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Design of Reduced-order Observers and Output Feedback Controllers for Sampled-data Strict-feedback Systems with Time-varying Sampling Intervals

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Abstract: We consider the design of reduced-order observers and output feedback stabilizing controllers for sampled-data strict-feedback systems with time-varying sampling intervals. We introduce a nominal sampling interval to construct the Euler model and we use it to design reduced-order observers, state feedback controllers, and observer-based output feedback controllers. Then we give the sufficient conditions that the designed observers and controllers achieve the desired control performance for the exact model of sampled-data systems.

Keywords: Sampled-data strict-feedback systems, time-varying sampling intervals, Reduced-order observers, Output feedback controllers.

1. INTRODUCTION

Modern and practical control systems use digital computers with zero-order holds and samplers to control plants. Such control systems are called sampled-data control systems. Recently, the use of wired or wireless communication networks, in which sensor data and control commands are communicated, becomes popular in the sampled-data control systems, because of many merits (for details see Hespanha et al (2007), Zhang et al (2001), and references therein). In such sampled-data control systems, sampling intervals become time-varying. It is well-known that the analysis and design of sampled-data systems with time-varying sampling intervals are more difficult than those for sampled-data systems with constant sampling intervals (Cloosterman et al (2010), Hetel et al (2017)).

The analysis and the design of linear sampled-data systems with time-varying sampling intervals have been widely discussed by many researchers (for details see Hetel et al (2017) and the references therein). For nonlinear sampled-data systems with time-varying sampling intervals, the emulation-like design of controllers and observers has been given (Nesic and Teel (2004), Postoyan and Nesic (2012a), Postoyan and Nesic (2012b)). But the design of controllers and observers based on approximate discrete-time models has not been actively discussed. Recently, the design frameworks of nonlinear sampled-data control systems with constant sampling intervals based on approximate discrete-time models (Arcak and Nesic (2004), Nesic, Teel, and Kokotovic (1999)) have been extended to those of nonlinear sampled-data systems with time-varying sampling intervals (van de Wouw et al (2012)). When state feedback controllers, which are designed based on approximate discrete-time models using a nominal sampling interval, globally asymptotically (GA) stabilize approximate models, the sufficient conditions on a nominal sampling interval and the upper and lower bounds of time-varying sampling intervals are given for the semiglobally practically asymptotic (SPA) stability of the exact model (van de Wouw et al (2012)).

In this paper we consider the sampled-data strict-feedback system with time-varying sampling intervals:

$$\dot{x}_c = f_1(x_c) + g(x_c)z_c,\ \dot{z}_c = f_2(x_c, z_c, u_c) \tag{1}$$

with $y(k) = x_c(s_k)$ where $x_c \in \mathbf{R}^{n_x}$ and $z_c \in \mathbf{R}^{n_z}$ are the states, $u_c \in \mathbf{R}^m$ is the control input given by $u_c(t) = u_c(s_k) =: u(k)$ for any $t \in [s_k, s_{k+1})$ and $k \in \mathbf{N}_0 := \{0, 1, 2, ...\}$, $y \in \mathbf{R}^{n_x}$ is the sampled observation, and $s_k \geq 0$ are monotone increasing sampling times satisfying $0 = s_0 < s_1 < \cdots < s_k < s_{k+1} < \cdots$. When sampling intervals are constant and known for the system (1), the design of reduced-order observers and output feedback controllers has been discussed (Katayama (2016)). Here we extend the results in Katayama (2016) to the same design problems for the sampled-data system (1). Following van de Wouw et al (2012), we first introduce a nominal sampling interval T^* to construct the Euler model of the system (1). We use Katayama (2016) to design reduced-order observers and give sufficient conditions that the design observers are semiglobal and practical in sampling intervals for the exact model of the sampled-data system (1). Then we design state feedback controllers that GA stabilize the Euler model and we give similar sufficient conditions for the SPA stability of the closed-loop exact model. Finally, we combine the designed observers and state feedback controllers to construct output feedback controllers and give sufficient conditions for the SPA stability of the closed-loop exact model. We also give numerical examples to illustrate the proposed design of reduced-order observers and output feedback controllers.

Notation: Let $|\cdot|$ be a norm of vectors and matrices, and $\mathbf{B}_r = \{x \in \mathbf{R}^n | |x| \leq r\}$. A function α is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$)

if it is continuous, zero at zero and strictly increasing. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function β: $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for any fixed $t \geq 0$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$ the function $\beta(s, \cdot)$ is decreasing to zero as its argument tends to infinity (Khalil (2002)). For simplicity of expression, we write $f(\eta_1(\cdot), \eta_2(\cdot)) = f(\eta_1, \eta_2)(\cdot)$.

2. PRELIMINARY RESULTS

For the sampled-data system (1), we assume that time-varying sampling intervals $T_k = s_{k+1} - s_k$ satisfy $T_k \in [T_m, T_M]$ for any $k \in \mathbf{N}_0$ where T_M and T_m are the maximal and minimal sampling intervals, respectively and they are design parameters that can be assigned arbitrarily. Since $u_c(t) = u(k)$ for any $t \in [s_k, s_{k+1})$, the exact (discrete-time) model of the system (1) is given by

$$x(k+1) = F^e_{1T_k}(x, z, u)(k), \; z(k+1) = F^e_{2T_k}(x, z, u)(k) \quad (2)$$

with $y(k) = x(k)$ where $F^e_{1T_k}(x, z, u)(k) = x(k) + \int_{s_k}^{s_k+T_k} [f_1(x_c) + g(x_c)z_c](s)ds$ and $F^e_{2T_k}(x, z, u)(k) = z(k) + \int_{s_k}^{s_k+T_k} f_2(x_c(s), z_c(s), u(k))ds$. Since $T_k \in [T_m, T_M]$ is unknown, the exact model (2) cannot be used for design purposes and we must use the approximate models to design observers and controllers. We now introduce a nominal sampling interval $T^* \in (T_m, T_M]$ and consider the following exact and Euler models of the system (1)

$$x(k+1) = F^e_{1T^*}(x, z, u)(k), \; z(k+1) = F^e_{2T^*}(x, z, u)(k) \quad (3)$$

and

$$x(k+1) = F^a_{1T^*}(x, z)(k), \; z(k+1) = F^a_{2T^*}(x, z, u)(k) \quad (4)$$

with $y(k) = x(k)$, respectively where $F^a_{1T^*}(x, z) = x + T^*[f_1(x) + g(x)z]$, $F^a_{2T^*}(x, z, u) = z + T^* f_2(x, z, u)$, and $(F^e_{1T^*}, F^e_{2T^*})$ is given by $(F^e_{1T_k}, F^e_{2T_k})$ with $s_k = kT^*$ and $T_k = T^*$. Let $\chi_c = [\, x_c^T \;\; z_c^T \,]^T$, $\chi = [\, x^T \;\; z^T \,]^T$, $f_2(\chi, u) = f_2(x, z, u)$, $f(\chi, u) = \begin{bmatrix} f_1(x) + g(x)z \\ f_2(\chi, u) \end{bmatrix}$ and $F^i_T(\chi, u) = F^i_T(x, z, u) = \begin{bmatrix} F^i_{1T}(x, z, u) \\ F^i_{2T}(x, z, u) \end{bmatrix}$ for $i = e, a$. Then the system (1) is rewritten by

$$\dot{\chi}_c = f(\chi_c, u_c), \quad y(k) = x_c(s_k). \quad (5)$$

The discrete-time models (2)-(4) are also rewritten, respectively by

$$\chi(k+1) = F^e_{T_k}(\chi, u)(k), \quad y(k) = x(k), \quad (6)$$
$$\chi(k+1) = F^e_{T^*}(\chi, u)(k), \quad y(k) = x(k), \quad (7)$$
$$\chi(k+1) = F^a_{T^*}(\chi, u)(k), \quad y(k) = x(k). \quad (8)$$

For given positive real numbers $(\Delta_x, \Delta_z, \Delta_u)$, let $\Omega = \mathbf{B}_{\Delta_x} \times \mathbf{B}_{\Delta_z} \times \mathbf{B}_{\Delta_u}$ and assume
A1: 1) f_1, f_2, and g are locally Lipshitz and there exist L_{f_2}, L_f, $l_g > 0$ satisfying $|f_2(x, z, u) - f_2(\bar{x}, \bar{z}, \bar{u})| \leq L_{f_2}(|x - \bar{x}| + |z - \bar{z}| + |u - \bar{u}|)$, $|f(\chi, u) - f(\bar{\chi}, \bar{u})| \leq L_f(|\chi - \bar{\chi}| + |u - \bar{u}|)$, and $|g(x)| \leq l_g$ for any (x, z, u), $(\bar{x}, \bar{z}, \bar{u}) \in \Omega$ where $\bar{\chi} = [\, \bar{x}^T \;\; \bar{z}^T \,]^T$.
2) $f_1(0) = 0$, $f_2(0, 0, 0) = 0$.

By **A1**, there exists $T_1^\# > 0$ such that $F^e_T(x, z, u)$ is well-defined for any $(x, z, u) \in \Omega$ and $T \in (0, T_1^\#)$. It is well-known that there exist $\hat{\gamma} \in \mathcal{K}$ and $T_2^\# \in (0, T_1^\#]$ satisfying $|F^e_T(x, z, u) - F^a_T(x, z, u)| \leq T\hat{\gamma}(T)$ for any $(x, z, u) \in \Omega$ and $T \in (0, T_2^\#)$, i.e., $F^a_T(x, z, u)$ is one step consistent with $F^e_T(x, z, u)$ (Nesic, Teel, and Kokotovic (1999)).

3. DESIGN OF REDUCED-ORDER OBSERVERS

Since $y(k) = x(k)$, we use the Euler model (4) to design discrete-time reduced-order observers that estimate $z(k)$ of the exact model (2). We first assume:
A2: On the compact set, $\phi(\cdot) = g^T g(\cdot) \in \mathbf{R}^{n_z \times n_z}$ is nonsingular and $\phi^{-1}(\cdot)$ is bounded, i.e., for given $\Delta_x > 0$, there exists $l_\phi > 0$ satisfying $|\phi^{-1}(x)| \leq l_\phi$ for any $|x| \leq \Delta_x$.

Since $z(k) = \phi^{-1}(y)g^T(y)\{(\rho y - y)/T^* - f_1(y)\}(k) =: \Psi_{T^*}(y, \rho y)(k)$, we can consider the system

$$\begin{aligned} \hat{z}(k+1) &= \hat{z}(k) + T^* f_2(y, \Psi_{T^*}(y, \rho y), u)(k) \\ &\quad + T^* H[\Psi_{T^*}(y, \rho y) - \hat{z}](k) \\ &= (I - T^* H)\hat{z}(k) + T^* \Pi_{T^*}(y, \rho y, u)(k) \quad (9) \\ &=: O_{T^*}(\hat{z}, y, \rho y, u)(k) \end{aligned}$$

where $(\rho y)(k) = y(k+1)$ and $\Pi_{T^*}(y, \rho y, u) = H\Psi_{T^*}(y, \rho y) + f_2(y, \Psi_{T^*}(y, \rho y), u)$ (Katayama (2016)). Let $e_z = z - \hat{z}$. Then we have $e_z(k+1) = (I - T^* H)e_z(k)$ and we assume:
A3: Let $\hat{T} > 0$ be given and $H = \mathrm{diag}\{h(1), ..., h(n_z)\}$ where $h(i) > 0$ satisfies $|1 - T^* h(i)| < 1$ for any $T^* \in (0, \hat{T}]$.

Remark 1. For any $T^* \in (0, \hat{T}]$, there exists a positive definite matrix P_{T^*} satisfying $(I - T^* H)^T P_{T^*}(I - T^* H) - P_{T^*} = -T^* I$. Then $P_{T^*} > 0$ is given by

$$P_{T^*} = \mathrm{diag}\{\frac{1}{h(1)[2 - T^* h(1)]}, \cdots, \frac{1}{h(n_z)[2 - T^* h(n_z)]}\}$$

and $q_1 \leq \lambda_{min}(P_{T^*}) \leq \lambda_{max}(P_{T^*}) \leq q_2$ where $q_1 = 1/(2h_{max})$, $q_2 = 1/(h_{min}[2 - \hat{T}h_{max}])$, $h_{min} = \min_{i=1,..,n_z} h(i)$, $h_{max} = \max_{i=1,..,n_z} h(i)$, and $\lambda_{min}(P_{T^*})$ and $\lambda_{max}(P_{T^*})$ are the minimal and maximal eigenvalues of P_{T^*}, respectively.

Let $F^i_T = F^i_T(x, z, u) = F^i_T(\chi, u)$, $F^i_{jT} = F^i_{jT}(x, z, u) = F^i_{jT}(\chi, u)$ for $i = e, a$ and $j = 1, 2$, $O^e_{T^*} = O_{T^*}(\hat{z}, y, F^e_{1T}, u)$, $O^a_{T^*} = O_{T^*}(\hat{z}, y, F^a_{1T^*}, u)$, $\Psi^e_{T^*} = \Psi_{T^*}(y, F^e_{1T})$, and $\Psi^a_{T^*} = \Psi_{T^*}(y, F^a_{1T^*})$. Let $V_{T^*}(z, \hat{z}) = e_z^T P_{T^*} e_z$. Then

$$q_1|e_z|^2 \leq V_{T^*}(z, \hat{z}) \leq q_2|e_z|^2, \quad (10)$$
$$V_{T^*}(F^a_{2T^*}, O^a_{T^*}) - V_{T^*}(z, \hat{z}) = -T^*|e_z|^2. \quad (11)$$

Let $T^\# = \min\{T_2^\#, \hat{T}\}$, $0 < T_m \leq T_M < T^\#$, $T^* = \epsilon T_m + (1 - \epsilon)T_M$, and $0 \leq \epsilon < 1$. Assume **A1**-**A3**. For given positive real numbers (D_e, d_e) and $(\Delta_x, \Delta_z, \Delta_u)$, let $R = q_2 D_e^2$, $r = q_1 d_e^2/2$, $\hat{\Delta}_z \geq \Delta_z + \sqrt{R/q_1} = \Delta_z + D_e\sqrt{q_2/q_1}$,

$$\begin{aligned} \Delta_{11} &= \sup_{(x,z,u)\in\Omega, |\hat{z}|\leq\hat{\Delta}_z} \max\{|F^e_{T^*}|, |F^a_{T^*}|, |\Psi^a_{T^*}|, |O^a_{T^*}|\}, \\ \Delta_{12} &= \sup_{(x,z,u)\in\Omega, |\hat{z}|\leq\hat{\Delta}_z, T\in(0,T^\#)} \max\{|F^e_T|, |\Psi^e_{T^*}|, |O^e_{T^*}|\}, \\ \Delta_1 &= \max\{\Delta_{11}, \Delta_{12}, \Delta_z, \hat{\Delta}_z\}. \quad (12) \end{aligned}$$

Remark 2. For any $|z_1|$, $|z_2|$, $|\hat{z}_1|$, $|\hat{z}_2| \leq \Delta_1$, there exists $L_V > 0$ satisfying

$$|V_{T^*}(z_1, \hat{z}_1) - V_{T^*}(z_2, \hat{z}_2)| \leq L_V(|z_1 - z_2| + |\hat{z}_1 - \hat{z}_2|). \quad (13)$$

Finally, let

$$\begin{aligned}\gamma_{ob} &= \gamma_{ob}(T^*, T_m, T_M) \\ &= L_V[1 + (h_{max} + L_{f_2})l_\phi l_g][\gamma_{un} + \hat{\gamma}(T^*)], \\ \gamma_{un} &= \gamma_{un}(T^*, T_m, T_M) \\ &= \frac{e^{L_f T^*}}{T^*}(\sqrt{\Delta_x^2 + \Delta_z^2} + \Delta_u)(e^{L_f M_T} - 1), \\ M_{ob} &= \min\{\frac{R-r}{T^*}, \frac{r}{T^*}, \frac{r}{2q_2}\} \end{aligned} \quad (14)$$

where $M_T = \max\{|T_M - T^*|, |T_m - T^*|\}$.

Theorem 3. Consider the exact model (2) and the reduced-order observer (9) with **A1-A3**. Assume that T^*, T_m, and T_M satisfy

$$\gamma_{ob}(T^*, T_m, T_M) \leq M_{ob} \quad (15)$$

for given $(\Delta_x, \Delta_z, \Delta_u)$ and (D_e, d_e). Then if $|e_z(0)| \leq D_e$ and $(x, z, u)(k) \in \Omega$ for any $k \in \mathbf{N}_0$,

$$|e_z(k)| \leq \sqrt{\frac{q_2}{q_1}} \exp\left(-\frac{1-\epsilon}{4q_2}kT_m\right)|e_z(0)| + d_e \quad (16)$$

for any $k \in \mathbf{N}_0$ where $e_z = z - \hat{z}$. Moreover, since (D_e, d_e) can be chosen arbitrarily, the reduced-order observer (9) is semiglobal and practical in T_k for the exact model (2).

To prove Theorem 3, we introduce the following result.

Lemma 4. Assume **A1-A3**. Let $(\Delta_x, \Delta_z, \Delta_u, \hat{\Delta}_z)$ be given. Then for any $|y| \leq \Delta_x$, $|z| \leq \Delta_z$, $|u| \leq \Delta_u$, $|\hat{z}| \leq \hat{\Delta}_z$, and $T \in (0, T^\#)$, we have

$$\frac{V_{T^*}(F^e_{2T}, O^e_{T^*}) - V_{T^*}(z, \hat{z})}{T_M} \leq -\frac{T^*}{T_M}(|e_z|^2 - \gamma_{ob}). \quad (17)$$

Proof. Let $\Delta V = [V_{T^*}(F^e_{2T}, O^e_{T^*}) - V_{T^*}(z, \hat{z})]/T_M$. Then by (11) and (13), we obtain

$$\Delta V \leq -\frac{T^*}{T_M}|e_z|^2 + \frac{L_V}{T_M}(|F^e_{2T} - F^a_{2T^*}| + |O^e_{T^*} - O^a_{T^*}|).$$

By direct calculation and **A1-A3**, we have

$$\Delta V \leq -\frac{T^*}{T_M}|e_z|^2 + \frac{L_V}{T_M}[1 + (h_{max} + L_{f_2})l_\phi l_g]|F^e_T - F^a_{T^*}|.$$

By the one-step consistency between $F^e_{T^*}$ and $F^a_{T^*}$, we obtain $|F^e_T - F^a_{T^*}| \leq |F^e_T - F^e_{T^*}| + T^*\hat{\gamma}(T^*)$. Let $T^* \leq T$ (we have the same result for $T^* > T$). By **A1** and $T^\# \leq T^\#_2$, the solutions $\chi_c(t)$ of $\dot{\chi}_c = f(\chi_c(t), u)$, $\chi_c(0) = \chi = [x^T \quad z^T]^T$ satisfy $|\chi_c(t)| \leq \Delta$ for any $t \in [0, T^\#)$ and we have $|F^e_T - F^e_{T^*}| \leq L_f \int_{T^*}^T [|\chi_c(s)| + |u|]ds$. By the Bellman-Gronwall's inequality (Khalil (2002)), $|F^e_T - F^e_{T^*}| \leq e^{L_f T^*}(|\chi| + |u|)(e^{L_f M_T} - 1)$. Since $(x, z, u) \in \Omega$, $|\chi| \leq \sqrt{\Delta_x^2 + \Delta_z^2}$ and

$$\begin{aligned}|F^e_T - F^a_{T^*}| \leq e^{L_f T^*}(\sqrt{\Delta_x^2 + \Delta_z^2} + \Delta_u)(e^{L_f M_T} - 1) \\ + T^*\hat{\gamma}(T^*).\end{aligned}$$

Hence we have (17).

Proof of Theorem 3. Let x and z be the states of the exact model (2) and assume $(x, z, u)(k) \in \Omega$, for any $k \in \mathbf{N}_0$. For simplicity of notation, let $T = T_k$, $(x, z, e_z) = (x, z, e_z)(k)$ and $(\rho x, \rho z) = (x, z)(k+1)$.

Similar to Katayama (2016), we can show that if $r \leq V_{T^*}(z, \hat{z}) \leq R$, then

$$\frac{V_{T^*}(F^e_{2T}, O^e_{T^*}) - V_{T^*}(z, \hat{z})}{T_M} \leq -\frac{T^*}{T_M}\frac{|e_z|^2}{2} \quad (18)$$

and if $V_{T^*}(z, \hat{z}) \leq r$, then $V_{T^*}(F^e_{2T}, O^e_{T^*}) \leq R$. Thus $V_{T^*}(z, \hat{z})(0) \leq R$ implies $V_{T^*}(z, \hat{z})(k) \leq R$ for any $k \in \mathbf{N}_0$.

Since $T \leq T_M$ and $T^* = \epsilon T_m + (1-\epsilon)T_M$, $T^*/T_M \geq 1 - \epsilon$ and we have

$$\frac{V_{T^*}(\rho z, \rho \hat{z}) - V_{T^*}(z, \hat{z})}{T} \leq -\frac{(1-\epsilon)|e_z(k)|^2}{2}$$

when $V_{T^*}(z, \hat{z}) \geq r$. Then similar to Arcak and Nesic (2004), we can show that if $V_{T^*}(z, \hat{z})(0) \leq R$, then

$$\begin{aligned}V_{T^*}(z, \hat{z})(k) \leq \max\{&\exp\left(-\frac{1-\epsilon}{2q_2}kT_m\right)V_{T^*}(z, \hat{z})(0), \\ &r + T^*\gamma_{ob}\}.\end{aligned} \quad (19)$$

If $q_2|e_z(0)|^2 \leq R$, then $V_{T^*}(z, \hat{z})(0) \leq R$ and we obtain

$$|e_z(k)| \leq \sqrt{\frac{q_2}{q_1}} \exp\left(-\frac{1-\epsilon}{4q_2}kT_m\right)|e_z(0)| + \sqrt{\frac{r + T^*\gamma_{ob}}{q_1}}.$$

By (10), (14), and (15), $T^*\gamma_{ob} \leq r$ and from the definitions of r and R, we have $|e_z(0)| \leq \sqrt{R/q_2} = D_e$ and $(r + T^*\gamma_{ob})/q_1 \leq 2r/q_1 = d_e^2$. Hence $|e_z(0)| \leq D_e$ implies (16).

Since $\frac{d}{dT}(e^{L_f T}/T) = e^{L_f T}(L_f T - 1)/T^2$, we have $L_f e \leq e^{L_f T}/T \leq \max\{e^{L_f T_m}/T_m,\ e^{L_f T_M}/T_M\}$. Also note

$$e^{L_f M_T} - 1 = L_f M_T \sum_{i=1}^{\infty} \frac{(L_f M_T)^{i-1}}{i!}$$

and $\hat{\gamma} \in \mathcal{K}$. This implies the existence of $0 < T_m < T^* \leq T_M$ satisfying $\gamma_{ob} \leq \nu$ for given $\nu > 0$. Hence there exist $0 < T_m < T^* \leq T_M$ satisfying $\gamma_{ob} \leq M_{ob}$ for given (D_e, d_e) and the reduced-order observer (9) is semiglobal and practical in $T_k \in [T_m, T_M]$ for the exact model (2).

4. DESIGN OF STATE FEEDBACK CONTROLLERS

Let state feedback controllers $u(k) = u_{T^*}(\chi(k))$ be designed based on the Euler model (4) and assume:

B1: There exist $W_{T^*}(\chi)$, α_1, $\alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{K}$, and $T^\#_3 > 0$ satisfying $\alpha_1(|\chi|) \leq W_{T^*}(\chi) \leq \alpha_2(|\chi|)$ and $W_{T^*}(F^a_{T^*}(\chi, u_{T^*}(\chi))) - W_{T^*}(\chi) \leq -T^*\alpha_3(|\chi|)$ for any $\chi \in \mathbf{R}^{n_x+n_z}$ and $T^* \in (0, T^\#_3)$.

B2: For given $\Delta_\chi > 0$, there exist L_W, $T^\#_4 > 0$ satisfying $|W_{T^*}(\chi) - W_{T^*}(\bar{\chi})| \leq L_W|\chi - \bar{\chi}|$ for any $|\chi|$, $|\bar{\chi}| \leq \Delta_\chi$ and $T^* \in (0, T^\#_4)$.

B3: For given $\Delta_\chi > 0$, there exist L_u, $T^\#_5 > 0$ satisfying $|u_{T^*}(\chi)| \leq L_u|\chi|$ for any $|\chi| \leq \Delta_\chi$ and $T^* \in (0, T^\#_5)$.

Remark 5. By **B1**, there always exist $T^\#_3 > 0$, $W_{T^*}(\chi)$, α_1, $\alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{K}$. Let (D_χ, d_χ) be given positive real numbers, Δ_x, $\Delta_z > 0$ satisfy $\Delta_x^2 + \Delta_z^2 \le D_\chi^2$, L_u, $T_5^\# > 0$ satisfy **B3** with $\Delta_\chi = D_\chi$, and $\Delta_u \ge L_u D_\chi$. Let $T_4^\#$, $L_W > 0$ satisfy **B2** with

$$\Delta_2 = \sup_{(x,z,u)\in\Omega, T\in(0,T_2^\#)} \max\{|F_T^e(\chi,u)|, |F_T^a(\chi,u)|, D_\chi\}. \quad (20)$$

Let $T^\# := \min\{T_2^\#, ..., T_5^\#\}$, $0 < T_m \le T_M < T^\#$, $r = \alpha_1(d_\chi)/4$, and $R = \alpha_2(D_\chi)$.

By Remark 5, we can choose a nominal sampling interval $T^* = \epsilon T_m + (1-\epsilon)T_M$ for some $0 \le \epsilon < 1$. We also obtain $|\chi| \le D_\chi \le \Delta_2$, $|u_{T^*}(\chi)| \le L_u D_\chi \le \Delta_u$, and $|F_T^e(\chi,u)|, |F_{T^*}^e(\chi,u)|, |F_{T^*}^a(\chi,u)| \le \Delta_2$ for any $(x,z,u) \in \Omega$. Finally, let

$$\begin{aligned}\gamma_{sf} &= \gamma_{sf}(T^*, T_m, T_M)\\ &= L_W[\frac{e^{L_f T^*}}{T^*}(1+L_u)D_\chi(e^{L_f M_T}-1) + \hat{\gamma}(T^*)],\\ M_{sf} &= \min\{\frac{1}{2}\alpha_3 \circ \alpha_2^{-1}(r), \frac{R-r}{T^*}, \frac{r}{T^*}\}.\end{aligned}$$

Then we have the following results. Their proofs are similar to those of Lemma 4 and Theorem 3, respectively.

Lemma 6. Let $D_\chi > 0$ be given and assume **A1**, **B1-B3**. Then

$$\frac{W_{T^*}(F_T^e(\chi, u_{T^*}(\chi))) - W_{T^*}(\chi)}{T_M} \le -\frac{T^*}{T_M}[\alpha_3(|\chi|) - \gamma_{sf}]$$

for any $|\chi| \le D_\chi$ and $T \in (0, T^\#)$.

Theorem 7. Consider the closed-loop exact model

$$\chi(k+1) = F_{T_k}^e(\chi(k), u_{T^*}(\chi(k))) \quad (21)$$

with **A1** and **B1-B3**. Assume that T^*, T_m, and T_M satisfy $\gamma_{sf}(T^*, T_m, T_M) \le M_{sf}$ for given (D_χ, d_χ). Then if $|\chi(0)| \le D_\chi$, there exists $\beta \in \mathcal{KL}$ satisfying $|\chi(k)| \le \beta(|\chi(0)|, kT_m) + d_\chi$ for any $k \in \mathbf{N}_0$. Moreover, since (D_χ, d_χ) can be chosen arbitrarily, the closed-loop exact model (21) is SPA stable.

5. DESIGN OF OUTPUT FEEDBACK CONTROLLERS

Let $\hat{z}$ be the state of the reduced-order observer (9) and $\hat{\chi} = [\,x^T \;\; \hat{z}^T\,]^T = [\,y^T \;\; \hat{z}^T\,]^T$. Consider the output feedback controller

$$u(k) = u_{T^*}(\hat{\chi}(k)),\; \hat{z}(k+1) = O_{T^*}(\hat{z}, y, \rho y, u)(k) \quad (22)$$

that is obtained by combining $u = u_{T^*}(\chi)$ and the reduced-order observer (9). Let $e_z = z - \hat{z}$. Then $\hat{\chi} = \chi - [\,0 \;\; e_z^T\,]^T$ and the controller (22) is rewritten as

$$\begin{aligned}u(k) &= u_{T^*}(\chi - [\,0 \;\; e_z^T\,]^T)(k), \quad (23)\\ e_z(k+1) &= z(k+1) - O_{T^*}(\hat{z}, y, \rho y, u)(k).\end{aligned}$$

Let $\mu = [\,\chi^T \;\; e_z^T\,]$. Then the closed-loop systems of the models (6)-(8) and the controller (23) are given by

$$\begin{aligned}\mu(k+1) &= \tilde{F}_{T_k}^e(\mu(k)) \quad (24)\\ &= \begin{bmatrix} F_{T_k}^e(\chi,\hat{u}) \\ F_{2T_k}^e(\chi,\hat{u}) - O_{T^*}(\hat{z}, y, F_{1T_k}^e(\chi,\hat{u}), \hat{u})\end{bmatrix}(k),\\ \mu(k+1) &= \tilde{F}_{T^*}^i(\mu(k))\\ &= \begin{bmatrix} F_{T^*}^i(\chi,\hat{u}) \\ F_{2T^*}^i(\chi,\hat{u}) - O_{T^*}(\hat{z}, y, F_{1T^*}^i(\chi,\hat{u}), \hat{u})\end{bmatrix}(k)\end{aligned}$$

for $i = e, a$, respectively where $\hat{u} = u_{T^*}(\hat{\chi})$.

Remark 8. Assume **A1-A3** and **B1-B3**. Then by **B1**, there exist $T_3^\# > 0$, $W_{T^*}(\chi)$, α_1, $\alpha_2 \in \mathcal{K}_\infty$, and $\alpha_3 \in \mathcal{K}$. Let (D, d) be given positive real numbers, (D_χ, D_e) positive real numbers such that $D_\chi^2 + D_e^2 \le D^2$, and (Δ_x, Δ_z) positive real numbers such that $\Delta_x^2 + \Delta_z^2 \le D_\chi^2$. Let L_u, $T_5^\# > 0$ satisfy **B3** with $\Delta_\chi = D_\chi + D_e$, $L_u(D_\chi + D_e) \le \Delta_u$, and $\Omega = \mathbf{B}_{\Delta_x} \times \mathbf{B}_{\Delta_z} \times \mathbf{B}_{\Delta_u}$. Then $|u_{T^*}(\chi)|$, $|u_{T^*}(\hat{\chi})| \le \Delta_u$ for any $|\mu| \le D$. Let $T_4^\#$, $L_W > 0$ satisfy **B2** with $\Delta_\chi = \Delta_2$. Let $T^\# := \min\{\hat{T}, T_2^\#, ..., T_5^\#\}$, $0 < T_m \le T_M < T^\#$, $\hat{\Delta}_z \ge \Delta_z + D_e$, and $T^* \in (T_m, T_M]$. Let Δ_1 and Δ_2 be defined by (12) and (20), respectively and $\Delta = \max\{\Delta_1, \Delta_2\}$. Then $|F_T^e(\chi,u)|$, $|F_T^e(\chi,\hat{u})|$, $|F_{T^*}^e(\chi,u)|$, $|F_{T^*}^e(\chi,\hat{u})|$, $|F_{T^*}^a(\chi,u)|$, $|F_{T^*}^a(\chi,\hat{u})| \le \Delta$ for any $|\mu| \le D$ and $T \in (0, T^\#)$ where $u = u_{T^*}(\chi)$ and $\hat{u} = u_{T^*}(\hat{\chi})$.

We now replace **B3** by the following slightly stronger assumption.

C1: $u_{T^*}(\chi)$ is continuous and $u_{T^*}(0) = 0$, i.e., there exists $L_u > 0$ satisfying $|u_{T^*}(\chi) - u_{T^*}(\bar{\chi})| \le L_u|\chi - \bar{\chi}|$ for any $|\chi|$, $|\bar{\chi}| \le \Delta$.

Let c be a positive real number and $U_{T^*}(\mu) = W_{T^*}(\chi) + cV_{T^*}(z, \hat{z})$ as a candidate of Lyapunov functions that guarantees the SPA stability of the closed-loop exact model (24). Then by (10) and **B1**, we have $\alpha_1(|\chi|) + cq_1|e_z|^2 \le U_{T^*}(\mu) \le \alpha_2(|\chi|) + cq_2|e_z|^2$ and by Katayama (2014), there exist α_{U1} $\alpha_{U2} \in \mathcal{K}_\infty$ satisfying $\alpha_{U1}(|\mu|) \le \alpha_1(|\chi|) + cq_1|e_z|^2$ and $\alpha_{U2}(|\mu|) \ge \alpha_2(|\chi|) + cq_2|e_z|^2$. Thus we obtain

$$\alpha_{U1}(|\mu|) \le U_{T^*}(\mu) \le \alpha_{U2}(|\mu|). \quad (25)$$

Let $T = T_k$, $u = u_{T^*}(\chi)$, $\hat{u} = u_{T^*}(\hat{\chi})$, and $\Delta U = [U_{T^*}(\tilde{F}_T^e(\mu)) - U_{T^*}(\mu)]/T_M$. Then $\Delta U = \Delta W + c\Delta V$ where $\Delta W = [W_{T^*}(F_T^e(\chi,\hat{u})) - W_{T^*}(\chi)]/T_M$, $\Delta V = [V_{T^*}(F_{2T}^e(x,z,\hat{u}), O_{T^*}(\hat{z}, y, F_{1T}^e(x,z,\hat{u}), \hat{u}) - V_{T^*}(z,\hat{z})]/T_M$. By Lemma 4, $\Delta V \le -(T^*/T_M)(|e_z|^2 - \gamma_{ob})$ and by Lemma 6, we have

$$\begin{aligned}\Delta W \le\ & \frac{W_{T^*}(F_T^e(\chi,u)) - W_{T^*}(\chi)}{T_M}\\ &+ \frac{|W_{T^*}(F_T^e(\chi,\hat{u})) - W_{T^*}(F_T^e(\chi,u))|}{T_M}\\ \le & -\frac{T^*}{T_M}[\alpha_3(|\chi|) - \gamma_{sf}]\\ &+ \frac{L_W}{T_M}|F_T^e(\chi,\hat{u}) - F_T^e(\chi,u)|.\end{aligned}$$

Since $|F_T^e(\chi,\hat{u}) - F_T^e(\chi,u)| \le L_f\int_0^T[|\tilde{\chi}_c(s) - \chi_c(s)| + |\hat{u} - u|]ds$, we use the Bellman-Gronwall's inequality to have $|\tilde{\chi}_c(t) - \chi_c(t)| \le |\hat{u} - u|(e^{L_f t} - 1)$ and $|F_T^e(\chi,\hat{u}) - F_T^e(\chi,u)| \le T^*|\hat{u} - u|(e^{L_f T_M} - 1)/T_m$. By **C1** we have

$$\Delta W \le -\frac{T^*}{T_M}\alpha_3(|\chi|) + \frac{T^*}{T_M}[\gamma_{sf} + \Theta(T_m, T_M)|e_z|],$$
$$\Theta(T_m, T_M) = L_W L_u \frac{e^{L_f T_M} - 1}{T_m}.$$

Let ε be a sufficiently small positive real number and

$$c = 1 + \frac{\Theta(T_m, T_M)}{\tilde{c}}, \quad \tilde{c} = \frac{\varepsilon}{\Theta(T_m, T_M)}. \tag{26}$$

Then we have

$$\Delta U \le -\frac{T^*}{T_M}[|e_z|^2 + \alpha_3(|\chi|)] + \frac{T^*}{T_M}(\gamma_{sf} + c\gamma_{ob}) + \frac{T^*}{T_M}\Theta(T_m, T_M)|e_z|\left(1 - \frac{|e_z|}{\tilde{c}}\right).$$

If $|e_z| < \tilde{c}$, then

$$\Theta(T_m, T_M)|e_z|\left(1 - \frac{|e_z|}{\tilde{c}}\right) \le \Theta(T_m, T_M)|e_z| \le \varepsilon$$

and if $|e_z| \ge \tilde{c}$, then $\Theta(T_m, T_M)|e_z|\left(1 - \frac{|e_z|}{\tilde{c}}\right) \le 0$. Hence we have

$$\Delta U \le -\frac{T^*}{T_M}[|e_z|^2 + \alpha_3(|\chi|)] + \frac{T^*}{T_M}\gamma_{of}$$

where $\gamma_{of}(T^*, T_m, T_M) = \gamma_{sf} + c\gamma_{ob} + \varepsilon$. By Katayama (2014), there exists $\alpha_{U3} \in \mathcal{K}$ satisfying $\alpha_{U3}(|\mu|) \le |e_z|^2 + \alpha_3(|\chi|)$ and we obtain

$$\Delta U \le -\frac{T^*}{T_M}\alpha_{U3}(|\mu|) + \frac{T^*}{T_M}\gamma_{of}(T^*, T_m, T_M). \tag{27}$$

For given positive real numbers (D, d), let $r = \alpha_{U1}(d)/4$, $R = \alpha_{U2}(D)$, and

$$M_{of} = \min\{\frac{1}{2}\alpha_{U3} \circ \alpha_{U2}^{-1}(r), \frac{R - r}{T^*}, \frac{r}{T^*}\}. \tag{28}$$

Then we have the following result. Its proof is similar to the proof of Theorem 3.

Theorem 9. Consider the closed-loop exact model (24) with **A1-A3**, **B1**, **B2**, and **C1**. For given positive real numbers (D, d), assume that T^*, T_m, and T_M satisfy $\gamma_{of}(T^*, T_m, T_M) \le M_{of}$. Then if $|\mu(0)| \le D$, there exists $\beta \in \mathcal{KL}$ satisfying $|\mu(k)| \le \beta(|\mu(0)|, kT_m) + d$ for any $k \in \mathbf{N}_0$. Furthermore, since (D, d) can be chosen arbitrarily, the closed-loop exact model (24) is SPA stable.

6. A NUMERICAL EXAMPLE

Example 1. Consider

$$\dot{x}_c = z_c, \ \dot{z}_c = -x_c + 0.01z_c(1 - x_c^2) \tag{29}$$

with $y(k) = x_c(s_k)$. The system (29) satisfies **A1** and **A2**. Let $\hat{T} = 1$ and $H = 0.5$. Then **A3** is satisfied and a reduced-order observer is given by

$$\hat{z}(k+1) = (1 - T^*H)\hat{z}(k) + T^*[\Psi_{T^*} - y + 0.01\Psi_{T^*}(1 - y^2)](k) \tag{30}$$

where $\Psi_{T^*} = (\rho y - y)/T^*$. Let $(x_c, z_c)(0) = (0.4, 0.4)$ and $\hat{z}(0) = 0$. Then we have $|x_c(t)|$, $|z_c(t)| \le 0.8$ for any $t \in [0, 30]$ and we obtain $L_{f2} = 1.0128$ and $L_f = 1.749$. By

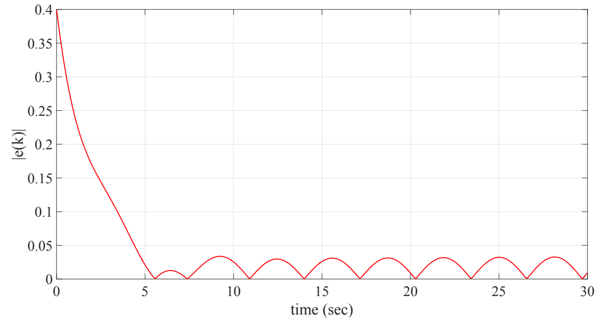

Fig. 1. Time response of $|e(k)|$ for $T_k = 0.1$

direct calculation, we also have $\hat{\gamma}(T) = 1.1314(\exp(L_f T) - L_f T - 1)/T$ and $\hat{\gamma}(0.1) = 0.1836$.

First we assume $T_k = 0.1$ (s). Then from the simulation result (Fig 1), the offset $d_e = d_e(0.1)$ in (16) is less than 0.05 and there is a gap between 0.05 and $d_e(0.1)$ for $t \ge 5$. Let $T_M = 0.1$ (s), $T^* = (T_M + T_m)/2$, and consider three cases $T_m = 0.09$, 0.08, and 0.07 (s). Then we have $(T_m, \gamma_{un}) = \{(0.09, 0.1235), (0.08, 0.2596), (0.07, 0.4105)\}$. We use the values of γ_{un} and $\hat{\gamma}(0.1)$, and $|0.05 - d_e(0.1)|$ to expect that (16) with $d_e = 0.05$ is satisfied for $(T_M, T_m) = (0.1, 0.09)$ and it is not satisfied for $(T_M, T_m) = (0.1, 0.07)$. In fact, Figs 2 and 3 show the time responses of $|e_z(k)|$ for $T_m = 0.09$ and $T_m = 0.07$ (s), respectively where the black, red, and blue lines correspond to different sequences of sampling intervals. As we see Figs 2 and 3, we have a desired result.

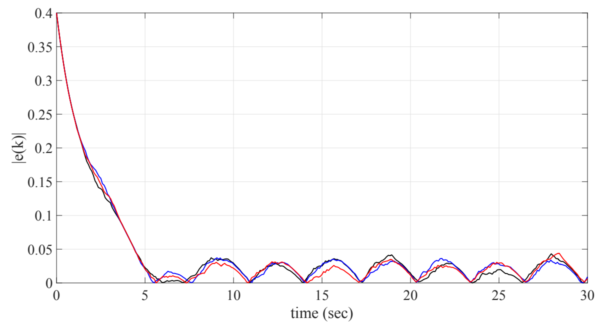

Fig. 2. Time response of $|e_z(k)|$ for $T_k \in [0.09, 0.1]$

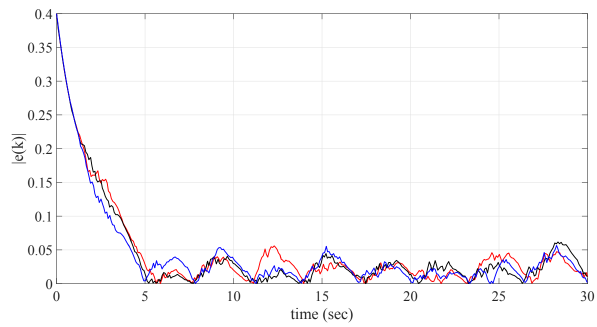

Fig. 3. Time response of $|e_z(k)|$ for $T_k \in [0.07, 0.1]$

Example 2. Consider

$$\dot{x}_c = x_c^3 - x_c + z_c, \ \dot{z}_c = u_c, \ y(k) = x_c(s_k) \tag{31}$$

where $u_c(t) = u(k)$ for any $t \in [s_k, s_{k+1})$. The system (31) satisfies **A1** and **A2**. Let $\hat{T} = 1$ and $T^* \in (0, \hat{T})$ a nominal sampling interval. Then the Euler model of the system (31) is given by

$$x(k+1) = r_{T^*}(k),\ \ z(k+1) = z(k) + T^* u(k), \qquad (32)$$

and $y(k) = x(k)$ where $r_{T^*} = r_{T^*}(x,z) = x + T^*(x^3 - x + z)$.

Let $H = 0.8$. Then **A3** is satisfied for any $T^* \in (0, \hat{T})$ and the reduced-order observer is given by

$$\hat{z}(k+1) = (1 - T^* H)\hat{z}(k) + T^*(\Psi_{T^*} + u)(k) \qquad (33)$$

where $\Psi_{T^*} = (\rho y - y)/T^* - y^3 + y$. Consider the subsystem $x(k+1) = r_{T^*}(x,z)(k)$ where z is a virtual input. Then the state feedback controller that GA stabilizes this subsystem is given by $z = \kappa_{T^*}(x) = -x^3$ and we have $r^{\kappa}_{T^*}(x) = r_{T^*}(x, \kappa_{T^*}(x)) = (1 - T^*)x$. Let $W_{1T^*}(x) = x^2$. Then we have $W_{1T^*}(r^{\kappa}_{T^*}) - W_{1T^*}(x) = -T^* x^2 - T^*(1 - T^*)x^2 \le -T^* x^2$ for any $T^* \in (0, \hat{T})$. All conditions of Theorem 3 in Nesic and Teel (2006) are satisfied and there exist $(u_{T^*}(\chi), W_{T^*}(\chi))$ satisfying **B1-B3** for the Euler model (32). Moreover, $u_{T^*}(\chi)$ is given by

$$u_{T^*}(\chi) = \frac{\Delta\kappa_{T^*}}{T^*} + \frac{\Delta\varrho}{T^*}\xi - \frac{1}{2}\varrho(|r_{T^*}|)\xi \qquad (34)$$

where $\Delta\kappa_{T^*} = \kappa_{T^*}(r_{T^*}) - \kappa_{T^*}(x)$, $\Delta\varrho = \varrho(|r_{T^*}|) - \varrho(|x|)$, $\xi = [x - \kappa_{T^*}(x)]/\varrho(|x|)$, $\varrho(s) = 1/[2\omega(s)(1+s)]$, and $\omega(s) = 1 + 2s[1 + \hat{T}(3s^2 + s + 1)]$. We can also show that the state feedback controller (34) satisfies **C1**. Then the output feedback controller is given by

$$\begin{aligned} u(k) &= u_{T^*}(\hat{\chi}(k)), \\ \hat{z}(k+1) &= (1 - T^* H)\hat{z}(k) + T^*(\Psi_{T^*} + u)(k) \end{aligned} \qquad (35)$$

where $\hat{\chi} = [\,y \quad \hat{z}\,]^T$.

Let $(x_c, z_c)(0) = (1.5, 0)$. First we assume $T_k = T^*$, i.e., T_k is constant. Then numerical simulations show that the SPA stability of the closed-loop exact model is guaranteed for any $T^* \in (0, 0.169]$ (s). Next we assume $T_k \in [T_m, T_M]$ for any $k \in \mathbf{N}_0$. Let $T_m = 0.1$ and $T^* = (T_m + T_M)/2$. From numerical simulations, we have $T_M = 0.127$ (s), i.e., the SPA stability of the closed-loop system is guaranteed for any $T_k \in [0.1, 0.127]$ (s). Figure 4 shows the state trajectories (x, z) of the closed-loop system where the black, blue, and red lines express the state trajectories for the sampling interval sequences $(T_0, T_1, T_2, \cdots) = (0.1245, 0.1034, 0.1247, \cdots)$, $(T_0, T_1, T_2, \cdots) = (0.1116, 0.1050, 0.1244, \cdots)$, and $(T_0, T_1, T_2, \cdots) = (0.1228, 0.1093, 0.1211, \cdots)$, respectively. As we see Fig 4, the designed output feedback controller (35) achieves the SPA stability of the closed-loop system.

7. CONCLUSION

In this paper we have considered the design of semiglobal and practical reduced-order observers and SPA stabilizing output feedback controllers for the exact model of sampled-data strict-feedback systems with time-varying sampling intervals. We have given the sufficient conditions that the reduced-order observers and controllers designed based on the Euler model achieve the desired control performance for the exact model.

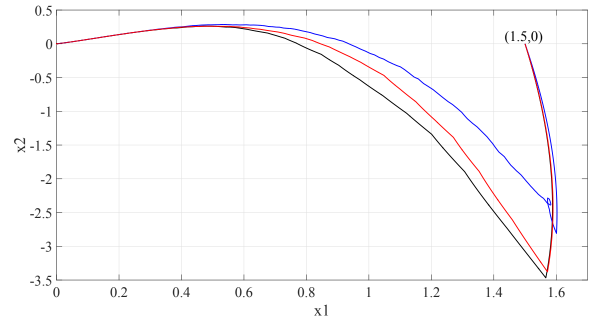

Fig. 4. State trajectories (x, z) of the closed-loop system

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