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Limit cycle global asymptotic stability of continuous-time switched affine systems ⋆

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Abstract: This paper treats global asymptotic stability of a limit cycle, determined by the designer, for continuous-time switched affine systems, taking into account sampled and non-sampled switching rules. More specifically, our goal is to design a state-dependent switching function assuring global asymptotic stability of a limit cycle, which is determined from criteria of interest related to the system steady-state response. The conditions, expressed in terms of differential linear matrix inequalities, are based on a time-varying quadratic Lyapunov function and take into account a guaranteed cost, which assures a suitable performance level for the system transient response. These conditions can be converted into two linear matrix inequalities, making the problem simple-to-solve. An academic example is used for validation and comparison.

Keywords: Switched affine systems, sampled-data switching function, limit cycle global asymptotic stability, continuous-time.

1. INTRODUCTION

Switched affine systems are composed of a set of subsystems and a switching function (rule) responsible to activate one of them at each instant of time. These systems are very common in several engineering areas, mainly in the power electronics domain, see Cardim et al. (2009), Egidio et al. (2017), Sferlazza et al. (2019) and Beneux et al. (2019), among others. They present several equilibrium points composing a region in the state space. One of the control problems of great interest is to design a state or output dependent switching function in order to govern the state trajectories to a desired equilibrium point inside this region. References Deaecto et al. (2010), Deaecto (2016), Trofino et al. (2009) and Seatzu et al. (2006) have accomplished this goal by means of conditions that assure global asymptotic stability. Generally, the desired equilibrium point is not common to any subsystem and, consequently, an arbitrarily high switching frequency is required to maintain the trajectories at this point. In this case, the switching rule may be impossible to be implemented due to physical limitations, as response time of switches or sampling period in embedded systems. Moreover, a high switching frequency yields chattering, an undesired phenomenon that may cause equipment damage and impair control performance. To circumvent this inconvenient, several contributions deal with sampled-data switching function design or study switched affine systems in the discrete-time domain, where the switching rate is naturally limited by the discrete nature of the system. References Sferlazza et al. (2019), Hauroigne et al. (2011), Hetel and Fridman (2013), Sanchez et al. (2019), in the continuous-time domain, and Deaecto and Geromel (2017), Egidio and Deaecto (2019), in the discrete-time domain, are some examples. In all these cases, practical stability is taken into account, where the trajectories are orchestrated by the switching function to a region as small as possible containing the desired equilibrium point. Unfortunately, nothing can be concluded about the steady-state behaviour of the state trajectories once inside this region.

This paper deals with global asymptotic stability of a limit cycle for continuous-time switched affine systems. The limit cycle of interest is designed based on criteria chosen by the designer related to system steady-state behaviour. As a first step, conditions for global asymptotic stability of a desired equilibrium point are provided based on a convex time-varying Lyapunov function, which are less conservative than other methodologies available in the literature based on quadratic Lyapunov functions. Afterwards, these conditions are generalized to cope with asymptotic stability of a desired limit cycle, considering sampled and non-sampled switching functions. The literature presents few results dealing with this topic. Very recently, reference Benmiloud et al. (2019) has approached this theme but assuring only local stability and in Egidio et al. (2020) global asymptotic stability of a limit cycle defined in the discrete-time domain has been studied. Unlike the latter, the proposed continuous-time approach adopts conditions described in terms of differential linear matrix inequalities (DLMIs), which can be rewritten as the solution of simple linear matrix inequalities (LMIs), permitting to employ

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readily available tools for the design step. Additionally, these conditions allow for a guaranteed cost that assures a suitable performance level for the transient response. An illustrative example is used for comparison and validation.

The notation is standard. For real vectors or matrices, $(')$ refers to their transpose. For symmetric matrices, $(\bullet)$ denotes each of their symmetric blocks. The symbols $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. The set $\mathbb{K} = \{1, \cdots, N\}$ is composed of the N first positive natural numbers. For a symmetric matrix, $X > (<)\ 0$ denotes a positive (negative) definite matrix. The unit simplex Λ is composed of all nonnegative vectors $\lambda \in \mathbb{R}^N$, such that $\sum_{j\in\mathbb{K}} \lambda_j = 1$. The convex combination of matrices $\{X_1, \cdots, X_N\}$ is denoted by $X_\lambda = \sum_{i\in\mathbb{K}} \lambda_i X_i$, $\lambda \in \Lambda$. For a vector $w \in \mathbb{R}^m$, norms are denoted as $\|w\|_2 = \sqrt{w'w}$ and $\|w\|_\infty = \max_{1\le i\le m} |w_i|$. Finally, the greatest integer less or equal to a is denoted as $\lfloor a \rfloor$. In general, but not always, we denote $f(t_n) = f[n]$ where $0 \le t_n \in \mathbb{R}$ and $n \in \mathbb{N}$.

2. PROBLEM STATEMENT

Consider a switched affine system with realization

$$\dot{x}(t) = A_{\sigma(t)} x(t) + b_{\sigma(t)},\ x(0) = x_0 \tag{1}$$

$$z(t) = E_{\sigma(t)} x(t) \tag{2}$$

defined for all $t \ge 0$, where $x \in \mathbb{R}^{n_x}$ is the state, $z \in \mathbb{R}^{n_z}$ is the controlled output and $\sigma(t) : \mathbb{R} \to \mathbb{K}$ is the switching function that selects one of the subsystems (A_i, b_i, E_i) as active at each switching instant. The signal $\sigma(t)$ is piecewise constant being modelled as follows

$$\sigma(t) = \sigma[n], \forall t \in [t_n, t_{n+1}) \tag{3}$$

where t_n and t_{n+1} are two successive switching instants such that $t_0 = 0$ and $t_{n+1} - t_n = h > 0, \forall n \in \mathbb{N}$.

Let us assume that system (1)-(2) admits a periodic solution $x_e(t)$ with period $T = n_h h > 0$, for some $n_h \in \mathbb{N}$ chosen by the designer. The switching signal along this trajectory is a periodic sequence $c[0], c[1], \cdots$ with period n_h meaning that $c[n+n_h] = c[n]$ for all $n \in \mathbb{N}$, which yields $\sigma(t) = c[n],\ \forall t \in [t_n, t_{n+1})$. The limit cycle associated to this switching function is given by

$$\mathcal{X}_e(c) = \{x_e(t)\ :\ x_e(t) = x_e(t+T),\ t \ge 0\} \tag{4}$$

where $x_e(t)$ for an arbitrary $t \in [t_n, t_{n+1})$ is given by

$$x_e(t) = e^{A_{c[n]}(t-t_n)} x_e(t_n) + \int_{t_n}^{t} e^{A_{c[n]}(t-\tau)} d\tau\ b_{c[n]} \tag{5}$$

valid for all $n \in \mathbb{N}$ and respecting the boundary condition $x_e(t_0) = x_e(t_{n_h})$. This solution clearly satisfies equation (1).

Defining the auxiliary state variable $\xi(t) = x(t) - x_e(t)$, we obtain the following system

$$\dot{\xi}(t) = A_{\sigma(t)} \xi(t) + \ell_{\sigma(t)}(t),\ \xi(0) = \xi_0 \tag{6}$$

$$z_e(t) = E_{\sigma(t)} \xi(t) \tag{7}$$

where $\ell_i(t) = A_i x_e(t) + b_i - \dot{x}_e(t)$ for all $i \in \mathbb{K}$, $z_e(t) = z(t) - E_{\sigma(t)} x_e(t)$ and $\xi_0 = x(0) - x_e(0)$. Notice that, $\dot{x}_e(t)$ is well defined in each time interval $[t_n, t_{n+1})$ for each $n \in \mathbb{N}$. Whenever $\xi(t) = 0$ the limit cycle $x_e(t)$ is reached by the solution of system (1)-(2). Moreover, for $\sigma(t)$ such that $\ell_{\sigma(t)}(t) = 0$, we obtain the limit cycle (4).

Our main goal is to design a state-dependent switching function $\sigma(t) = u(x(t)) : \mathbb{R}^{n_x} \to \mathbb{K}$ to assure global asymptotic stability of the limit cycle $\mathcal{X}_e(c)$. In other words, we want to design $\sigma(t) = u(x(t))$ in order to guarantee that the origin $\xi = 0$ of (6)-(7) is a globally asymptotically stable equilibrium point, assuring a suitable upper bound for the cost

$$J = \int_0^\infty z_e(t)' z_e(t) dt \tag{8}$$

The design conditions are based on the following convex time-varying Lyapunov function

$$v(\xi(t), t) = \xi(t)' P(t) \xi(t) \tag{9}$$

with $P(t) = P(t+T) > 0$. Among other possibilities provided in Gonçalves et al. (2019), we consider that $P(t)$ is a piecewise linear function. Indeed, taking into account that the time interval $[0, T)$ is split into n_T subintervals of length $\eta = T/n_T$ and that n_T is a multiple of n_h, we have

$$P(t) = P_p + \frac{P_{p+1} - P_p}{\eta}(t - p\eta) \tag{10}$$

valid in the time segment $t \in [p\eta, (p+1)\eta)$ for each $p = 0, \cdots, n_T - 1$ with matrices P_p to be determined. Notice that this function is continuous and piecewise differentiable but may not be continuously differentiable at the isolated points $t_p = p\eta$. Moreover, the function (9) has been recently adopted in several contexts that deal with DLMIs, see for instance, Gonçalves et al. (2019), Allerhand and Shaked (2013) and Baldi et al. (2018) where the last two references treat the problem of switched systems stabilization.

3. MAIN RESULTS

This section presents our main results. Firstly, a methodology to generate a set of limit cycle candidates is provided employing criteria based on the desired system steady-state behaviour. Afterwards, the design conditions assuring asymptotic stability of an equilibrium point are presented, which are generalized to cope with limit cycle global asymptotic stability taking into account an associated guaranteed cost.

3.1 Limit cycle generation

The key point to determine the continuous-time trajectory $x_e(t)$, associated to the sequence $c[n] \in \mathbb{K}$, is to find the subset of points $x_e(t_n)$ at which the switching events occur. The remaining points of $x_e(t)$ are obtained directly from (5). Hence, evaluating the limit cycle (5) only at the switching instants t_n, we can define the following equivalent discrete-time switched affine system

$$x_e[n+1] = F_{c[n]} x_e[n] + g_{c[n]} \tag{11}$$

where $x_e[n] = x_e(t_n),\ \forall n \in \mathbb{N}$, and

$$F_i = e^{A_i h},\ g_i = \int_0^h e^{A_i \tau} d\tau\ b_i \tag{12}$$

Before continuing, let us provide some definitions borrowed from Deaecto and Geromel (2018). For a given positive value $n_h \in \mathbb{N}$, which defines the number of switching events inside the period T, let $\mathfrak{C}(n_h) = \mathbb{K}^{n_h}$ be the set obtained from the Cartesian product of $\mathbb{K}$ by itself n_h times. It contains N^{n_h} elements $c \in \mathfrak{C}(n_h)$, which are finite sequences $c = (c[0], \cdots, c[n_h - 1])$. Associated to each $c \in \mathfrak{C}(n_h)$ there exists a limit cycle $\mathcal{X}_e(c)$ with points $x_e(t_n)$ obtained from the linear equation

$$\tilde{F}(c)\tilde{x}_e = -\tilde{g}(c) \tag{13}$$

where $\tilde{x}_e = [x_e[0]' \ x_e[1]' \ \cdots \ x_e[n_h-1]']'$, matrix

$$\tilde{F}(c) = \begin{bmatrix} F_{c[0]} & -I & 0 & \cdots & 0 \\ 0 & F_{c[1]} & -I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -I & 0 & 0 & \cdots & F_{c[n_h-1]} \end{bmatrix} \tag{14}$$

and $\tilde{g}(c) = [g'_{c[0]} \ g'_{c[1]} \ \cdots \ g'_{c[n_h-1]}]'$. Notice that the boundary condition $x_e(t_0) = x_e(t_{n_h})$ has been taken into account in (13). After determining $x_e(t_n)$ for $n = \{0, \cdots, n_h - 1\}$, the points between two successive switching events can be obtained from (5), completing the entire trajectory $x_e(t)$. The N^{n_h} different sequences of $\mathfrak{C}(n_h)$ allows us to determine a family of possible limit cycles as being

$$\mathfrak{X} = \{\mathcal{X}_e(c) \ : \ c \in \mathfrak{C}(n_h)\} \tag{15}$$

The search for a suitable limit cycle $\mathcal{X}_e^* \in \mathfrak{X}$ can be constrained to a subset $\mathfrak{X}_s \subseteq \mathfrak{X}$, which is related to some criterion specified by the designer. Given a reference point x_*, a possible one can be expressed as

$$\mathfrak{X}_s = \left\{ \mathcal{X}_e \in \mathfrak{X} \ : \ \frac{1}{T}\int_0^T \|\Gamma(x_e(t) - x_*)\|_2 \, dt < \alpha \right\} \tag{16}$$

which contains limit cycles whose medium distance between $\Gamma x_e(t)$ and Γx_* over $t \in [0, T)$ is smaller than α, with the matrix Γ and the positive scalar α being provided by the designer. Alternatively, when the goal is to bound the ripple amplitudes of the state trajectories, another option to be adopted is the subset

$$\mathfrak{X}_s = \left\{ \mathcal{X}_e \in \mathfrak{X} \ : \ \max_{t \in [0,T)} \|\Gamma(x_e(t) - x_*)\|_\infty < \alpha \right\} \tag{17}$$

with, as in the former case, the matrix Γ and the scalar α are provided by the designer. Notice that the criterion choice depends exclusively on the interest of the designer with respect to the steady-state behaviour of the system trajectories leading to different sets $\mathfrak{X}_s$. The set of sequences c associated with each $\mathcal{X}_e(c) \in \mathfrak{X}_s$ is defined as $\mathfrak{C}_s(n_h) \subseteq \mathfrak{C}(n_h)$. The next subsection deals specifically with the design of a state-dependent switching function.

3.2 Stability and guaranteed cost

After obtaining the subset $\mathfrak{X}_s \in \mathfrak{X}$ containing all the limit cycles candidates satisfying the criteria chosen by the designer, the idea is to determine a state-dependent switching function to govern the state trajectories of the system (1)-(2) asymptotically towards the limit cycle $\mathcal{X}_e(c^*) \in \mathfrak{X}_s$ that minimizes an upper bound for the cost defined in (8).

Non-sampled switching function

At this first moment, let us consider a simpler case where the switching frequency is arbitrary and the goal is to assure global asymptotic stability of the equilibrium point $x_e \in X_e$ with

$$X_e = \{x_e \ : \ x_e = -A_\lambda^{-1} b_\lambda, \ \lambda \in \Lambda\} \tag{18}$$

known as the set of attainable equilibrium points, see Deaecto et al. (2010). This problem has already been treated in the literature adopting different Lyapunov functions, as for instance, the quadratic one (Deaecto et al. (2010), Deaecto (2016)), the max-type one (Scharlau et al. (2014)), among others. The next theorem provides a solution based on the time-varying Lyapunov function given in (9).

Theorem 1. Consider system (1)-(2), let $T > 0$ and the equilibrium point $x_e \in X_e$ with its associated vector $\lambda \in \Lambda$ be given. Defining $Q_i = E_i' E_i$, if there exists $P(t) > 0$ satisfying the differential linear matrix inequality

$$\dot{P}(t) + A_\lambda' P(t) + P(t) A_\lambda + Q_\lambda < 0 \tag{19}$$

for all $t \in [0, T)$ with the boundary condition $P(0) = P(T) > 0$, then the state-dependent switching function $\sigma(t) = u(\xi(t), t)$ with

$$u(\xi, t) = \arg\min_{i \in \mathbb{K}} \xi'(2P(t)A_i + Q_i)\xi + 2\ell_i' P(t)\xi \tag{20}$$

where $\xi(t) = x(t) - x_e$, assures global asymptotic stability of the equilibrium point $x_e \in X_e$. Moreover, the following guaranteed cost

$$J < \xi_0' P(0) \xi_0 \tag{21}$$

corresponding to the initial condition $\xi(0) = \xi_0 \neq 0$ is satisfied.

Proof: To ease the notation the time dependency of some variables is omitted. The time derivative of the Lyapunov function $v(\xi, t)$ provided in (9) along an arbitrary trajectory of system (6)-(7), in the time interval $t \in [0, T)$, yields

$$\begin{aligned} \dot{v}(\xi, t) &= \dot{\xi}' P \xi + \xi' P \dot{\xi} + \xi' \dot{P} \xi \\ &= \xi'(A_\sigma' P + P A_\sigma + Q_\sigma + \dot{P})\xi + 2\ell_\sigma' P \xi - z_e' z_e \\ &= \min_{i \in \mathbb{K}} \xi'(A_i' P + P A_i + Q_i + \dot{P})\xi + 2\ell_i' P \xi - z_e' z_e \\ &= \min_{\lambda \in \Lambda} \xi'(A_\lambda' P + P A_\lambda + Q_\lambda + \dot{P})\xi + 2\ell_\lambda' P \xi - z_e' z_e \\ &\le \xi'(A_\lambda' P + P A_\lambda + Q_\lambda + \dot{P})\xi + 2\ell_\lambda' P \xi - z_e' z_e \\ &< -z_e' z_e \end{aligned} \tag{22}$$

where the third equality comes from the switching function (20), the fourth one is due to a known property of the minimum operator and the last inequality is a consequence of (19) and the fact that $\ell_\lambda = A_\lambda x_e + b_\lambda = 0$ because $x_e \in X_e$. Due to the boundary condition $P(0) = P(T) > 0$ and the periodic continuation $P(t) = P(t + T)$, inequality (22) holds for all $t \ge 0$ proving the asymptotic stability of the equilibrium point $x_e \in X_e$. Now, integrating both sides of (22) from $t = 0$ until $t \to \infty$, we obtain the guaranteed cost (21), concluding thus the proof. □

The fact that $\dot{P}(t)$ is sign undefined contributes to reduce the conservatism of the conditions compared with others available in the literature, see Deaecto and Santos (2015), Sanchez et al. (2019), and Sferlazza et al. (2019) as some examples.

At this point, an important remark is how to solve the differential linear matrix inequality (19). As it has been proposed in Gonçalves et al. (2019), there are different manners depending on the class of functions considered for $P(t)$. In this paper, we have adopted $P(t)$ piecewise linear as in (10), which implies that (19) can be converted into two LMIs as presented in the next lemma borrowed from Gonçalves et al. (2019).

Lemma 1. Let $n_T \ge 1$ be given. The piecewise linear function $P(t) : [0, T) \to \mathbb{R}^{n_x \times n_x}$ given in (10) is a solution of the differential linear matrix inequality

$$\mathcal{L}(\dot{P}(t), P(t)) < 0 \tag{23}$$

if and only if for each $p = 0, \cdots, n_T - 1$, the LMIs

$$\mathcal{L}\left(\frac{P_{p+1} - P_p}{\eta}, P_p\right) < 0 \tag{24}$$

$$\mathcal{L}\left(\frac{P_{p+1} - P_p}{\eta}, P_{p+1}\right) < 0 \tag{25}$$

with $\eta = T/n_T$ are feasible.

This lemma will be very useful to solve the DLMIs presented throughout this paper. Applying Lemma 1, Theorem 1 can be solved by the solution of the following convex optimization problem

$$\inf_{\mathcal{P}} \xi_0' P_0 \xi_0 \tag{26}$$

where $\mathcal{P} = \{P_0, \cdots, P_{n_T}\}$, subject to $P_p > 0$ and

$$A_\lambda' P_p + P_p A_\lambda + Q_\lambda + \frac{P_{p+1} - P_p}{\eta} < 0 \tag{27}$$

$$A_\lambda' P_{p+1} + P_{p+1} A_\lambda + Q_\lambda + \frac{P_{p+1} - P_p}{\eta} < 0 \tag{28}$$

for all $p = 0, 1, \cdots, n_T - 1$ and $\eta = T/n_T$ and the boundary condition $P_0 = P_{n_T} > 0$. These conditions are certainly less conservative than the available ones based on a single Lyapunov matrix. However, when the state trajectories reach the equilibrium point, the switching frequency is arbitrarily high in order to maintain them at this point, causing an undesired phenomenon known as chattering, see Hauroigne et al. (2011) and Sanchez et al. (2019). In order to avoid the occurrence of this phenomenon during the steady state response of the system, instead of considering a single equilibrium point let us govern the state trajectories towards a suitable limit cycle $\mathcal{X}_e(c) \in \mathfrak{X}_s$. The next theorem presents design conditions regarding this goal.

Theorem 2. Consider system (6)-(7) with $\xi(0) = \xi_0 \neq 0$ and let $T > 0$, $n_h \in \mathbb{N}$ and the set of limit cycle candidates $\mathcal{X}_e(c) \in \mathfrak{X}_s$ with the correspondent $c \in \mathfrak{C}_s(n_h)$ be given. Define $Q_i = E_i' E_i$, if there exists $P(t) > 0$ satisfying the optimization problem

$$\min_{\mathcal{X}_e(c) \in \mathfrak{X}_s} \inf_{P(t)>0} \xi_0' P(0) \xi_0 \tag{29}$$

subject to the differential linear matrix inequality

$$\dot{P}(t) + A_{c(t)}' P(t) + P(t) A_{c(t)} + Q_{c(t)} < 0 \tag{30}$$

in the time interval $t \in [0, T)$, $c(t) = c[n] \in \mathbb{K}$, $\forall t \in [t_n, t_{n+1})$, with $c \in \mathfrak{C}_s(n_h)$, $n = 0, \cdots, n_h - 1$ and the boundary condition $P(0) = P(T) > 0$, then the state-dependent switching function $\sigma(t) = u(\xi(t), t)$ with

$$u(\xi, t) = \arg\min_{i \in \mathbb{K}} \xi'(2P(t)A_i + Q_i)\xi + 2\ell_i(t)' P(t)\xi \tag{31}$$

assures the global asymptotic stability of the limit cycle $\mathcal{X}_e(c^*)$ and the optimal solution of (29), associated to c^*, is a suitable upper bound for the cost (8).

Proof: Performing the time derivative of $v(\xi, t)$ along a trajectory of (6)-(7) in the time interval $t \in [0, T)$, and taking into account the switching function $\sigma(t) = u(\xi(t), t)$ given in (20), we obtain

$$\begin{aligned} \dot{v}(\xi, t) &= \min_{i \in \mathbb{K}} \xi'(A_i' P + P A_i + Q_i + \dot{P})\xi + 2\ell_i' P \xi - z_e' z_e \\ &\leq \xi'(A_c' P + P A_c + Q_c + \dot{P})\xi + 2\ell_c' P \xi - z_e' z_e \\ &< -z_e' z_e \end{aligned} \tag{32}$$

where the first inequality is a consequence of the minimum operator and the last one comes directly from the validity of (30) and the fact that $\ell_{c(t)}(t) = 0$ since $\mathcal{X}_e(c) \in \mathfrak{X}_s$. Finally, the guaranteed cost follows the same reasoning as in Theorem 1. □

Although chattering still can occur during the transient response, Theorem 2 avoids its occurrence during the steady-state response, making it more amenable for practical implementations when compared with Theorem 1. About the solution of Theorem 2, it is important to remark that the time interval $[0, T)$ has been split into n_T subintervals for the Lyapunov function and in n_h switching events $c(t) \in \mathbb{K}$. Nevertheless, it is not necessary that n_T be a multiple of n_h, this makes the problem easier to be implemented. Indeed, considering that $n_T \geq n_h$ and using Lemma 1, Theorem 2 can be solved as the solution of the following convex optimization problem

$$\min_{\mathcal{X}_e(c) \in \mathfrak{X}_s} \inf_{\mathcal{P}} \xi_0' P_0 \xi_0 \tag{33}$$

where $\mathcal{P} = \{P_0, \cdots, P_{n_T}\}$, subject to $P_p > 0$ and

$$A_{c[n]}' P_p + P_p A_{c[n]} + Q_{c[n]} + \frac{P_{p+1} - P_p}{\eta} < 0 \tag{34}$$

$$A_{c[n]}' P_{p+1} + P_{p+1} A_{c[n]} + Q_{c[n]} + \frac{P_{p+1} - P_p}{\eta} < 0 \tag{35}$$

where $\eta = T/n_T$, $n = \lfloor p/\theta \rfloor$ with $\theta = n_T/n_h$, for all $p = 0, 1, \cdots, n_T - 1$, $c[n] \in \mathfrak{C}_s(n_h)$ and the boundary condition $P_0 = P_{n_T}$. The next topic treats the case where the switching function is piecewise constant, which is our main concern in this paper.

Sampled-data switching function

Consider now that the switching function satisfies the constraint (3) for all $t \geq 0$, avoiding chattering occurrence during the transient and steady-state responses. In this situation, global asymptotic stability of a single equilibrium point is generally impossible to be assured. Most of the results available in the literature deals with practical stability, where the trajectories are guided to a region as small as possible containing the equilibrium point, see Hetel and Fridman (2013), Hauroigne et al. (2011), Sferlazza et al. (2019), among others. However, nothing is imposed to the state trajectories once inside this region. In order to control the steady-state response of the system, the idea is to govern the trajectories asymptotically towards a desired limit cycle. Very recently, reference Benmiloud et al. (2019) has treated this problem using an approach based on the hybrid Poincaré map, but assuring only local stability. For a periodic switching function, the next corollary presents conditions for the global asymptotic stability of a desired limit cycle.

Corollary 1. The conditions of Theorem 2 remain valid whenever the switching function (31) is replaced by $\sigma(t) = c(t)$ with $c(t) = c[n]$, $\forall t \in [t_n, t_{n+1})$, $c \in \mathfrak{C}_s(n_h)$ and $c[n + n_h] = c[n]$.

Proof: The proof is very simple and is based on the fact that the periodic switching function $\sigma(t) = c(t)$ is known for all $t \in [0, T)$. Indeed, the time derivative of $v(\xi, t)$ becomes

$$\begin{aligned} \dot{v}(\xi, t) &= \xi'(A_c' P + P A_c + Q_c + \dot{P})\xi + 2\ell_c' P \xi - z_e' z_e \\ &< -z_e' z_e \end{aligned} \tag{36}$$

where the last inequality comes from (30) and the fact that $\ell_{c(t)}(t) = 0$ since $\mathcal{X}_e(c) \in \mathfrak{X}_s$. The proof is concluded. $\square$

Based on this result, the idea is to obtain a sampled-data switching function that provides a true cost smaller than the one of Corollary 1. In this sense, we can adopt a different switching function, which is state-dependent, and consists in choosing at each switching instant the subsystem that guarantees stability and enhances the performance with respect to the periodic switching function.

Corollary 2. Considering $c(t) = c[n]$, $\forall t \in [t_n, t_{n+1})$ with $c \in \mathfrak{C}_s(n_h)$ and $c[n+n_h] = c[n]$, the conditions of Theorem 2 remain valid whenever the switching function (31) is replaced by $\sigma(t) = \sigma[n] = \nu(\xi(t_n))$, $\forall t \in [t_n, t_{n+1})$ and $n \in \mathbb{N}$ with

$$\nu(\xi(t_n)) = \arg\min_{i\in\mathbb{K}} \left\{ \int_{t_n}^{t_{n+1}} \xi(t)' Q_i \xi(t) dt + \xi(t_{n+1})' P(t_{n+1}) \xi(t_{n+1}) \right\} \tag{37}$$

where $\xi(t_{n+1}) = F_i \xi(t_n) + \hat{g}_{ic[n]}$ with $F_i = e^{A_i h}$,

$$\hat{g}_{ic[n]} = \int_{t_n}^{t_{n+1}} e^{A_i(t_{n+1}-\tau)} \hat{\ell}_{ic[n]}(\tau) d\tau \tag{38}$$

and $\hat{\ell}_{ic[n]}(t) = (A_i - A_{c[n]})x_e(t) + (b_i - b_{c[n]})$ and $h = T/n_h$.

Proof: Theorem 2 assures the existence of $P(t)$ solution of the DLMI (30) in the time interval $t \in [0, T)$. Hence $v(\xi, t)$ satisfies the inequality

$$v(\xi(t_n), t_n) \geq \int_{t_n}^{t_{n+1}} \xi(t)' Q_{c[n]} \xi(t) dt + v(\xi(t_{n+1}), t_{n+1}) \tag{39}$$

with $c[n] \in \mathbb{K}$. In particular, it holds for $c[n] = c^*[n] \in \mathbb{K}$ where $c = c^* \in \mathfrak{C}_s(n_h)$ and, consequently, we have

$$\begin{aligned} v(\xi(t_n), t_n) &\geq \int_{t_n}^{t_{n+1}} \xi(t)' Q_{c[n]} \xi(t) dt + v(\xi(t_{n+1}), t_{n+1}) \\ &\geq \int_{t_n}^{t_{n+1}} \xi(t)' Q_{\sigma[n]} \xi(t) dt + v(\xi(t_{n+1}), t_{n+1}) \end{aligned} \tag{40}$$

The conclusion is that the sampled-data switching strategy (37) assures global asymptotic stability and the same guaranteed cost as c^* does. The proof is concluded. $\square$

It is important to stress that the guaranteed cost provided by Theorem 2 is not reduced by the sampled-data switching control strategy (37). However, the true cost is, in general, smaller due to the minimization performed with respect to all $i \in \mathbb{K}$. For completeness, let us point out how to calculate the sampled-data switching function (37). Defining

$$\mathcal{A}_{ic[n]} = \begin{bmatrix} A_i & 0 & b_i \\ 0 & A_{c[n]} & b_{c[n]} \\ 0 & 0 & 0 \end{bmatrix}$$

for $t \in [t_n, t_{n+1})$, $n \in \mathbb{N}$ we have

$$\xi(t) = \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix}' e^{\mathcal{A}_{ic[n]}(t-t_n)} \begin{bmatrix} x(t_n) \\ x_e(t_n) \\ 1 \end{bmatrix}$$

Then, the switching function (37) can be rewritten as

$$\nu(\xi(t_n)) = \arg\min_{i\in\mathbb{K}} \begin{bmatrix} x(t_n) \\ x_e(t_n) \\ 1 \end{bmatrix}' \mathcal{Q}_{ic[n]} \begin{bmatrix} x(t_n) \\ x_e(t_n) \\ 1 \end{bmatrix} \tag{41}$$

where

$$\mathcal{Q}_{ic[n]} = \int_0^h e^{\mathcal{A}'_{ic[n]}\tau} \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} Q_i \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix}' e^{\mathcal{A}_{ic[n]}\tau} d\tau + e^{\mathcal{A}'_{ic[n]}h} \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} P(t_{n+1}) \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix}' e^{\mathcal{A}_{ic[n]}h} \tag{42}$$

The next example illustrates the proposed techniques.

4. NUMERICAL EXAMPLE

In this section, we evaluate the previous results by means of a numerical example whose data were borrowed from Egidio et al. (2020). Consider a discrete-time switched affine system (1)-(2) defined by matrices

$$A_1 = \begin{bmatrix} -4 & 3 \\ -3 & 2.5 \end{bmatrix}, A_2 = \begin{bmatrix} 4 & -1 \\ 1 & -2 \end{bmatrix}, b_1 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \tag{43}$$

that describes two unstable subsystems. The goal is to design a switching function capable of governing the state trajectories to some adequate steady-state behaviour such that the mean value of the first state component is as close as possible to -9. However, the switching frequency must not exceed 10 Hz, which can be fulfilled by adopting a switching function of the form (3) with $h = 0.1$ s. In order to obtain a globally asymptotically stable limit cycle with period $T = 1$ s, we choose $n_h = 10$. The adoption of $\Gamma = [1 \ 0]$, $x_* = [9 \ r]'$ where r is of no account and $\alpha = 1$ leads to a set of candidate limit cycles $\mathfrak{X}_s$ as in (16) with 10 candidates. Solving the optimisation problem (33) for $n_T = 50$ an optimal solution $\mathcal{P}$ along with $c^* = (1, 1, 1, 1, 1, 1, 1, 1, 1, 2)$ was found. This solution provides the optimal limit cycle $\mathcal{X}_e^* = \mathcal{X}_e(c^*)$ and allows the implementation of the sampled-data min-type switching function (37) given in Corollary 2. By numerical simulation we have obtained the system response starting from $x_0 = [5 \ 5]'$ that is shown in continuous-blue in Figure 1, on the state space and in Figure 2 as a function of time along with the resulting switching signal $\sigma(t)$. For comparison, a periodical switching function $\sigma(t) = c(t)$ and a non-sampled one, given in (20), were adopted and the resulting curves are given in Figure 1, respectively in dot-dashed red and dashed green. Moreover, the integral (8) was calculated for the trajectories obtained from the min-type sampled, periodic and non-sampled switching functions, providing $J = 387.53$, $J = 754.06$ and $J = 411.26$, respectively. Notice that all of them respected the obtained guaranteed cost $\xi_0' P_0 \xi_0 = 1{,}215.85 > J$ and that, as it can be seen, the actual cost related to the min-type sampled switching function was smaller than the non-sampled one.

5. CONCLUSION

Throughout this paper, sufficient conditions for the global asymptotic stability of a limit cycle $\mathcal{X}_e^*$, satisfying a desired steady-state criterion, are provided for continuous-time switched affine systems. The conditions are based on a convex time-varying Lyapunov function and described in terms of DLMIs, which are easy to solve, since they can be converted as the solution of a finite set of LMIs. Moreover, they taken into account a guaranteed cost that assure a good performance for the transient response. An

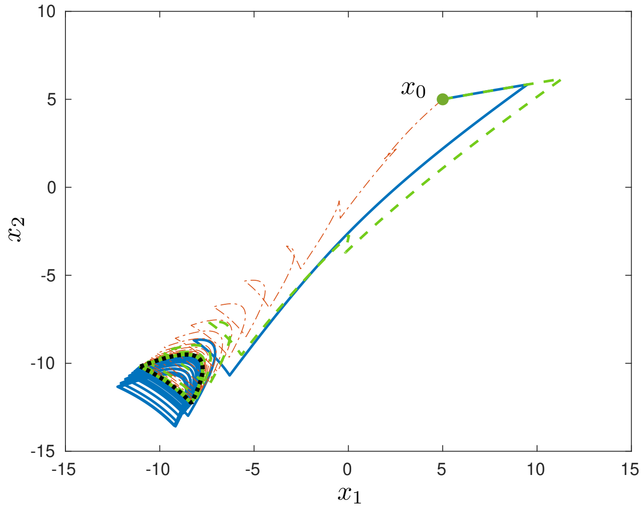

Fig. 1. System trajectories $x(t)$ with sampled switching function (37) (solid blue), non-sampled one (20) (dashed-green) and periodic one (dot-dashed red) along with the desired limit cycle $\mathcal{X}_e^*$ (dashed-black)

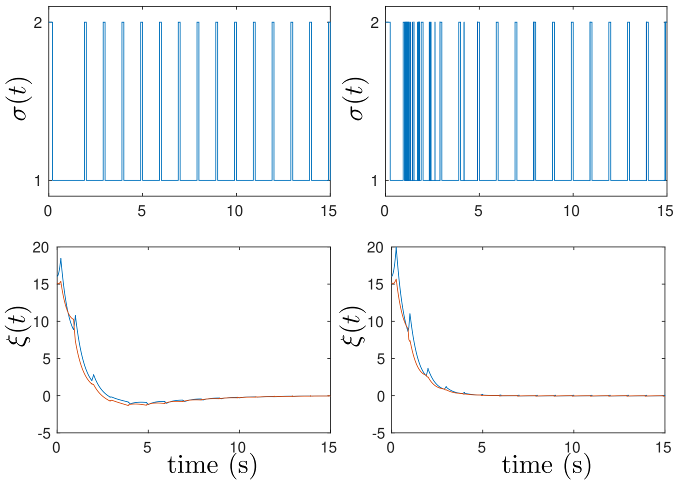

Fig. 2. Translated system response $\xi(t)$ and associated switching signal $\sigma(t)$ over time for the sampled switching function (left) and the non-sampled one (right).

academical example was used to validate the proposed control technique.

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