) \\ & + \omega^T(t)\mathcal{X}^T\tilde{P}\xi(t) + (\frac{\kappa_2}{\theta} - \gamma - c_0)\eta(t) - \gamma\omega^T(t)\omega(t) \\ & - (1+\kappa_2)\delta^T(t)N\delta(t) \triangleq \mu_2^T(t)\Lambda\mu_2(t) \end{aligned} \tag{36}$$

with $\mu_2(t) \triangleq [\xi^T(t), \delta^T(t), \omega^T(t), \sqrt{\eta(t)}]^T$.

By Schur complement, it can be shown that the expression (31) can promise $J_2 < 0$, from which we obtain

$$\dot{V}_2(t) < \gamma V_2(t) + \gamma\omega^T(t)\omega(t) \tag{37}$$

Then, multiplying $e^{-\gamma t}$ for both sides of the expression (37) and integrating from T^* to t with $t \in [T^*, T]$ yields:

$$\begin{aligned} & e^{-\gamma t}V_2(t) \\ < & e^{-\gamma T^*}V_2(T^*) + \gamma\int_{T^*}^{t} e^{-\gamma t}\omega^T(\upsilon)\omega(\upsilon)\mathrm{d}\upsilon \end{aligned} \tag{38}$$

Note that one has from (20):

$$V_2(T^*) < \bar{\lambda}c^* + \eta_0 e^{-(c_0+\frac{1}{\theta})T^*} \tag{39}$$

Resulting from (38) and (39), we have

$$\begin{aligned} & e^{-\gamma t}\underline{\lambda}\xi^T(t)R\xi(t) + e^{-\gamma t}\eta(t) \\ < & \bar{\lambda}c^* + \eta_0 e^{-(c_0+\frac{1}{\theta})T^*} + \gamma\bar{\omega}^2\tilde{T} \end{aligned} \tag{40}$$

For $\eta(t) > 0$, the condition (33) can promise $\xi^T(t)R\xi(t) < c_2$ for all $t \in [T^*, T]$, which implies the sliding motion FTB within $[T^*, T]$. ∎

3.4 Zeno phenomena

The following Theorem ensures the lower bound of the inter-event time be a positive value, which effectively avoid the Zeno phenomena.

Theorem 4. Let t_k $(k = 1, \ldots, \infty)$ be the triggering sequences generated by the event-trigger rule (2) and the inter-event time as $T_{in} \triangleq t_{k+1} - t_k$, one has:

$$T_{in} \ge \frac{1}{\mathcal{G}_1}\ln\Big(1 + \frac{\mathcal{G}_1\big(\eta(t_{k+1}) + \theta\alpha y^T(t_{k+1})My(t_{k+1})\big)}{\theta\lambda_{\max}(N)\big(\mathcal{G}_2\|y(t_k)\|^2 + \zeta_0\big)}\Big) \tag{41}$$

with

$$\begin{aligned} \zeta_0 \triangleq & (\|\bar{\mathcal{F}}\| + \|\tilde{\mathcal{F}}\|\|H\|)\frac{c_2}{\lambda_{\min}(R)} + \|CD\|\bar{\omega}^2 + \rho m\|CB\| \\ \mathcal{G}_1 \triangleq & \|\bar{\mathcal{F}}\| + \|\tilde{\mathcal{F}}\|\|H\| + \|CB(GB)^{-1}GL\| \\ & + \rho\|CB\| + \|CD\| \\ \mathcal{G}_2 \triangleq & \|CB(GB)^{-1}GL\|,\ \tilde{\mathcal{F}} \triangleq \big[CE,\ CE\big] \\ \bar{\mathcal{F}} \triangleq & \big[C(A + BK + B(GB)^{-1}GLC),\ CA\big] \end{aligned}$$

Proof. For all $t \in [t_k, t_{k+1})$, we have the differential of event-trigger error:

$$\begin{aligned} \dot{\delta}(t) =& -C\dot{x}(t) \\ =& -\bar{\mathcal{F}}\xi(t) - \breve{\mathcal{F}}\xi(t) + CB(GB)^{-1}GLy(t_k) \\ & + \rho CB\mathrm{sgn}(s(t)) - CD\omega(t) \end{aligned} \tag{42}$$

with $\breve{\mathcal{F}} \triangleq \big[C\Delta A,\ C\Delta A\big]$. Thus, one has

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\|\delta(t)\|^2 \le 2\|\delta(t)\|\|\dot{\delta}(t)\| \\ \le & \|\bar{\mathcal{F}}\|(\|\xi(t)\|^2 + \|\delta(t)\|^2) + \|\tilde{\mathcal{F}}\|\|H\|(\|\xi(t)\|^2 + \|\delta(t)\|^2) \\ & + \|CB(GB)^{-1}GL\|(\|y(t_k)\|^2 + \|\delta(t)\|^2) \\ & + \rho\|CB\|(m + \|\delta(t)\|^2) + \|CD\|(\bar{\omega}^2 + \|\delta(t)\|^2) \\ = & (\|\bar{\mathcal{F}}\| + \|\tilde{\mathcal{F}}\|\|H\|)\|\xi(t)\|^2 + (\|\bar{\mathcal{F}}\| + \|\tilde{\mathcal{F}}\|\|H\| \\ & + \|CB(GB)^{-1}GL\| + \rho\|CB\| + \|CD\|)\|\delta(t)\|^2 \\ & + \|CB(GB)^{-1}GL\|\|y(t_k)\|^2 + \|CD\|\bar{\omega}^2 \\ & + \rho m\|CB\| \end{aligned} \tag{43}$$

Due to $\xi^T(t)R\xi(t) < c_2$, we have for $t \in [0, T]$

$$\|\xi(t)\|^2 < \frac{c_2}{\lambda_{\min}(R)} \tag{44}$$

Then from the expression (43), it yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta(t)\|^2 \le \mathcal{G}_1\|\delta(t)\|^2 + \mathcal{G}_2\|y(t_k)\|^2 + \zeta_0 \tag{45}$$

Integrating (45) from t_k to t_{k+1}, and noting at the trigger moment we have $\|\delta(t_k) = 0\|$, thus

$$\|\delta(t_{k+1})\|^2 \le \frac{\mathcal{G}_2\|y(t_k)\|^2 + \zeta_0}{\mathcal{G}_1}\big(e^{\mathcal{G}_1 T_{in}} - 1\big) \tag{46}$$

Besides, by the event-trigger rule (2), we have

$$\|\delta(t_{k+1})\|^2 \ge \frac{\eta(t_{k+1}) + \theta\alpha y^T(t_{k+1})My(t_{k+1})}{\theta\lambda_{\max}(N)} \tag{47}$$

Combining (46) and (47), we can achieve the lower bound of the inter-event time (43). ∎

3.5 Solving algorithm

A sufficient condition based on the linear matrix inequalities (LMI) will be given to promise the nonlinear conditions in Theorem 2 and 3 hold simultaneously.

Theorem 5. Given the parameters $(c_1, c_2, T, R, \bar{\omega})$ and the adjustable parameters $\tau > 0$, $\kappa > 0$, γ and ϵ, if there exist positive matrix $Q > 0$, real matrices $\mathcal{K}$, $\mathcal{L}$, $\mathcal{U}$, and positive scalars c^*, ς, ε_1, ε_2 and χ satisfying the following LMIs:

$$\Pi \triangleq \begin{bmatrix} \Pi_1^1 & \Pi_2^1 & \Pi_3^1 \\ * & \Pi_2^2 & 0 \\ * & * & \Pi_3^3 \end{bmatrix} < 0 \tag{48}$$

$$\begin{bmatrix} -\chi I & * \\ CQ - \mathcal{U}C & -I \end{bmatrix} < 0 \tag{49}$$

$$\varsigma R^{-1} < \tilde{Q} < \tau R^{-1} \tag{50}$$

$$\begin{bmatrix} \eta_0 + \gamma\bar{\omega}^2 T + \gamma m \rho^2 T - \frac{1}{\tau} e^{-\gamma T} c^* & \sqrt{c_1} \\ * & -\varsigma \end{bmatrix} < 0 \tag{51}$$

$$c^* + \varsigma\eta_0 + \varsigma\gamma\bar{\omega}^2 T - \frac{1}{\tau} e^{-\gamma T} \varsigma c_2 < 0 \tag{52}$$

$$c_1 < c^* < c_2 \tag{53}$$

where

$$\Pi_1^1 \triangleq \begin{bmatrix} \mathrm{He}(AQ + B\mathcal{K}) - \gamma Q & \tilde{G}\mathcal{L}C \\ * & \mathrm{He}(AQ + \mathcal{L}C) - \gamma Q \end{bmatrix},$$

$$\Pi_2^1 \triangleq \begin{bmatrix} \tilde{G}\mathcal{L}CC^T & 0 & -B & 0 \\ -L\tilde{G}\mathcal{L}CC^T & D & 0 & 0 \end{bmatrix},$$

$$\Pi_3^1 \triangleq \begin{bmatrix} 0 & QH^T & 0 & 0 & \hat{\kappa}\alpha QC^T \\ \varepsilon_1 E & 0 & \varepsilon_2 E & QH^T & \hat{\kappa}\alpha QC^T \end{bmatrix},$$

$$\Pi_2^2 \triangleq \mathrm{diag}\{\epsilon^2 \hat{\kappa}^{-1} N^{-1} - 2\epsilon CQC^T, -\gamma I, -\gamma I, \varpi\},$$

$$\Pi_3^3 \triangleq \mathrm{diag}\{-\varepsilon_1 I, -\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_2 I, -\hat{\kappa}\alpha M^{-1}\},$$

$$\tilde{Q} \triangleq \mathrm{diag}\{Q, Q\}, \ \varpi \triangleq \frac{\kappa}{\theta} - \gamma - c_0, \ \hat{\kappa} \triangleq 1 + \kappa.$$

Then the closed-loop system (9) is FTB with respect to $(c_1, c_2, [0, T], R, \bar{\omega})$, and the control gain $K = \mathcal{K}P$ and the observer gain $L = \mathcal{L}\mathcal{U}^{-1}$.

Proof. Let $Q \triangleq P^{-1}$. By left and right multiplying $\mathrm{diag}\{Q, Q, CQC^T, I, I, I\}$ and the Schur's complement, as well as letting $KQ = \mathcal{K}$, $L\mathcal{U} = \mathcal{L}$ and the linear equality $CQ = \mathcal{U}C$, it can be easily verified that (48) can promise the condition (16) and (31) simultaneously, with selecting $\kappa = \kappa_1 = \kappa_2$. Meanwhile, the linear equality $CQ = \mathcal{U}C$ is solved by the approximated inequality (49).

From the condition (50), we have

$$\varsigma I < R^{\frac{1}{2}} \tilde{Q} R^{\frac{1}{2}} < \tau I \tag{54}$$

Together with

$$\lambda_{\max}(R^{\frac{1}{2}} \tilde{Q} R^{\frac{1}{2}}) = \frac{1}{\underline{\lambda}}, \ \lambda_{\min}(R^{\frac{1}{2}} \tilde{Q} R^{\frac{1}{2}}) = \frac{1}{\overline{\lambda}} \tag{55}$$

we have

$$\underline{\lambda} > \frac{1}{\tau}, \ \overline{\lambda} < \frac{1}{\varsigma} \tag{56}$$

Then, the expression (17) and (33) can be ensured by (51) and (52). From the above analysis, the Theorem 2 and 3 can be guaranteed simultaneously, which implied the closed-loop system (9) is FTB $wrt(c_1, c_2, [0, T], R, \bar{\omega})$. ■

4. NUMERICAL SIMULATION

Consider the system (1) with the following parameters:

$$A = \begin{bmatrix} -2.6 & 2 \\ 1 & -2 \end{bmatrix}, \ B = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \ C = [\,4 \ \ 6\,],$$

$$D = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \ E = \begin{bmatrix} 1 \\ -0.8 \end{bmatrix}, \ H = [\,0.5 \ \ 0.9\,],$$

$$F(t) = \cos(5t), \ \omega(t) = 0.5\cos(3t).$$

Then, the dynamic event-triggered rule (2)-(3) is set as $\theta = 0.8$, $\alpha = 0.1$, $M = 6$, $N = 30$, $c_0 = 0.5$, $\eta_0 = 0.2$. Let $c_1 = 0.43$, $c_2 = 2.5$, $T = 5$, $R = I$, $\beta = 0.2$, the matrix $X = I$ and the initial conditions for $x(0) = [0.8, -0.2]^T$ and $\hat{x}(0) = [0.6, -0.1]^T$. Thus, it can be calculated that $\rho > 0.0667$ according to Theorem 1. By solving the LIMs in Theorem 5 with $\rho = 0.1$, we obtain:

$$K = [\,19.16 \ \ -47.32\,], \ L = [\,-0.0457 \ \ -0.0176\,],$$

with $c^* = 1.1847$.

Thus, the simulation results are shown in Figs.1-6. Figs.1-2 show that the system trajectories of closed-loop system (9) does not exceed c_2. Fig.3 shows that $\hat{x}(t)$ can be driven to the sliding surface $s(t) = 0$ within a finite time T^*, where T^* can be obtained $T^* \leq 3.2$ from (15). Fig.5 shows that the dynamic ETS case has 27 times released instants, and it is much less than the static ETS case ($\eta(t) = 0$) 222 times. Fig.6 gives the evolution of the internal dynamical variable $\eta(t)$, with $\eta(t) > 0$ convergence for $t \in [0, T]$.

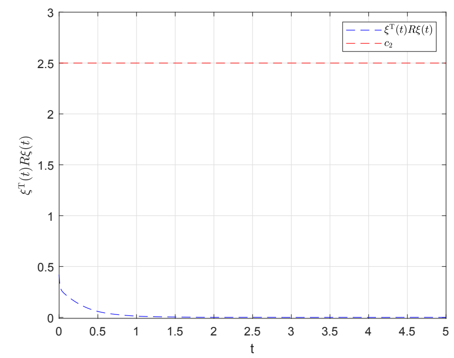

Fig. 1. The evolution of $\xi^T(t)R\xi(t)$

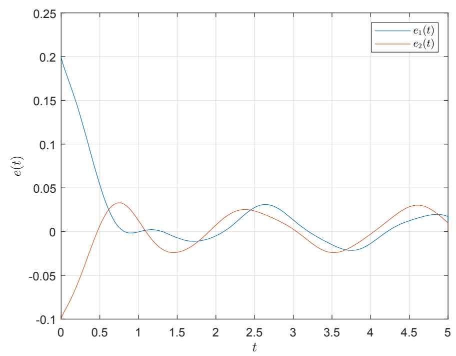

Fig. 2. The observer error $e(t)$

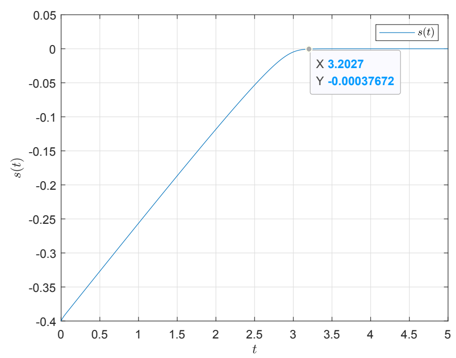

Fig. 3. The sliding variable $s(t)$

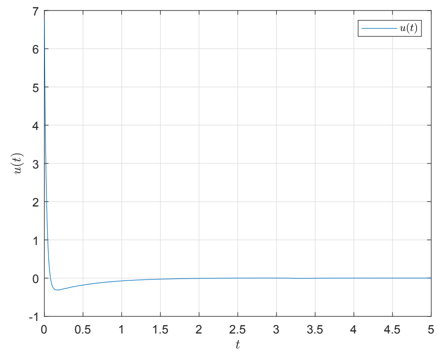

Fig. 4. The control input $u(t)$

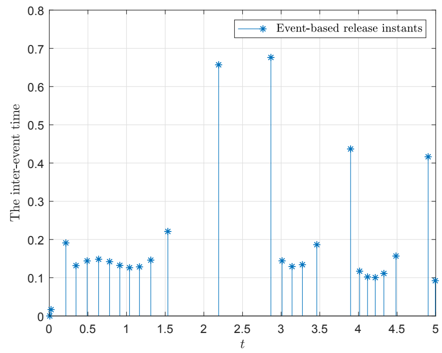

Fig. 5. The release instants and intervals

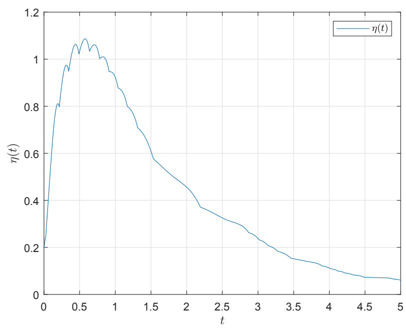

Fig. 6. The evolution of dynamic variable $\eta(t)$

5. CONCLUSION

This paper has investigated the observer-based finite-time SMC problem for the continuous-time dynamic ETS. The partitioning strategy was adopted to promise the reachability of sliding surface within $[0, T^*]$ and the sliding motion within $[T^*, T]$. Besides, the minimum inter-event time has been given, and it can be found that the lower-bound inter-event time of dynamic ETS is strict smaller than the static ETS. Moreover, compared with the discrete-time dynamic ETS, $\eta(t) > 0$ in continuous-time case is promised by the different parameters selection.

REFERENCES

Amato, F., Ariola, M., and Dorato, P. (2001). Finite-time control of linear systems subject to parametric uncertainties and disturbances. *Automatica*, 37(9), 1459–1463.

Cao, Z., Niu, Y., and Song, J. (2019). Finite-time sliding mode control of markovian jump cyber-physical systems against randomly occurring injection attacks. *IEEE Transactions on Automatic Control*, Doi:10.1109/TAC.2019.2926156.

Dolk, V.S., Borgers, D.P., and Heemels, W.P.M.H. (2017). Output-based and decentralized dynamic event-triggered control with guaranteed $\mathcal{L}_p$- gain performance and zeno-freeness. *IEEE Transactions on Automatic Control*, 62(1), 34–49.

ElBsat, M.N. and Yaz, E.E. (2013). Robust and resilient finite-time bounded control of discrete-time uncertain nonlinear systems. *Automatica*, 49(7), 2292–2296.

Girard, A. (2015). Dynamic triggering mechanisms for event-triggered control. *IEEE Transactions on Automatic Control*, 60(7), 1992–1997.

Gu, Z., Yue, D., and Tian, E. (2018). On designing of an adaptive event-triggered communication scheme for nonlinear networked interconnected control systems. *Information Sciences*, 422, 257–270.

Kumari, K., Bandyopadhyay, B., Kim, K.S., and Shim, H. (2019). Output feedback based event-triggered sliding mode control for delta operator systems. *Automatica*, 103, 1–10.

Li, J. and Niu, Y. (2019). Sliding mode control subject to Rice channel fading. *IET Control Theory and Applications*, Doi:10.1049/iet-cta.2019.0130.

Peng, C. and Li, F. (2018). A survey on recent advances in event-triggered communication and control. *Information Sciences*, 457-458, 113–125.

Polyakov, A. and Fridman, L. (2014). Stability notions and lyapunov functions for sliding mode control systems. *Journal of the Franklin Institute*, 351(4), 1831–1865.

Song, J. and Niu, Y. (2019). Dynamic event-triggered sliding mode control: Dealing with slow sampling singularly perturbed systems. *IEEE Transactions on Circuits and Systems II: Express Briefs*, Doi:10.1109/TCSI.2019.2929554.

Song, J., Niu, Y., and Zou, Y. (2017). Finite-time stabilization via sliding mode control. *IEEE Transactions on Automatic Control*, 62(3), 1478–1483.

Tabuada, P. (2007). Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9), 1680–1685.

Wu, L., Gao, Y., Liu, J., and Li, H. (2017). Event-triggered sliding mode control of stochastic systems via output feedback. *Automatica*, 82, 79–92.

Zhao, H. and Niu, Y. (2019). Finite-time sliding mode control of switched systems with one-sided lipschitz nonlinearity. *Journal of the Franklin Institute*, Doi:10.1016/j.jfranklin.2019.05.019.

Zhao, H., Niu, Y., and Zhao, J. (2019). Event-triggered sliding mode control of uncertain switched systems under denial-of-service attacks. *Journal of the Franklin Institute*, Doi:10.1016/j.jfranklin.2019.03.04.