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On-line estimation of the Caputo fractional derivatives with application to $PI^{\mu}D^{\nu}$ control ⋆

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Abstract: This paper proposes new procedures for calculation of the Caputo derivative of model-free measured signals. The evaluation of the non-integer derivative is realized by integrating a set of ordinary differential equations and convolution. The derivative of order ν $(0 < \nu < 2)$ is seen as an output of a linear-time-varying system driven by a time-dependent known signal. Two procedures are proposed depending on the variation range of the non-integer differentiation order. The proposed formulations facilitate the estimation of the fractional derivatives when they are associated to dynamical systems represented by integer-order differential equations. The efficiency of the developed numerical procedures are validated and compared to exact fractional derivatives for different values of ν. It is shown that $PI^{\mu}D^{\nu}$ controllers can be easily realized by system augmentation and convolution. The advantages, the straightforwardness and the main features of the proposed design are given.

Keywords: Fractional-order derivatives; Estimation; Time-varying Systems; System Theory, $PI^{\mu}D^{\nu}$ control.

1. INTRODUCTION

Fractional calculus is recognized as a branch of mathematical analysis that exposes various possibilities of defining the powers of the differentiation operator. The generalization of the derivative notion consists of letting the power of the derivative take a real or a complex value. Extensive results and definitions were exposed in Zwillinger (2014) and the reader can find a good introduction to the subject in Annaby and Mansour (2012), Oustaloup (2014), Oustaloup (1995), Sabatier et al. (2007), Samko et al. (1987). The author is also referred to the references Farges et al. (2010), Trigeassou et al. (2011), Sabatier et al. (2007), Sabatier et al. (2015), Li et al. (2010), Ibrir (2017), Ibrir and Bettayeb (2015) for some results on control of fractional-order linear systems of commensurate type.

The ν-derivative of a given signal has a local property when the derivative of order ν is a positive integer number. Actually, when the order ν takes non-integer values, the ν-derivative of a signal becomes dependent on the signal past values; which means that, the non-integer derivative has a memory or it has simply a non-local property. In fact, many real systems are better modeled with fractional differential equations like systems with long-term "memory" and systems exhibiting chaotic behaviors. Referring to the literature, it was found that fractional-order systems are suitable for characterization of the anomalous behavior of dynamical systems and more representative of complex dynamics that are slower or faster than exponential functions. In those cases, the solutions are best represented by Mittag-Leffler functions. Moreover, the utilization of fractional-order derivatives and integrals in closed-loop control systems has shown outstanding performances, see e.g., Luo and Chen (2009), Sabatier et al. (2015), Duma (2012).

There are numerous attempts to realise and implement fractional-order differentiators. The design of limited-bandwidth fractional-order differentiators has been investigated in Serrier et al. (2007). In Liu et al. (2017) and Wei et al. (2019), the authors propose fractional-order differentiator to estimate the Riemann–Liouville fractional derivatives of the system output in discrete noisy environment. Fractional-order differentiation through polynomial integration has been proposed in Liu et al. (2015). In Tolba et al. (2019) the authors present an implementation of the Grünwald-Letnikov differentiator on Field-Programmable Gate Arrays (FPGAs). Roughly speaking, the methods to evaluate fractional-order signals are basically classified into three main groups. The first group gathers computational methods based on the analytic definitions of the different types of fractional derivatives. These methods necessitate the well knowledge of the explicit forms of the signal and its derivatives which are often difficult to obtain except for some types of signals that are of polynomial shape, see for example Samadi et al. (2004). In the second group, we find all types of fractional-derivative approximations by rational functions or transfer functions describing frequency-domain realizations of the operator D^{ν} in continuous time, see e.g., Oustaloup (1983), Oustaloup (1995), Oustaloup (2014). Numerical methods based on approximation of the fractional-derivative operators by discrete transfer functions constitute the third group of

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methods; however, these techniques are essentially based upon finite series approximation which makes the derivative very sensible to the signal frequencies. Moreover, if the series has a large number of coefficients, more processing time is required to implement the procedures in real time.

In this paper, new computational procedures for estimation of the Caputo fractional-order derivatives are proposed. The proposed methods consist of transforming the fractional order operator into the solution of a set of ordinary differential equations followed by a convolution operation. The overall design is seen as an output of a Linear-Time-Varying (LTV) system that can be easily integrated/solved in real time. The speed of convergence of to the exact fractional derivative is controlled by a positive design parameter. The new proposed methods facilitate the implementation of $PI^{\mu}D^{\nu}$ controllers for systems described by ordinary differential equations where the dynamics of the controller is simply seen as an augmented subsystem to the system to be controlled. More importantly, the proposed design is not alerted to the size or the frequency of the signal to be differentiated. In contrast to the differentiators proposed in Wei et al. (2019), and Liu et al. (2017), the signals are not required to be dependent on the dynamics of well-known dynamical systems which makes the proposed design classified as a model-free technique.

Throughout this paper, we note by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$, the set of real numbers and the set of positive real numbers, respectively. The notation $\dot{x}(t)$ stands for the time-derivative of the function $x(t)$, that is $\mathrm{d}x/\mathrm{d}t$. The notation $\mathscr{C}^n(\mathbb{R})$ refers to the set of continuous functions being n-times continuously differentiable on $\mathbb{R}$. For given integrable functions $f(t)$ and $g(t)$, the notation $f(t) \star g(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$ denotes the convolution of f and g. The notation $f[k]$ stands for the discrete signal which is available only at regular instants.

2. LTV SYSTEM APPROACH TO THE CALCULATION OF THE CAPUTO FRACTIONAL-ORDER DERIVATIVE

2.1 Preliminaries

Let $f(t) \in \mathscr{C}^n(\mathbb{R})$ be a time-dependent signal where ν is a non-integer number and let $a \in \mathbb{R}$. Define $n = \mathrm{ceil}(\nu)$ as a natural number verifying $n-1 \leq \nu < n$. The Caputo fractional derivative of $f(t)$ of order ν is defined for $t > a$ as

$$ {}_a^C D_t^{\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \int_a^t (t-\tau)^{n-\nu-1} \frac{d^n}{d\tau^n} f(\tau)\, d\tau, \quad (1)$$

where $\Gamma(z)$ is the Gamma function defined as

$$\Gamma(z) = \int_0^{+\infty} e^{-x} x^{z-1}\, dx. \quad (2)$$

Notice that the Riemann-Liouville fractional derivative of $f(t) \in \mathscr{C}^n(\mathbb{R})$ is defined for $t > a$ by:

$$ {}_a^{RL} D_t^{\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\nu-1} f(\tau)\, d\tau. \quad (3)$$

Referring to Samko et al. (1987), the fractional integral of a function $f(t) \in \mathscr{C}(\mathbb{R})$ is defined by:

$$I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, \;\; t > 0 \quad (4)$$

where $\mu \in \mathbb{R}_{>0}$ stands for the fractional-integration order. In fact, the evaluation of the integral necessitates the knowledge of the function $f(t)$ which is not always available as an explicit function of time. Additionally, even the function $f(t)$ is known, the convolution term that involves the fractional derivatives (i.e., the Caputo and the Riemann Liouville) are not always easy to find except for some functions like polynomial functions. When the function $f(t)$ is given numerically as a set of values in $\mathbb{R}$, the formulas (1) and (3) shall be discretized to have an estimate of the fractional derivatives. The objective of this section is to provide the Caputo derivatives through the solution of an ordinary differential equations when the function or the signal $f(t) \in \mathbb{R}$ is measured in continuous or discrete manners. More importantly, it is assumed that the model that generates $f(t)$ is not necessarily known.

2.2 The first procedure: case $0 < \nu < 1$

In this part of the paper, the numerical procedure is limited to the calculation of the Caputo fractional derivatives in the particular case where $0 < \nu < 1$. Notice that the Caputo derivative is exactly the convolution of the signal $g(t) = \frac{t^{n-\nu-1}}{\Gamma(n-\nu)}$ and the first derivative of $f(t)$; $\nu \in \mathbb{R}_{>0}$. Having the exact first derivative of $f(t)$ for all $t > 0$ renders the evaluation of the Caputo fractional derivative a simple convolution operation of two time-dependent functions. Based on this fact, and using some preliminary results given in Ibrir (2003), the procedure is divided into two steps: the first step consists of evaluating the exact time derivative of $f(t)$ and in the second step, the convolution function $g(t) \star \frac{df}{dt}(t)$ is calculated over the time interval $(a,\ t]$.

Theorem 1. Let $f(t) \in \mathscr{C}^2(\mathbb{R}_{\geq 0})$ be a continuously measured signal. Define $x_1(t)$ and $x_2(t)$ as the trajectories of the following ordinary differential equations:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\beta^2 t^2 \big(x_1(t) - \tan^{-1}(f(t))\big) - 2\beta\, t\, x_2(t), \\ &\qquad \beta > 0, x_2(0) = 0, \\ y &= \frac{t^{-\nu}}{\Gamma(1-\nu)} \star ((1+f^2(t))x_2(t)), \quad t > a, \\ &\qquad y(s) = 0 \;\text{ for } s \leq a. \end{aligned} \quad (5)$$

Then, for sufficiently large value of $a > 0$,

$$\lim_{t\to+\infty} {}_a^C D_t^{\nu} f(t) = \lim_{t\to+\infty} y(t), \quad 0 < \nu < 1. \quad (6)$$

Proof. Let $z(t) = x_1(t)$ for all time. To prove the result of Theorem 1, it is sufficient to prove that $z(t)$ will converge to the signal $u(t) = \tan^{-1}(f(t))$. The z variable verifies the following differential equation:

$$\ddot{z} + 2\beta\, t\dot{z} + t^2\beta^2(z-u) = 0, \quad \forall t > 0, z(0) = z_0, \dot{z}(0) = 0. \quad (7)$$

For all $t \geq 0$, the solution of (7) is explicitly given by:

$$
\begin{aligned}
z(t) &= \left(\frac{1}{2}\, e^{-\frac{1}{2}\,t\left(\beta\, t-2\,\sqrt{\beta}\right)} + \frac{1}{2}\, e^{-\frac{1}{2}\,t\left(\beta\, t+2\,\sqrt{\beta}\right)}\right) z_0 \\
&+ \frac{1}{2}\,\beta^{\frac{3}{2}}\left(e^{2\,t\sqrt{\beta}}\int_0^t s^2 u\,(s)\, e^{\frac{s}{2}\,\left(\beta\, s-2\,\sqrt{\beta}\right)}\, ds\right. \\
&\left.- \int_0^t s^2 u\,(s)\, e^{\frac{s}{2}\,\left(\beta\, s+2\,\sqrt{\beta}\right)}\, ds\right) e^{-\frac{t}{2}\left(\beta\, t+2\,\sqrt{\beta}\right)}.
\end{aligned} \tag{8}
$$

In Eq. (8), the term involving z_0 vanishes to zero when time tends to infinity; therefore, the behavior of $z(t)$ at infinity is only dependent on the remaining terms, i.e.,

$$
\begin{aligned}
&\lim_{t\to+\infty} z(t) = \\
&\lim_{t\to+\infty} \frac{1}{2}\,\beta^{3/2}\left(e^{2\,t\sqrt{\beta}}\int_0^t s^2 u\,(s)\, e^{\frac{s}{2}\,\left(\beta\, s-2\,\sqrt{\beta}\right)}\, ds\right. \\
&\left.- \int_0^t s^2 u\,(s)\, e^{\frac{s}{2}\,\left(\beta\, s+2\,\sqrt{\beta}\right)}\, \mathrm{d}s\right) e^{-\frac{t}{2}\left(\beta\, t+2\,\sqrt{\beta}\right)}.
\end{aligned} \tag{9}
$$

Define

$$
\begin{aligned}
V_1(t) &= \int_0^t s^2 u\,(s)\, e^{\frac{s}{2}\left(\beta\, s-2\,\sqrt{\beta}\right)}\, \mathrm{d}s, \\
V_2(t) &= \int_0^t s^2 u\,(s)\, \mathrm{e}^{\frac{s}{2}\left(\beta\, s+2\,\sqrt{\beta}\right)}\, \mathrm{d}s.
\end{aligned} \tag{10}
$$

Using integration by parts, we have

$$
\begin{aligned}
V_1(t) &= \frac{\left(t\beta^{\frac{3}{2}}+\beta\right) e^{-\frac{t}{2}\left(-\beta\, t+2\,\sqrt{\beta}\right)} u\,(t)}{\beta^{\frac{5}{2}}} - \frac{u\,(0)}{\beta^{\frac{3}{2}}} \\
&- \int_0^t \frac{\left(s\beta^{\frac{3}{2}}+\beta\right) e^{-\frac{s}{2}\left(-\beta\, s+2\,\sqrt{\beta}\right)} \dot{u}\,(s)}{\beta^{\frac{5}{2}}}\, \mathrm{d}s,
\end{aligned} \tag{11}
$$

and

$$
\begin{aligned}
V_2(t) &= \frac{\left(t\beta^{\frac{3}{2}}-\beta\right) e^{\frac{t}{2}\left(\beta\, t+2\,\sqrt{\beta}\right)} u\,(t)}{\beta^{\frac{5}{2}}} + \frac{u\,(0)}{\beta^{\frac{3}{2}}} \\
&- \int_0^t \frac{\left(s\beta^{\frac{3}{2}}-\beta\right) e^{\frac{s}{2}\left(\beta\, s+2\,\sqrt{\beta}\right)} \dot{u}\,(s)}{\beta^{\frac{5}{2}}}\, \mathrm{d}s.
\end{aligned} \tag{12}
$$

This gives,

$$
\begin{aligned}
&\lim_{t\to+\infty} z(t) = \\
&\lim_{t\to+\infty} \frac{1}{2}\,\beta^{\frac{3}{2}}\left(V_1(t)e^{2\sqrt{\beta}t} - V_2(t)\right) e^{-\frac{t}{2}\left(\beta\, t+2\,\sqrt{\beta}\right)} \\
&= \lim_{t\to+\infty} u(t) \\
&+ \lim_{t\to+\infty}\left[\left(-\frac{1}{2}\, e^{-\frac{1}{2}\,\beta\, t^2+t\sqrt{\beta}} - \frac{1}{2}\, e^{-\frac{1}{2}\,t\left(\beta\, t+2\,\sqrt{\beta}\right)}\right) u\,(0)\right. \\
&- \frac{1}{2} e^{-\frac{1}{2}\,\beta\, t^2+t\sqrt{\beta}} \int_0^t \left(\sqrt{\beta}s+1\right) e^{\frac{s}{2}\left(\beta\, s-2\,\sqrt{\beta}\right)} \dot{u}\,(s)\; \mathrm{d}s \\
&\left.+ \frac{1}{2} e^{-\frac{t}{2}\left(\beta\, t+2\,\sqrt{\beta}\right)} \int_0^t \left(\sqrt{\beta}s-1\right) e^{\frac{s}{2}\left(\beta\, s+2\,\sqrt{\beta}\right)} \dot{u}\,(s)\; \mathrm{d}s\right].
\end{aligned} \tag{13}
$$

This implies that

$$
\begin{aligned}
&\lim_{t\to+\infty} z(t) = \lim_{t\to+\infty}\left[u(t)\right. \\
&- \frac{1}{2} e^{-\frac{1}{2}\,\beta\, t^2+t\sqrt{\beta}} \int_0^t \left(\sqrt{\beta}s+1\right) e^{\frac{s}{2}\left(\beta\, s-2\,\sqrt{\beta}\right)} \dot{u}\,(s)\; \mathrm{d}s \\
&\left.+ \frac{1}{2} e^{-\frac{t}{2}\left(\beta\, t+2\,\sqrt{\beta}\right)} \int_0^t \left(\sqrt{\beta}s-1\right) e^{\frac{s}{2}\left(\beta\, s+2\,\sqrt{\beta}\right)} \dot{u}\,(s)\; \mathrm{d}s\right].
\end{aligned} \tag{14}
$$

Let

$$
\begin{aligned}
V_3(t) &= \int_0^t \left(\sqrt{\beta}s+1\right) e^{\frac{s}{2}\left(\beta\, s-2\,\sqrt{\beta}\right)} \dot{u}\,(s)\; \mathrm{d}s, \\
V_4(t) &= \int_0^t \left(\sqrt{\beta}s-1\right) e^{\frac{s}{2}\left(\beta\, s+2\,\sqrt{\beta}\right)} \dot{u}\,(s)\; \mathrm{d}s.
\end{aligned} \tag{15}
$$

Using integration by parts then, $V_3(t)$ takes the form (16) (see the top of the next page) where "i" is the pure complex number verifying $i^2 = -1$. Similarly, integrating $V_4(t)$ by parts gives (17), see the top of the next page, where

$$
\mathrm{erf}(s) = \frac{2}{\sqrt{\pi}}\int_0^s e^{-t^2}\mathrm{d}t, \quad -i\,\mathrm{erf}(i\,s) = \frac{2}{\sqrt{\pi}}\int_0^s e^{t^2}\mathrm{d}t. \tag{18}
$$

According to (16) and (17), one can write

$$
\begin{aligned}
\lim_{t\to+\infty} z(t) &= \lim_{t\to+\infty} u(t) \\
&- \frac{1}{2}\lim_{t\to+\infty} e^{-\frac{1}{2}\,\beta\, t^2+t\sqrt{\beta}} V_3(t) \\
&+ \frac{1}{2}\lim_{t\to+\infty} e^{-\frac{t}{2}\left(\beta\, t+2\,\sqrt{\beta}\right)} V_4(t).
\end{aligned} \tag{19}
$$

Since $u(t)$ and $\dot{u}(t)$ are bounded for all time $t \geq 0$ with $\lim_{t\to+\infty} e^{-t^2}\int_0^t e^{\tau^2}\, d\tau = 0$ and $\lim_{t\to+\infty} e^{-t^2}\int_0^t\int_0^t e^{\tau^2}\, d\tau = 0$ then,

$$
\begin{aligned}
&- \frac{1}{2}\lim_{t\to+\infty} e^{-\frac{1}{2}\,\beta\, t^2+t\sqrt{\beta}} V_3(t) \\
&+ \frac{1}{2}\lim_{t\to+\infty} e^{-\frac{t}{2}\left(\beta\, t+2\,\sqrt{\beta}\right)} V_4(t) = 0.
\end{aligned} \tag{20}
$$

Consequently,

$$
\lim_{t\to+\infty} z(t) = \lim_{t\to+\infty} u(t), \quad \lim_{t\to+\infty} x_2(t) = \lim_{t\to+\infty} \dot{u}(t). \tag{21}
$$

As a result: $\lim_{t\to+\infty} \dot{f}(t) = \lim_{t\to+\infty} \dot{u}(t)(1 + f^2(t))$. For a and β sufficiently large, the system output $y(t)$ will converge to the exact fractional-order derivative of $f(t)$. Remark that the transient of the differentiator is fast when β is large enough. This means that the rate of convergence is dependent on the free parameter β. This ends the proof.

Remark that system (5) is able to produce the first derivative of $\tan^{-1}(f)$ whatever the nature of f; i.e., being bounded or not bounded. Hence, the fractional-order derivative of order ν can be always estimated once $f \in \mathscr{C}^2(\mathbb{R}_{\geq 0})$. The coefficient β in system (5) regulates the transient behavior of the fractional-order derivative estimates. Obviously, fast transient behaviors are seen when β is large.

2.3 The second procedure: case $1 < \nu < 2$

When the non-integer differentiation order increases, the fractional derivatives will be dependent on the higher-order derivatives of the signal $f(t)$. In this subsection, the second algorithm deals with the case of ν between one and

$$
\begin{aligned}
V_3(t) = {} & \frac{\left(-i\, e^{-\frac{1}{2}}\, \mathrm{erf}\left(\frac{i}{2}\sqrt{2}\right) \pi \sqrt{2} - \sqrt{\pi}\right) \dot{u}\,(0)}{\sqrt{\pi}\sqrt{\beta}} + \frac{\left(-i e^{\frac{-1}{2}} \pi \sqrt{2}\, \mathrm{erf}\left(\frac{i}{2}\sqrt{2}\left(\sqrt{\beta} t - 1\right)\right) + \sqrt{\pi} e^{\frac{1}{2}t\left(t\beta - 2\sqrt{\beta}\right)}\right) \dot{u}\,(t)}{\sqrt{\pi}\sqrt{\beta}} \\
& + \frac{\int_0^t \left(i\sqrt{\pi} e^{-\frac{1}{2}}\sqrt{2}\, \mathrm{erf}\left(\frac{i}{2}\sqrt{2}\left(\sqrt{\beta} s - 1\right)\right) - e^{\frac{s}{2}\left(\beta\, s - 2\sqrt{\beta}\right)}\right) \ddot{u}(s)\, \mathrm{d}s}{\sqrt{\beta}}.
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
V_4(t) = {} & \frac{\left(-i\, \mathrm{e}^{-\frac{1}{2}}\, \mathrm{erf}\left(\frac{i}{2}\sqrt{2}\right) \pi \sqrt{2} - \sqrt{\pi}\right) \dot{u}\,(0)}{\sqrt{\pi}\sqrt{\beta}} + \frac{\left(i\, \mathrm{e}^{-\frac{1}{2}} \pi \sqrt{2}\, \mathrm{erf}\left(\frac{i}{2}\sqrt{2}\left(\sqrt{\beta} t + 1\right)\right) + \sqrt{\pi} e^{\frac{1}{2}t\left(t\beta + 2\sqrt{\beta}\right)}\right) \dot{u}\,(t)}{\sqrt{\pi}\sqrt{\beta}} \\
& - \frac{\int_0^t \left(i\sqrt{\pi} e^{-\frac{1}{2}}\sqrt{2}\, \mathrm{erf}\left(\frac{i}{2}\sqrt{2}\left(\sqrt{\beta} s + 1\right)\right) + e^{\frac{1}{2}s\left(\beta\, s + 2\sqrt{\beta}\right)}\right) \ddot{u}\,(s)\, \mathrm{d}s}{\sqrt{\beta}}.
\end{aligned} \tag{17}
$$

two where the signal to be differentiated is required to be differentiable up to the order three.

Theorem 2. Let $f(t) \in \mathscr{C}^3(\mathbb{R}_{\geq 0})$ be a continuously measured signal. Define $x_1(t)$, $x_2(t)$, and $x_3(t)$ as the trajectories of the following ordinary differential equations for all time $t \geq 0$:

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= x_3(t), \\
\dot{x}_3(t) &= -\beta^3 t^3 \big(x_1(t) - \tan^{-1}(f(t)\big) \\
&\quad - 3\beta^2 t^2 x_2(t) - 3\beta\, t\, x_3(t), \quad \beta > 0, \\
y &= \frac{t^{1-\nu}}{\Gamma(2-\nu)} \star \left[(1+f^2(t))\big[x_3(t) + 2f(t)x_2^2(t)\big]\right], \\
&\qquad t > a,
\end{aligned} \tag{22}
$$

with $y(s) = 0$ for $s \leq a$, $x_2(0) = x_3(0) = 0$. Then, for sufficiently large value of $a > 0$, we have

$$
\lim_{t\to+\infty} {}^C_a D^\nu_t f(t) = \lim_{t\to+\infty} y(t), \quad 1 < \nu < 2. \tag{23}
$$

Proof. The proof is omitted for space limitation.

3. DISCRETE-TIME IMPLEMENTATION AND ROBUSTNESS AGAINST ADDITIVE NOISE

3.1 Numerical discretization

The dynamics of systems (5) and (22) are sets of ordinary differential equations that can be solved by any numerical method like the Runge-Kutta, Dormand-Prince, Adams-Bashforth, or the predictor-corrector numerical procedures. However, the outputs of these systems can be approximated by the discrete convolution formulae that provides an integration error of order h; where h is the time sampling period. For $0 < \nu < 1$ and $a = 0$, the discrete output of the fractional derivative is approximated by:

$$
y[k] = y(k\, h) = h \sum_{j=1}^{k-1} \phi[j]\; \varphi[k-j], \quad y[0] = 0. \tag{24}
$$

where

$$
\phi[j] = \frac{t_j^{-\nu}}{\Gamma(1-\nu)}, \quad \varphi[j] = \big(1 + f^2[j]\big) x_2[j], \quad t_j = j\, h. \tag{25}
$$

Similarly, for $1 < \nu < 2$ and $a = 0$, the output of the fractional differentiator is estimated as

$$
\begin{aligned}
y[k] &= y(k\, h) = h \sum_{j=1}^{k-1} \psi[j]\; \vartheta[k-j], \quad y[0] = 0, \\
\psi[j] &= \frac{t_j^{1-\nu}}{\Gamma(2-\nu)},\ \vartheta[j] = \Big(\big(1 + f^2[j]\big)\big(x_3[j] + 2f[j]x_2^2[j]\big)\Big).
\end{aligned} \tag{26}
$$

Actually, the global error of approximation will be dependent on the numerical procedure used to integrate systems (5) and (22) and the error of integration of their outputs.

The developed algorithms can serve as potential candidate methods to realize $PI^\mu D^\nu$ control laws by integrating ordinary differential equations and then evaluating instantaneous convolutions. To clarify this claim, assume that a linear SISO system having a dynamics of the form $\dot{X} = A\, X + Bw$; $Y = CX$ is stabilizable by a feedback controller $w = k\, Y + k_d\, D^\nu Y$; where $0 < \nu < 1$. Then, the closed-loop system is realized as follows:

$$
\begin{aligned}
\dot{X}(t) &= A\, X(t) + Bw(t), \\
\dot{x}(t) &= G(t)\, x(t) + H(t) \tan^{-1}(Y(t)), \\
w(t) &= k\, Y(t) + K_d \left[\frac{t^{-\nu}}{\Gamma(1-\nu)} \star ((1 + Y^2(t))x_2(t))\right],
\end{aligned} \tag{27}
$$

where

$$
\begin{aligned}
x(t) &= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, G(t) = \begin{pmatrix} 0 & 1 \\ -\beta^2 t^2 & -2\beta\, t \end{pmatrix}, \\
H(t) &= \begin{pmatrix} 0 \\ \beta^2 t^2 \end{pmatrix}.
\end{aligned} \tag{28}
$$

3.2 Robustness

When the signal to be differentiated is corrupted by noise, it is quite recommended to saturate the time "t" when the first states of (5) and (22) enter in a small neighborhood of the input $u = \tan^{-1}(f)$. In case of noisy measurement, the *exact* asymptotic tracking of the systems' input is not recommended. Additionally, the value of the free parameter $\beta > 0$ must be selected to avoid signal error amplification. Since the signal $\tan^{-1}(f)$ is bounded for all time, the choice of a *reasonable* value of β is quite sufficient to estimate the fractional-order derivatives with bounded errors. The parameter β can be also tuned automatically to achieve this objective. The procedure of tuning the parameter β is not considered herein due to space limitation.

4. NUMERICAL SIMULATIONS

To illustrate the efficacy of the proposed algorithm, we have compared the exact fractional-order derivative of the sine wave with its counterpart given by numerical integration of the system of equations (5). By taking $\nu = \frac{1}{2}$, $a = 0$, and $f(t) = \sin(t)$, we have

$$\begin{aligned} {}_a^C D_t^\nu f(t) &= \sqrt{2}\mathscr{F}_c\left(\frac{\sqrt{2}\sqrt{t}}{\pi}\right)\cos(t) \\ &+ \sqrt{2}\mathscr{F}_s\left(\frac{\sqrt{2}\sqrt{t}}{\pi}\right)\sin(t), \end{aligned} \tag{29}$$

where $\mathscr{F}_c(\tau)$ and $\mathscr{F}_s(\tau)$ are respectively the Fresnel cosine integral and the Fresnel sine integral defined by:

$$\begin{aligned} \mathscr{F}_c(\tau) &= \int_0^\tau \cos\left(\frac{\pi t^2}{2}\right) dt, \\ \mathscr{F}_s(\tau) &= \int_0^\tau \sin\left(\frac{\pi t^2}{2}\right) dt. \end{aligned} \tag{30}$$

The system of equations (5) is integrated by the Dormand-Prince algorithm of order five with a uniform sampling period $\Delta t = 10^{-3}$ (sec), $\beta = 30$, $a = 0$. The system output of system (5) is approximated by the discrete convolution formulae and compared to the exact derivative (29), see Fig. 1. Actually, the value of the convolution term, provided by the system output y, could be improved by minimizing the sampling period Δt or by selecting another numerical scheme of signal integration.

Now, we shall assess the quality of estimation of the fractional derivative for $\nu = \frac{3}{2}$ and select the signal $f(t)$ as $\frac{1}{10} t \sin(t)$. The exact fractional derivative of f is

$$\begin{aligned} &{}_0^C D_t^{\frac{3}{2}} f(t) \\ &= \frac{1}{10\sqrt{\pi}}\Bigg[-\sqrt{2\pi}\left(t\sin(t) - \frac{3}{2}\cos(t)\right)\mathscr{F}_c\left(\sqrt{\frac{2t}{\pi}}\right) \\ &+ \sqrt{2\pi}\left(\cos(t)\,t + \frac{3}{2}\sin(t)\right)\mathscr{F}_s\left(\sqrt{\frac{2t}{\pi}}\right) + \sqrt{t}\Bigg]. \end{aligned} \tag{31}$$

The result of Theorem 2 is used to evaluate the instantaneous value of the fractional derivative as shown in Fig. 2 where $\beta = 20$, $a = 0$. System (22) is integrated by the Dormand-Prince method of order five with a sampling period $\Delta t = 10^{-3}$ (sec). It is noticed that even $a = 0$, the quality of the estimates of the fractional derivatives are quite good.

The second differentiator is tested again for a *noisy unbounded signal* $f(t) = \frac{1}{10}t\sin(t) + n(t)$ where $n(t)$ is a white noise, see Fig. 3. For $a = 0$, $\Delta t = 1$ ms, the fractional derivative of order $\frac{3}{2}$ is represented in Fig. 4. In this simulation, $\beta = 3$ while the time "t" in system (22) is saturated at 5 (sec). Notice that the output of system (22) is able to reproduce the exact derivative with a bounded error.

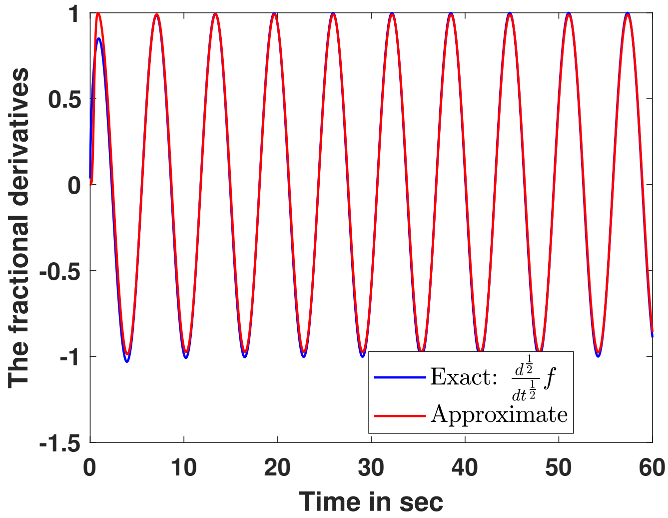

Fig. 1. The fractional derivatives of a bounded function, $\nu = \frac{1}{2}$ and $f(t) = \sin(t)$

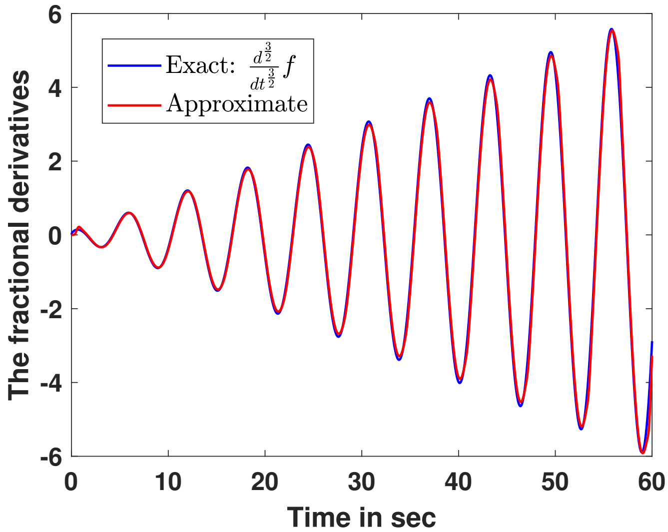

Fig. 2. The fractional derivatives of unbounded function, $\nu = \frac{3}{2}$, $f(t) = \frac{1}{10}t\sin(t)$

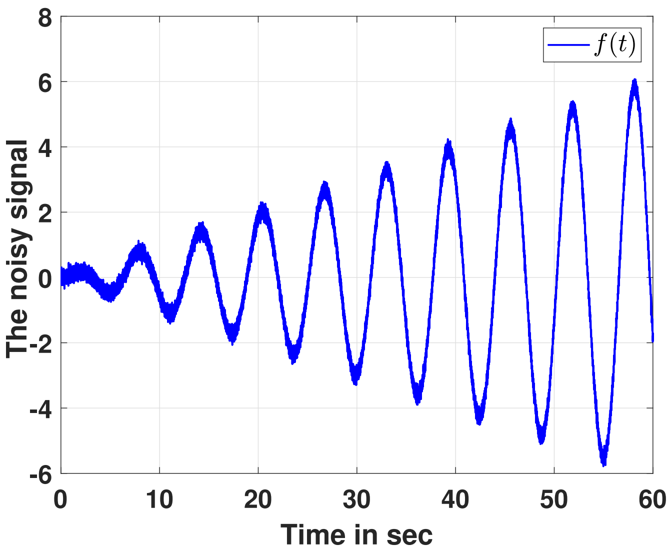

Fig. 3. The noisy signal

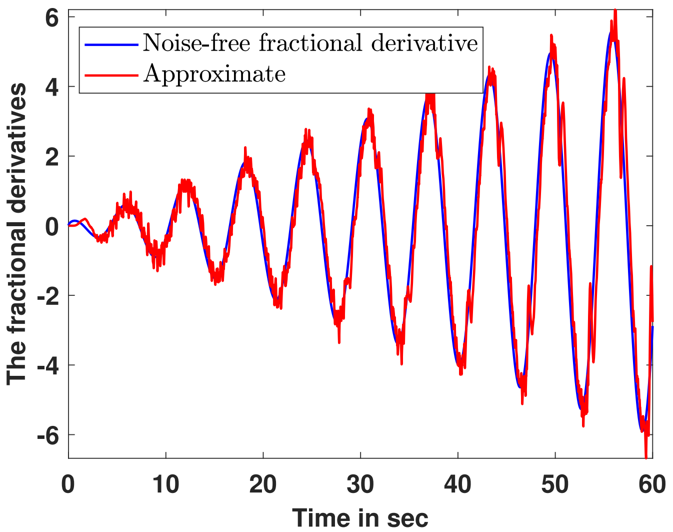

Fig. 4. The fractional derivatives of noisy unbounded function, $\nu = \frac{3}{2}$, $f(t) = \frac{1}{10} t \sin(t) + n(t)$

5. CONCLUSIONS

The Caputo derivative of a measured signal is transformed into a simple convolution of a time-dependent function with the solutions of a set of ordinary differential equations. This formulation favorites the realization of non-integer-order derivatives when combined with other system dynamics. The LTV-system approach to the calculation of the fractional derivatives is not dependent on the signal frequency and its form (bounded or not bounded). Moreover, the rate of convergence to the true derivatives is easily tunable by increasing or decreasing the value of a design parameter.

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