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Acceleration-Actuated Source Seeking Without Position and Velocity Sensing

Raik Suttner *, **Miroslav Krstić** **

* *Institute of Mathematics, University of Wuerzburg, Wuerzburg, Germany (e-mail: raik.suttner@mathematik.uni-wuerzburg.de)*
** *Department of Mechanical and Aerospace Engineering, University of California, San Diego, USA (e-mail: krstic@ucsd.edu)*

Abstract: We study the problem of source seeking for an acceleration-controlled unicycle. The objective is to asymptotically stabilize the unicycle around states where a smooth position-dependent signal (or cost) function attains a minimum value. An implementation of the proposed control strategy only requires measurements of the cost function value at the current position. We do *not* assume that the unicycle can measure its current positions or its current forward velocity. For this purpose, we extend a recently introduced approach to extremum seeking control, which is based on the approximation of symmetric products of vector fields. An additional high-gain observer is used to estimate the derivative of the sensed cost signal. The estimates provided by the observer compensate for the missing velocity measurements and allow a reduction of the kinetic energy without velocity-dependent damping. Under suitable assumptions on the cost function, the control law leads to semi-global practical asymptotic stability.

Keywords: extremum seeking, source seeking, autonomous vehicles, vibrational control, averaging, approximation of symmetric products

1. INTRODUCTION

The problem of source seeking describes the task for an autonomous agent to locate an extremum of a position-dependent scalar signal. For example, the agent might be a wheeled robot, the signal might be the strength of an electric or magnetic field, and the objective is to find a point at which the field strength attains an extreme value. It is assumed that the agent is equipped with a suitable sensor so that it can measure the signal at its current position at any given time. Additional information about the signal, like the gradient, is not available to the agent.

There are many different approaches in the literature to solve the source seeking problem for various types of agent models. For example, in Zhang et al. (2007b); Stanković and Stipanović (2009); Dürr et al. (2011) for single-integrator points, in Zhang et al. (2007a); Cochran and Krstić (2009); Matveev et al. (2011) for velocity-controlled unicycles, and in Zhang et al. (2007b); Stanković and Stipanović (2009); Scheinker (2018) for double-integrator points. In Suttner (2019a), there is a first source seeking control law for an acceleration-controlled unicycle. An implementation of the feedback law in Suttner (2019a) requires measurements of the source signal and the forward velocity. However, this means that the sensed variables (source signal and forward velocity) contain more information than the actively controlled variable (source signal). It is therefore natural to ask whether the source seeking problem for an acceleration-controlled unicycle can be also solved when no measurements of the forward velocity are available. The results in Zhang et al. (2007b) for double-integrator point agents indicate that information about the current velocity can be extracted from changes of the sensed source signal. A crucial assumption in Zhang et al. (2007b) is that the agent can be steered instantaneously in any desired direction at any given time. For a more realistic second-order agent model, like the unicycle model, there is no suitable approach in the literature so far. The contribution of this paper is to propose the first source seeking control law for the second-order unicycle model that does neither rely on measurements of the position nor on measurements of the forward velocity.

Source seeking with nonholonomic unicycles can be performed, in principle, in many different ways: by adjusting angular or forward velocity, by adjusting angular or forward acceleration, and by doing so around either zero or nonzero values. In Table 1, we survey the different methods that have appeared in the literature, based on which of the inputs are used for tuning. Generally speaking, methods that use tuning around zero angular velocity (Cochran and Krstić (2009), and its extensions Ghods and Krstić (2010); Raisch and Krstić (2017)) generate trajectories that approach the source in the most direct manner but have a problem with overshooting the source, whereas the methods that use tuning around the zero forward velocity or acceleration (Zhang et al. (2007b); Dürr et al. (2013), and the present paper) approach the source most inefficiently but are efficient at staying near the source. Methods that tune the angular velocity around a nonzero value (Scheinker and Krstić (2014); Dürr et al. (2017); Raisch and Krstić (2017) represent a tradeoff; they generate trajectories that drift towards the source in a circular manner and, in such a fashion, they have a less direct

	CK09	DKSE17	DSEJ13	GK10	RK17	SK14	ZAGSK07	present paper
angular velocity around zero	×			×	×			
angular velocity around nonzero		×			×	×		
forward velocity around zero			×				×	
forward velocity around nonzero				×				
angular acceleration around zero								
angular acceleration around nonzero								
forward acceleration around zero								×
forward acceleration around nonzero								
Krylov-Bogolyubov averaging	×			×	×		×	
Lie bracket approximation		×	×			×		
symmetric product approximation								×

Table 1. Collection of some of the existing papers on source seeking control for the unicycle model. The methods are categorized based on whether they perform tuning of the angular or longitudinal (forward) degree of freedom, whether they actuate velocity or acceleration, whether the tuning is around a zero value (with sinusoids) or a nonzero value (without sinusoids). The abbreviations of the references correspond to the last name initials and the years in the reference list. The last three rows indicate the averaging techniques, which are used in the analysis.

convergence to the source than the methods like Cochran and Krstić (2009) but are more efficient than the methods with back-and-forth arc-like motions as in Zhang et al. (2007b); Dürr et al. (2013) and the present paper.

As in Suttner (2019a), the control law in the present paper is based on an approximation of so-called *symmetric products* of vector fields. The underlying approximation property can be traced back to averaging results for mechanical systems under vibrational control in Bullo (2002). It is shown in Suttner (2019a); Suttner and Sun (2019) that symmetric products of suitably chosen vector fields can be used to get access to the gradient of a sensed objective (or cost) function. This approach is closely related to extremum seeking control by Lie bracket approximations as in Dürr et al. (2013); Scheinker and Krstić (2013); Suttner (2019b) for first-order kinematic systems. For second-order mechanical systems, an approximation of symmetric products has the practical advantage that the velocity of the closed-loop system remains bounded in the high-frequency, high-amplitude limit of the employed oscillatory inputs.

To overcome the problem of missing velocity measurements, we use a high-gain observer to estimate the derivative of the sensed scalar signal. A detailed averaging analysis reveals that the trajectories of the closed-loop system approximate the trajectories of a unicycle under a gradient-like control law. To be more precise, the gradient of the signal function in the averaged system is scaled in a suitable way by the derivative of the sensed signal. This additional scaling of the gradient induces a damping effect and consequently leads to a loss of kinetic energy. In particular, a velocity-induced damping as in Suttner (2019a) becomes obsolete. These findings might be also helpful for the purpose of sources seeking in environments with almost no friction; for example, when the agent is an autonomous underwater vehicle or a satellite in space.

2. PROBLEM STATEMENT AND CONTROL LAW

We consider the *second-order unicycle model*

$$\dot{p}_1 \;=\; v\,\cos\theta, \tag{1a}$$

$$\dot{p}_2 \;=\; v\,\sin\theta, \tag{1b}$$

$$\dot{v} \;=\; a, \tag{1c}$$

$$\ddot{\theta} \;=\; \tau, \tag{1d}$$

where $p = (p_1, p_2)^\top \in \mathbb{R}^2$ is the position, $\theta \in \mathbb{R}$ is an angle to describe the orientation, and $v \in \mathbb{R}$ is the forward velocity. It is assumed that the forward acceleration $a \in \mathbb{R}$ and the angular acceleration $\tau \in \mathbb{R}$ can be controlled. Moreover, we assume that the unicycle can constantly measure the value

$$y \;=\; \psi(p) \tag{2}$$

of a smooth, position-dependent cost function $\psi\colon \mathbb{R}^2 \to \mathbb{R}$. One can interpret (2) as a position-dependent signal that is sensed by the unicycle. We do *not* demand that the unicycle can measure its current position p or its current forward velocity v. Only real-time measurements of (2) can be used for a feedback law. The control objective is to steer the unicycle to a state at which the cost function attains a minimum value. The gradient of ψ is treated as an unknown quantity.

To solve the above problem, we propose the subsequent control law. We assume that the angle θ as a function of the time parameter t is of the form

$$\theta(t) \;=\; \theta_0 + C\,\cos(\Omega t + \Phi) \tag{3}$$

with some arbitrary but fixed constants $\theta_0, \Phi \in \mathbb{R}$ and $C, \Omega > 0$. This can be ensured by applying a suitable periodic torque $t \mapsto \tau(t)$. The actual values of $\theta_0, \phi, C, \Omega$ and the current value of θ do not need to be known to implement the control law for the forward acceleration, which is described below. Roughly speaking, the time-varying alignment in (3) ensures that the unicycle can explore changes of the sensed signal along different directions in the plane. It is also possible to allow a different time dependence of θ. A comment on the particular choice (3) for the angle is given in Remark 4 in Section 4. Since the angle θ is given by (3), the alignment of the unicycle can be described by the vector

$$e_{\shortparallel}(t) \;:=\; \begin{pmatrix}\cos\theta(t)\\ \sin\theta(t)\end{pmatrix}$$

at any time $t \in \mathbb{R}$.

To state the control law for the forward acceleration a, we introduce several control parameters and functions.

Choose constants $\varphi \in \mathbb{R}$ and $c > 0$. For every $\omega > 0$, define $u^\omega\colon \mathbb{R} \to \mathbb{R}$ by

$$u^\omega(t) \;:=\; c\,\omega\,\cos(\omega t + \varphi). \tag{4}$$

The oscillatory signal u^ω has the purpose to induce an approximation of symmetric products of vector fields as in Bullo (2002) for sufficiently large values of ω. Next, we introduce two suitable design functions α_1, α_2. A detailed averaging analysis in Section 3 will reveal that α_1, α_2 have the purpose to get access to a suitably scaled gradient of ψ through the symmetric product approximations. This step in the construction of the control law is similar to the choice of suitable design functions in the context of extremum seeking control by Lie bracket approximations as in Dürr et al. (2013); Scheinker and Krstić (2013); Grushkovskaya et al. (2018); Suttner and Sun (2018). The purpose of α_1, α_2 is further explained after the definition of the control law. First, choose a smooth function $\alpha_1\colon \mathbb{R} \to \mathbb{R}$ such that

- $(\alpha_1\alpha_1')(0) > 0$,
- α_1 and its derivative are bounded on $\mathbb{R}$.

For example, we may define α_1 by

$$\alpha_1(\xi) \;:=\; 1 + \tanh(\xi). \tag{5}$$

Second, choose a smooth function $\alpha_2\colon \mathbb{R} \to \mathbb{R}$ such that

- $\alpha_2(0) \neq 0$,
- $\big(\alpha_2^2(\xi) - \alpha_2^2(0)\big)\,\xi > 0$ for every $\xi \neq 0$,
- α_2 and its derivative are bounded on $\mathbb{R}$.

For example, we may define α_2 by

$$\alpha_2(\xi) \;:=\; \sqrt{2 + \tanh(\xi)}. \tag{6}$$

Choose constants $h_1, h_2 > 0$. We propose the control law

$$a \;=\; u^\omega(t)\,\alpha_1(y - z_1)\,\alpha_2(z_2) \tag{7}$$

for the forward acceleration, where z_1, z_2 are the state variables of an ω-dependent high-gain observer

$$\dot{z}_1 \;=\; z_2 + h_1\,\omega^{1/4}\,(y - z_1), \tag{8a}$$

$$\dot{z}_2 \;=\; h_2\,\omega^{1/2}\,(y - z_1). \tag{8b}$$

Note that an implementation of (7) requires no other information than measurements of (2). It is known from Khalil (2002) that, with increasing observer gains $h_1\,\omega^{1/4}$ and $h_2\,\omega^{1/2}$, the variable z_1 approximates y and the variable z_2 approximates the derivative $\dot{y}$. In other words, an extra integration of the sensed signal y results in an approximate differentiation of y for large values of ω. The transfer function from y to $y - z_1$ and from y to z_2 is shown in Figure 1. Before we begin with the averaging and stability analysis for the proposed control law, we briefly indicate the purpose of the terms $\alpha_1(y - z_1)$ and $\alpha_2(z_2)$ in (7) and the above assumptions on α_1 and α_2.

The averaging analysis in Section 3 will show that, in the limit $\omega \to \infty$, the term $\alpha_1(y - z_1)$ in (7) leads to the contribution

$$-\,(\alpha_1\alpha_1')(0)\nabla\psi(p)^\top e_{\shortparallel}(t) \tag{9}$$

in the forward acceleration of the averaged system, which is derived later in equation (15). Because of our goal to minimize the value of ψ, we are interested in the negative gradient direction $-\nabla\psi(p)$. This explains the assumption $(\alpha_1\alpha_1')(0) > 0$ on α_1: it ensures that the unicycle is forced into a descent direction of ψ along its current alignment. However, a pure negative gradient force would lead to

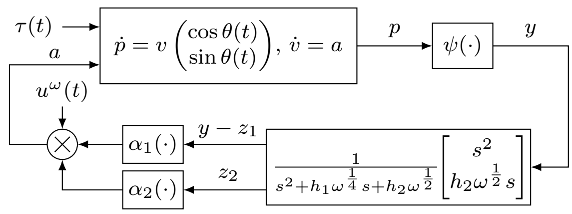


Fig. 1. Extremum seeking control for the unicycle model. With increasing parameter $\omega > 0$, the sinusoid u^ω induces a symmetric product approximation, and the variables z_1 and z_2 approximate y and $\dot{y}$, respectively.

undamped oscillations around minima of ψ. To circumvent this problem, a damping effect is induced by the following feature. In the limit $\omega \to \infty$, the term $\alpha_2(z_2)$ in (7) leads to the additional scaling factor

$$\alpha_2^2(\dot{y}) \tag{10}$$

of the negative gradient in (9). The assumption $\alpha_2(0) \neq 0$ ensures that the scaling factor (10) is positive for $\dot{y} = 0$. On the other hand, we also demand that α_2 satisfies $(\alpha_2^2(\xi) - \alpha_2^2(0))\,\xi > 0$ for every $\xi \neq 0$. This has the purpose to reduce the kinetic energy in the following sense. If the unicycle drives into a descent direction of ψ, then $\dot{y}$ is negative and the scaling factor (10) becomes smaller. Conversely, if the unicycle drives into an ascent direction of ψ, then $\dot{y}$ is positive and the scaling factor (10) becomes larger. This means that the negative gradient force (9) is reduced if the unicycle drives into a descent direction of ψ, and amplified if the unicycle drives into an ascent direction of ψ. The stability analysis in Section 4 will show that the scaled gradient force leads to a loss of kinetic energy in the averaged system. Because of the approximation property, the same is also true for the closed-loop system if ω is sufficiently large. Finally, we comment on the choice of the exponents $1/4$ and $1/2$ of ω in (8). This ensures two properties. On the one hand, it ensures that the observer gains grow with increasing ω. On the other hand, the factors $\omega^{1/4}$ and $\omega^{1/2}$ grow slower than the amplitude ω of the sinusoid u^ω in (4), which is important to ensure a good approximation of the averaged system.

For the sake of a compact notation, we define a function $\alpha\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ to be the product

$$\alpha(y, z) \;:=\; \alpha_1(y - z_1)\,\alpha(z_2)$$

on the right-hand side of (7). Moreover, to represent the linear system (8), we introduce the matrices

$$A^\omega \;:=\; \begin{pmatrix} -h_1\omega^{1/4} & 1 \\ -h_2\omega^{1/2} & 0 \end{pmatrix} \quad\text{and}\quad B^\omega \;:=\; \begin{pmatrix} h_1\omega^{1/4} \\ h_2\omega^{1/2} \end{pmatrix}. \tag{11}$$

Now, using the above notation, the closed-loop system is the ω-dependent, time-varying system

$$\dot{p} \;=\; v\,e_{\shortparallel}(t), \tag{12a}$$

$$\dot{v} \;=\; u^\omega(t)\,\alpha(\psi(p), z), \tag{12b}$$

$$\dot{z} \;=\; A^\omega z + B^\omega\psi(p) \tag{12c}$$

on $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$. In the next section, we study the behavior of (12) in the limit $\omega \to \infty$.

3. AVERAGING ANALYSIS

Our goal is to apply the averaging approach from Bullo (2002) to the closed-loop system (12). For every $z \in \mathbb{R}^2$, define a time-varying vector field $f(\cdot, \cdot, z)$ on $\mathbb{R}^2$ by

$$f(t,p,z) \;:=\; \alpha(\psi(p),z)\, e_{\shortparallel}(t).$$

Then, equation (12b) can be written as

$$\dot v \;=\; u^{\omega}(t)\, f(t,p,z)^{\top} e_{\shortparallel}(t). \tag{13}$$

Next, for every $z \in \mathbb{R}^2$, the so-called *(fixed-time) symmetric product* of $f(\cdot,\cdot,z)$ with itself is the time-varying vector field $\langle f\!:\!f\rangle(\cdot,\cdot,z)$ on $\mathbb{R}^2$ that is given by

$$\langle f\!:\!f\rangle(t,p,z) \;:=\; 2\,\frac{\partial f}{\partial p}(t,p,z) f(t,p,z),$$

where $\frac{\partial f}{\partial p}(t,p,z)$ denotes the derivative of $f(t,\cdot,z)$ at p. A direct computation shows that

$$\langle f\!:\!f\rangle(t,p,z) \;=\; 2\left(\frac{\partial \alpha}{\partial y}\right)(\psi(p),z)\,\big(\nabla\psi(p)^{\top} e_{\shortparallel}(t)\big) e_{\shortparallel}(t),$$

where $\frac{\partial \alpha}{\partial y}(y,z)$ denotes the derivative of $\alpha(\cdot,z)$ at y and $\nabla\psi(p) \in \mathbb{R}^2$ denotes gradient vector of ψ at p. It is known from Bullo (2002); Bullo and Lewis (2004) that the particular choice of u^{ω} in (4) leads to an approximation of $\langle f\!:\!f\rangle$ for sufficiently large ω. Roughly speaking, in the limit $\omega \to \infty$, we may replace (13) by

$$\dot v \;=\; -\Lambda\, \langle f\!:\!f\rangle(t,p,z)^{\top} e_{\shortparallel}(t),$$

where $\Lambda := (c/2)^2$. On the other hand, also the differential equation (12c) of the observer state z depends on ω. It is know from Khalil (2002) that high observer gains lead to an approximation of the signal (2) and its derivative. Note that the derivative of ψ along (12a) is given by $v\,\nabla\psi(p)^{\top} e_{\shortparallel}(t)$. For this reason, we define $\bar z\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^2$ for every $t \in \mathbb{R}$ and every $x = (p^{\top}, v)^{\top}$ with $p \in \mathbb{R}^2$ and $v \in \mathbb{R}$ to be the vector

$$\bar z(t,x) \;:=\; \begin{pmatrix} \psi(p) \\ v\,\nabla\psi(p)^{\top} e_{\shortparallel}(t) \end{pmatrix} \tag{14}$$

of the signal (2) and its derivative. Roughly speaking, in the limit $\omega \to \infty$, we may replace the observer state z by the vector $\bar z(t,x)$. A direct computation shows that

$$\begin{aligned} &\langle f\!:\!f\rangle(t,p,\bar z(t,x)) \\ &\quad= 2(\alpha_1\alpha_1')(0)\,\alpha_2^2(v\,\nabla\psi(p)^{\top} e_{\shortparallel}(t))\,\big(\nabla\psi(p)^{\top} e_{\shortparallel}(t)\big)\, e_{\shortparallel}(t). \end{aligned}$$

This leads us to the time-varying system

$$\dot{\bar p} = \bar v\, e_{\shortparallel}(t), \tag{15a}$$

$$\dot{\bar v} = -\frac{c^2}{2}\,(\alpha_1\alpha_1')(0)\,\alpha_2^2(\bar v\,\nabla\psi(\bar p)^{\top} e_{\shortparallel}(t))\,\nabla\psi(\bar p)^{\top} e_{\shortparallel}(t) \tag{15b}$$

on $\mathbb{R}^2\times\mathbb{R}$ as a candidate for the averaged system of (12). Indeed, the following approximation result holds.

Theorem 1. Assume that $\bar P_0 \subseteq \bar P \subseteq \mathbb{R}^2$ and $\bar V_0 \subseteq \bar V \subseteq \mathbb{R}$ are compact sets such that, for every $t_0 \in \mathbb{R}$ and every maximal solution $(\bar p, \bar v)$ of (15) with $\bar p(t_0) \in \bar P_0$, $\bar v(t_0) \in \bar V_0$, we have $\bar p(t) \in \bar P$, $\bar v(t) \in \bar V$ for every $t \geq t_0$. Then, for every compact set $\tilde Z_0 \subseteq \mathbb{R}^2$, every $\varepsilon > 0$, and every $T > 0$, there exists $\omega_0 > 0$ such that, for every $\omega \geq \omega_0$, every $t_0 \in \mathbb{R}$, every maximal solution (p,v,z) of (12), every maximal solution $(\bar p,\bar v)$ of (15) with $\bar p(t_0) = p(t_0) \in \bar P_0$, $\bar v(t_0) = \tilde v(t_0) \in \bar V_0$, $\tilde z(t_0) \in \tilde Z_0$, we have

$$\|\bar p(t) - p(t)\| \;\leq\; \varepsilon, \tag{16a}$$

$$|\bar v(t) - \tilde v(t)| \;\leq\; \varepsilon, \tag{16b}$$

$$\|\tilde z(t) - \mathrm{e}^{(t-t_0)A^{\omega}}\tilde z(t_0)\| \;\leq\; \varepsilon \tag{16c}$$

for every $t \geq t_0$, where

$$U^{\omega}(t) \;:=\; c\sin(\omega t + \varphi), \tag{17}$$

$$\tilde v(t) \;:=\; v(t) - U^{\omega}(t)\,\alpha(\psi(p(t)), z(t)), \tag{18}$$

$$\tilde z(t) \;:=\; z(t) - \bar z(t, \tilde x(t)), \tag{19}$$

and $\tilde x(t) := (p(t)^{\top}, \tilde v(t))^{\top}$.

The proof of Theorem 1 is similar to the proof of Theorem 1 in Suttner (2019a) but more technical because of the peaking phenomenon in the observer (see Remark 2 below). An outline of the main arguments is given in Appendix A.

Remark 2. In Theorem 1, the symbol $\mathrm{e}^{(t-t_0)A^{\omega}}$ denotes the matrix exponential of $(t-t_0)A^{\omega}$. A direct computation shows that the entries $(\mathrm{e}^{tA^{\omega}})_{ij}$ of $\mathrm{e}^{tA^{\omega}}$ are given by

$$\begin{aligned} (\mathrm{e}^{tA^{\omega}})_{11} &= \mathrm{e}^{-\frac{h_1}{2}\omega^{\kappa} t}\big(\cos(\Delta\omega^{\kappa} t) - \frac{h_1}{2}\omega^{\kappa} t\,\mathrm{sinc}(\Delta\omega^{\kappa} t)\big), \\ (\mathrm{e}^{tA^{\omega}})_{12} &= \mathrm{e}^{-\frac{h_1}{2}\omega^{\kappa} t}\,\frac{1}{\omega^{\kappa}}\,\omega^{\kappa} t\;\mathrm{sinc}(\Delta\omega^{\kappa} t), \\ (\mathrm{e}^{tA^{\omega}})_{21} &= \mathrm{e}^{-\frac{h_1}{2}\omega^{\kappa} t}\,\omega^{\kappa}\,(\omega^{\kappa} t)\;\mathrm{sinc}(\Delta\omega^{\kappa} t), \\ (\mathrm{e}^{tA^{\omega}})_{22} &= \mathrm{e}^{-\frac{h_1}{2}\omega^{\kappa} t}\big(\cos(\Delta\omega^{\kappa} t) + \frac{h_1}{2}\omega^{\kappa} t\,\mathrm{sinc}(\Delta\omega^{\kappa} t)\big), \end{aligned}$$

where $\kappa := 1/4$ and $\Delta := \sqrt{h_2 - (h_1/2)^2}$ is a complex number with real part $< h_1/2$. Thus, we have $\mathrm{e}^{tA^{\omega}} \to 0$ as $t \to \infty$. Note, however, that $(\mathrm{e}^{tA^{\omega}})_{21}$ leads to a transient peak of height $\sim \omega^{\kappa}$ if the initial state of z_1 deviates significantly from $\psi(p(t_0))$. This behavior is known as the *peaking phenomenon*; cf. Khalil (2002). It is therefore favorable (but not necessary) to initialize z_1 through the first measurement of the signal (2). The assumed boundedness of α_1, α_2 and their derivatives in Section 2 shields the state of the unicycle from a possible peaking phenomenon in the observer.

In the next section, we investigate stability properties of the closed-loop system (12) and its averaged system (15).

4. STABILITY ANALYSIS

To prove global uniform asymptotic stability for the averaged system (15), we make the same assumptions on the signal function $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ as in Suttner (2019a):

A1 ψ is smooth,
A2 ψ has a unique critical point $p^* \in \mathbb{R}^2$,
A3 $\psi(p) \to \infty$ as $\|p\| \to \infty$.

The above assumptions are sufficient to derive the following stability result for the averaged system (15).

Theorem 3. Suppose that ψ satisfies assumptions A1-A3. Then, the point $(p^*, 0) \in \mathbb{R}^3$ is globally uniformly asymptotically stable for (15).

Proof. We combine the position $\bar p \in \mathbb{R}^2$ and the velocity $\bar v \in \mathbb{R}$ to one vector $\bar x = (\bar p^{\top}, \bar v)^{\top} \in \mathbb{R}^3$. Note that (15) is a periodic system. Our goal is to prove the stability result by applying LaSalle's invariance principle for periodic systems (see, e.g., Theorem 5.26 in Sastry (1999)). For this purpose, we introduce the total energy function $E\colon \mathbb{R}^3 \to \mathbb{R}$ as the sum

$$E(\bar x) \;:=\; \frac{1}{2}\bar v^2 + \frac{c^2}{2}\,(\alpha_1\alpha_1')(0)\,\alpha_2^2(0)\,\psi(\bar p) \tag{20}$$

of kinetic and potential energy. Because of the assumptions on α_1, α_2, and ψ, the function E is (up to an additive constant) positive definite around $x^* := ((p^*)^{\top}, 0)^{\top} \in \mathbb{R}^3$ and its only critical point is x^*. A direct computation shows that the derivative $\dot{E}\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ of E along (15) is given by

$$\dot{E}(t,\bar{x}) \;=\; -\frac{c^2}{2}\,(\alpha_1\alpha_1')(0)\,\big(\alpha_2^2(\bar{z}_2(t,\bar{x})) - \alpha_2^2(0)\big)\,\bar{z}_2(t,\bar{x}),$$

where $\bar{z}_2(t,\bar{x})$ denotes the second component of (14). Because of the assumptions on α_1, α_2 we conclude that $\dot{E}$ only takes non-positive values. For the rest of the proof, suppose that $\bar{x} = (\bar{p}^\top, \bar{v})^\top\colon \mathbb{R}\to\mathbb{R}^3$ is a solution of (15) such that $\dot{E}(t,\bar{x}(t)) = 0$ for every $t\in\mathbb{R}$. To conclude that x^* is globally uniformly asymptotically stable for (15), we have to show that $\bar{x}$ is identically equal to x^*. Define $\bar{y}\colon \mathbb{R}\to\mathbb{R}$ by $\bar{y}(t) := \psi(\bar{p}(t))$. Note that

$$\dot{E}(t,\bar{x}(t)) \;=\; -\frac{c^2}{2}\,(\alpha_1\alpha_1')(0)\,\big(\alpha_2^2(\dot{\bar{y}}(t)) - \alpha_2^2(0)\big)\,\dot{\bar{y}}(t)$$

for every $t\in\mathbb{R}$. Because of the assumptions on α_1, α_2, the condition $\dot{E}(t,\bar{x}(t)) = 0$ for every $t\in\mathbb{R}$ implies that $\dot{\bar{y}}$ is identically equal to 0. Therefore, $\bar{y}$ is constant, which means that $\bar{p}$ runs in one of the compact level sets of ψ. Moreover,

$$(\bar{v}^2)\dot{}(t) \;=\; -c^2\,(\alpha_1\alpha_1')(0)\,\alpha_2^2(\dot{\bar{y}}(t))\,\dot{\bar{y}}(t) \;=\; 0$$

for every $t\in\mathbb{R}$ implies that $\bar{v}$ is identically equal to some $\bar{v}_0\in\mathbb{R}$. We make a case analysis. First, we suppose that $\bar{v}_0 = 0$. Then $\bar{p}$ is identically equal to some $\bar{p}_0\in\mathbb{R}^2$. Using (15b), it follows that $\nabla\psi(\bar{p}_0)^\top e_{\shortparallel}(t) = 0$ for every $t\in\mathbb{R}$. This in turn implies $\nabla\psi(\bar{p}_0) = 0$ and therefore $\bar{p}_0 = p^*$. The proof is complete if we can rule out the case $\bar{v}_0 \neq 0$.

Suppose for the sake of contradiction that $\bar{v}_0 \neq 0$. Recall that $\dot{\bar{p}}$ is defined by the differential equation (15a) and that the alignment vector $e_{\shortparallel}(t)$ is determined by the particular choice of the angle $\theta(t)$ in (3). After a suitable shift of the time parameter and a suitable rotation of the coordinate system, we may assume that

$$\dot{\bar{p}}(t) \;=\; \bar{v}_0\begin{pmatrix}\cos(C\,\sin(\Omega t))\\ \sin(C\,\sin(\Omega t))\end{pmatrix} \tag{21}$$

for every $t\in\mathbb{R}$. If we would have $\bar{p}(2\pi/\Omega) \neq \bar{p}(0)$, then the periodicity of the right-hand side of (21) would imply that $\bar{p}(2\pi k/\Omega) = k\big(\bar{p}(2\pi/\Omega) - \bar{p}(0)\big)$ for every integer k, which in turn would contradict the fact that $\bar{p}$ runs in one of the compact level sets of ψ. Thus, we have $\bar{p}(2\pi/\Omega) = \bar{p}(0)$. By integrating both sides of (21), we obtain that

$$\bar{p}_0 \;:=\; \bar{p}\big(\tfrac{\pi}{2\Omega}\big) \;=\; \bar{p}\big(\tfrac{3\pi}{2\Omega}\big)$$

and also

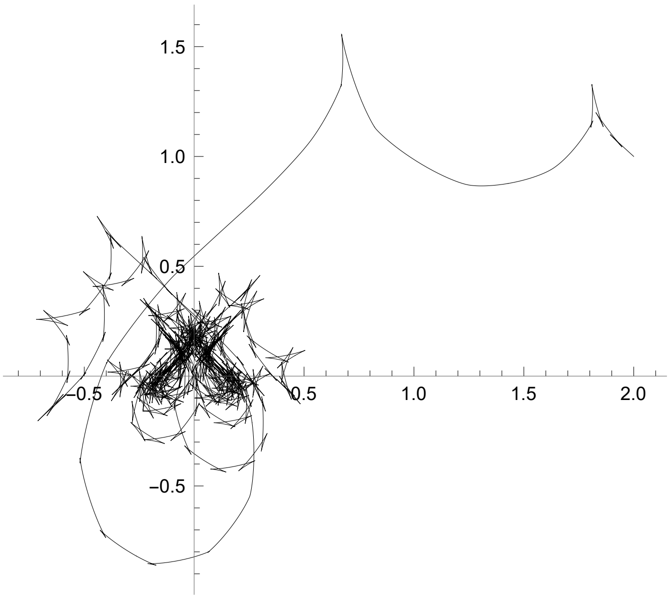
$$\int_0^{\pi/(2\Omega)} \cos(C\,\sin(\Omega t))\,\mathrm{d}t \;=\; 0. \tag{22}$$

Equation (22) and the results in Watson (1944) on the roots of Bessel functions imply that C is not an integer multiple of $\pi/2$. It follows that $\dot{\bar{p}}\big(\frac{\pi}{2\Omega}\big)$ and $\dot{\bar{p}}\big(\frac{3\pi}{2\Omega}\big)$ are linearly independent vectors. Since $\nabla\psi(\bar{p}(t))^\top\dot{\bar{p}}(t) = \dot{\bar{y}}(t)/\bar{v}_0 = 0$ for every $t\in\mathbb{R}$, we conclude that $\nabla\psi(\bar{p}_0) = 0$ and therefore $\bar{p}_0 = p^*$. Since $\bar{p}$ runs in a level set of ψ, this implies that $\bar{p}$ is identically equal to p^*, which contradicts the assumption $\bar{v}_0 \neq 0$. This completes the proof. □.

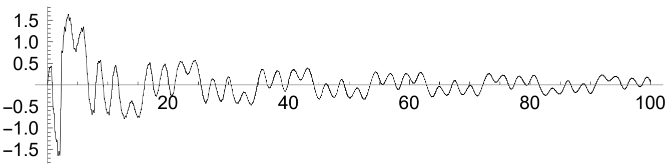
Remark 4. The statement of Theorem 3 is, in general, not true if we would replace the choice (3) of the angle θ as in Suttner (2019a) by

$$\theta(t) \;=\; \Omega t + \Phi \tag{23}$$

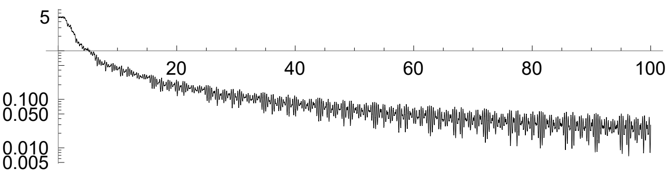
for every $t\in\mathbb{R}$. To verify this by an example, suppose that the circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$ of radius one centered at the origin is a level set of the cost function ψ. For instance, this happens if ψ is the standard quadratic function $p \mapsto \|p\|^2$. If we initialize the averaged system (15) with $\bar{p}(0) = \big(\begin{smallmatrix}\sin\Phi\\ -\cos\Phi\end{smallmatrix}\big)$ and $\bar{v}(0) = 1$ and use (23) instead of (3), then $\bar{p}$ runs along $\mathbb{S}^1$ and $\bar{v}$ remains identically equal to 1. In other words, $\bar{p}$ is "trapped" in a level set of ψ. One possible (but not unique) way to overcome this problem is to choose an alignment as in (3). The proof of Theorem 3 shows that, for θ as in (3), the curve $\bar{p}$ cannot run along a level set of ψ except for the case in which $\bar{p}$ is identically equal to the optimal point p^*.



(a) Position $t \mapsto p(t)$ on the time interval $[0, 100]$.



(b) Transformed velocity $t \mapsto \tilde{v}(t)$ with $\tilde{v}$ as in (18).



(c) Logarithmic plot of the "total energy" $t \mapsto E(\tilde{x}(t))$ with E defined in (20).

Fig. 2. Results for the situation as described in Section 5.

Next, we introduce suitable notions of stability for the closed-loop system (12). As explained in Remark 2, we cannot expect practical stability for the entire system because of the possible peaking phenomenon in the observer shortly after initialization. Once the transient peak is over, the observer state approximates the sensed signal and its derivative. To circumvent this problem, the subsequent notions of stability only take the position and the velocity of the unicycle into account.

Definition 5. The solutions of (12) are said to be *practically uniformly bounded* if, for all compact sets $P_0 \subseteq \mathbb{R}^2$,

$V_0 \subseteq \mathbb{R}$, and $Z_0 \subseteq \mathbb{R}^2$, there exist $\omega_0 > 0$ and compact supersets P of P_0 and V of V_0 such that, for every $\omega \geq \omega_0$, every $t_0 \in \mathbb{R}$, and every maximal solution (p, v, z) of (12) with $p(t_0) \in P_0$, $v(t_0) \in V_0$, and $z(t_0) \in Z_0$, we have $p(t) \in P$ and $v(t) \in V$ for every $t \geq t_0$.

Definition 6. For given $p^* \in \mathbb{R}^2$ and $\hat{v} > 0$, the compact subset $\{p^*\} \times [-\hat{v}, \hat{v}]$ of $\mathbb{R}^3$ is said to be *practically uniformly stable* for (12) if, for all $\rho, \delta > 0$ and every compact set $Z_0 \subseteq \mathbb{R}^2$, there exist $r, d > 0$ and $\omega_0 > 0$ such that, for every $\omega \geq \omega_0$, every $t_0 \in \mathbb{R}$, and every maximal solution (p, v, z) of (12) with $\|p(t_0) - p^*\| \leq r$, $|v(t_0)| \leq \hat{v} + d$, and $z(t_0) \in Z_0$, we have $\|p(t) - p^*\| \leq \rho$ and $|v(t)| \leq \hat{v} + \delta$ for every $t \geq t_0$.

Definition 7. For given $p^* \in \mathbb{R}^2$ and $\hat{v} > 0$, the compact subset $\{p^*\} \times [-\hat{v}, \hat{v}]$ of $\mathbb{R}^3$ is said to be *semi-globally practically uniformly attractive* for (12) if, for all $r, d > 0$, every compact set $Z_0 \subseteq \mathbb{R}^2$, and all $\rho, \delta > 0$, there exist $T > 0$ and $\omega_0 > 0$ such that, for every $\omega \geq \omega_0$, every $t_0 \in \mathbb{R}$, and every maximal solution (p, v, z) of (12) with $\|p(t_0) - p^*\| \leq r$, $|v(t_0)| \leq \hat{v} + d$, and $z(t_0) \in Z_0$, we have $\|p(t) - p^*\| \leq \rho$ and $|v(t)| \leq \hat{v} + \delta$ for every $t \geq t_0 + T$.

Theorem 8. Define $\hat{v} := c\,|\alpha_1(0)|\,|\alpha_2(0)| > 0$. Suppose that ψ satisfies assumptions A1-A3. Then, the solutions of (12) are practically uniformly bounded and the compact subset $\{p^*\} \times [-\hat{v}, \hat{v}]$ of $\mathbb{R}^3$ is semi-globally practically uniformly attractive for (12).

Proof. The statement follows from the stability result for the averaged system (Theorem 3) and the approximation result for the trajectories of the closed-loop system (Theorem 1). The arguments are the same as in the proof of Theorem 7 in Suttner (2019a). A similar proof can be also found in Dürr et al. (2013). □

Remark 9. Note that Theorem 8 does *not* guarantee practical uniform stability for the closed-loop system. As explained in Remark 8 in Suttner (2019a), the position of the unicycle displays a transient behavior that violates the definition of practical stability if we have $U^\omega(t_0) \neq 0$ for the initial time $t_0 \in \mathbb{R}$, where $U^\omega \colon \mathbb{R} \to \mathbb{R}$ is defined by (17). However, we do have practical stability for every initial time $t_0 \in \mathbb{R}$ with $U^\omega(t_0) = 0$, because then the transformed initial velocity $\tilde{v}(t_0)$ in (18) coincides with the actual initial velocity of the closed-loop system.

5. SIMULATION RESULTS

Finally, we provide numerical data for the following situation. The cost function in (2) is given by $\psi(p) := \|p\|^2$. The parameters in (3) are given by $\theta_0 := \Phi := 0$, $\Omega := 1$, and C is chosen as the smallest positive real number that satisfies (22). The parameters in (4) are given by $\varphi := 0$, $c := 1$, and $\omega := 20$. The functions α_1 and α_2 are given by (5) and (6), respectively. The parameters in (8) are given by $h_1 := 2$ and $h_2 := 6$. Since $\varphi = 0$, we know from Remark 9 that the closed-loop system (12) is semi-globally practically asymptotically stable if we start the implementation of the control law at an initial time $t_0 \in \mathbb{R}$ with $\sin(\omega t_0) = 0$. Figure 2 shows the results for the initial condition $p(0) = (2, 1)^\top$, $v(0) = 0$, $\bar{z}(0) = (5, 0)^\top$.

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Appendix A. PROOF OF THEOREM 1

Throughout the proof, we consider the closed-loop system (12) in the coordinates $(p, \tilde{v}, \tilde{z})$ that are given by (18) and (19). When we use the notation $\tilde{x} = (p^\top, \tilde{v})^\top$, then we mean that $\tilde{x}$ is a vector in $\mathbb{R}^3$ with components $p \in \mathbb{R}^2$ and $\tilde{v} \in \mathbb{R}$. We define $\tilde{a}\colon \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\tilde{a}(t, \tilde{x}, \tilde{z}) \;:=\; -\frac{c^2}{2}\,(\alpha_1\alpha_1)'(\tilde{z}_1)\,\alpha_2^2(\bar{z}_2(t,\tilde{x})+\tilde{z}_2)\,\nabla\psi(p)^\top e_{\shortparallel}(t),$$

where $\bar{z}_2(t,\tilde{x})$ denotes the second component of (14). Next, we define $F\colon \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$ by

$$F(t, \tilde{x}, \tilde{z}) \;:=\; \begin{pmatrix} \tilde{v}\, e_{\shortparallel}(t) \\ \tilde{a}(t, \tilde{x}, \tilde{z}) \end{pmatrix}.$$

Lemma 10. For each compact subset $\tilde{X}$ of $\mathbb{R}^3$, there exist $b > 0, L_x > 0, L_z > 0$ such that

$$\begin{aligned} \|F(t,\tilde{x},\tilde{z})\| &\le b, \\ \|F(t,\bar{x},0) - F(t,\tilde{x},0)\| &\le L_x\,\|\bar{x}-\tilde{x}\|, \\ \|F(t,\tilde{x},\tilde{z}) - F(t,\tilde{x},0)\| &\le L_z\,\|\tilde{z}\| \end{aligned}$$

for every $t \in \mathbb{R}$, all $\tilde{x}, \bar{x} \in \tilde{X}$, and every $\tilde{z} \in \mathbb{R}^2$.

Proof. The statement follows immediately from the smoothness of F, the periodicity of F with respect to time, and the boundedness of α_1, α_2 and their derivatives. □

We define $\tilde{\alpha}\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\tilde{\alpha}(\bar{z}_2, \tilde{z}) \;:=\; -\alpha_1(-\tilde{z}_1)\,\alpha_2(\bar{z}_2 + \tilde{z}_2).$$

For each $\omega > 0$, we define $W^\omega\colon \mathbb{R} \to \mathbb{R}$ by

$$W^\omega(t) \;:=\; U^\omega(t)^2 - \frac{c^2}{2} \;=\; -\frac{c}{2}\,\cos(2\omega t + 2\varphi)$$

and $\beta^\omega\colon \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} \beta^\omega(t,\tilde{x},\tilde{z}) := {} & \tilde{a}(t,\tilde{x},\tilde{z})\,\nabla\psi(p)^\top e_{\shortparallel}(t) + \tilde{v}^2\, e_{\shortparallel}(t)^\top \nabla^2\psi(p) e_{\shortparallel}(t) \\ & + \tilde{v}\,\nabla\psi(p)^\top \dot{e}_{\shortparallel}(t) + W^\omega(t)\,\tilde{a}(t,\tilde{x},\tilde{z})\,\nabla\psi(p)^\top \dot{e}_{\shortparallel}(t) \\ & + U^\omega(t)\,\tilde{\alpha}(\bar{z}_2(t,\tilde{x}),\tilde{z})\,\tilde{v}\,e_{\shortparallel}(t)^\top \nabla^2\psi(p) e_{\shortparallel}(t), \end{aligned}$$

where $\nabla^2\psi(p) \in \mathbb{R}^{2\times 2}$ denotes the Hessian matrix of ψ at p and $\dot{e}_{\shortparallel}(t)$ denotes the derivative of $e_{\shortparallel}$ at t. Next, we define $\Gamma\colon \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Gamma(t,\tilde{x},\tilde{z}) \;:=\; \nabla\psi(p)^\top e_{\shortparallel}(t) \begin{pmatrix} \tilde{\alpha}(\bar{z}_2(t,\tilde{x}),\tilde{z}) \\ \frac{\partial\tilde{\alpha}}{\partial\tilde{z}}(\bar{z}_2(t,\tilde{x}),\tilde{z}) A^\omega \tilde{z} \end{pmatrix},$$

where $\frac{\partial\tilde{\alpha}}{\partial\tilde{z}}(\bar{z}_2,\tilde{z})$ denotes the derivative of $\frac{\partial\tilde{\alpha}}{\partial\tilde{z}}(\bar{z}_2,\cdot)$ at $\tilde{z}$, and $A^\omega \in \mathbb{R}^{2\times 2}$ is defined in (11). For each $\omega > 0$, define $R^\omega\colon \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\begin{aligned} R^\omega(t,\tilde{x},\tilde{z}) \;:=\; & U^\omega(t) \begin{pmatrix} \tilde{\alpha}(\bar{z}_2(t,\tilde{x}),\tilde{z})\, e_{\shortparallel}(t) \\ \frac{\partial\tilde{\alpha}}{\partial\tilde{z}}(\bar{z}_2(t,\tilde{x}),\tilde{z}) A^\omega \tilde{z} \end{pmatrix} \\ & + W^\omega(t) \begin{pmatrix} 0 \\ \tilde{a}(t,\tilde{x},\tilde{z}) \end{pmatrix}. \end{aligned}$$

A direct computation shows that, in the coordinates (18) and (19), system (12) reads

$$\dot{\tilde{x}} \;=\; F(t,\tilde{x},\tilde{z}) + R^\omega(t,\tilde{x},\tilde{z}), \tag{A.1a}$$

$$\dot{\tilde{z}} \;=\; A^\omega\tilde{z} - \begin{pmatrix} 0 \\ \beta^\omega(t,\tilde{x},\tilde{z}) \end{pmatrix} - U^\omega(t)\,\Gamma(t,\tilde{x},\tilde{z}). \tag{A.1b}$$

Lemma 11. Let $\tilde{X} \subseteq \mathbb{R}^3$ and $\tilde{Z}_0 \subseteq \mathbb{R}^2$ be compact sets, and let $\varepsilon_z > 0$ and $T > 0$. Then, there exists $\omega_0 > 0$ such that, for every $\omega \ge \omega_0$, every $t_0 \in \mathbb{R}$, every maximal solution $(\tilde{x},\tilde{z})$ of (A.1) with $\tilde{z}(t_0) \in \tilde{Z}_0$, the following implication holds: if $\tilde{x}(t) \in \tilde{X}$ for every $t \in [t_0, t_0+T]$, then, for every $t \in [t_0, t_0+T]$, we have

$$\|\tilde{z}(t) - \mathrm{e}^{(t-t_0)A^\omega}\tilde{z}(t_0)\| \;\le\; \varepsilon_z.$$

Proof. By the variation of constant formula, we have

$$\tilde{z}(t_1) - \mathrm{e}^{(t-t_0)A^\omega}\tilde{z}(t_0) \tag{A.2a}$$

$$= -\int_{t_0}^{t_1} \beta^\omega(t,\tilde{x}(t),\tilde{z}(t))\,\mathrm{e}^{(t_1-t)A^\omega}\begin{pmatrix}0\\1\end{pmatrix}\mathrm{d}t \tag{A.2b}$$

$$- \int_{t_0}^{t_1} U^\omega(t)\,\mathrm{e}^{(t_1-t)A^\omega}\,\Gamma(t,\tilde{x}(t),\tilde{z}(t))\,\mathrm{d}t \tag{A.2c}$$

for every t_1 in the domain of $\tilde{z}$. A similar argument as in the proof of Lemma 10 shows that β is bounded by a constant on $\mathbb{R} \times \tilde{X} \times \mathbb{R}^2$. Moreover, using the formulas in Remark 2 for the matrix exponential of A^ω, one can verify that there exists $\mu > 0$ such that

$$\int_{t_0}^{t_1} \left\| \mathrm{e}^{(t_1-t)A^\omega}\begin{pmatrix}0\\1\end{pmatrix} \right\| \mathrm{d}t \;\le\; \frac{\mu}{\omega^{1/4}}$$

for every $\omega \ge 1$ and every $t_1 \ge t_0$. Thus, the integral in (A.2b) tends uniformly to zero for $\omega \to \infty$ as long as $\tilde{x}$ runs in $\tilde{X}$. For the integral in (A.2c), we apply integration by parts. Note that $t \mapsto -\frac{c}{\omega}\cos(\omega t + \varphi)$ is an antiderivative of U^ω. A lengthy computation reveals that the factor $\frac{1}{\omega}$ from the antiderivative of U^ω compensates the powers of ω from the observer gains as well as the impact of a transient peak of the observer state with increasing ω. This way, one can show that also the integral in (A.2c) tends uniformly to zero for $\omega \to \infty$ as long as $\tilde{x}$ runs in $\tilde{X}$. □

Lemma 12. Let $\tilde{X} \subseteq \mathbb{R}^3$ and $\tilde{Z}_0 \subseteq \mathbb{R}^2$ be compact sets, and let $\varepsilon_x > 0$ and $T > 0$. Then, there exists $\omega_0 > 0$ such that, for every $\omega \ge \omega_0$, every $t_0 \in \mathbb{R}$, and every maximal solution $(\tilde{x},\tilde{z})$ of (A.1) with $\tilde{z}(t_0) \in \tilde{Z}_0$, the following implication holds: if $\tilde{x}(t) \in \tilde{X}$ for every $t \in [t_0, t_0+T]$, then, for every $t \in [t_0, t_0+T]$, we have

$$\left\| \int_{t_0}^{t_1} R^\omega(t,\tilde{x}(t),\tilde{z}(t))\,\mathrm{d}t \right\| \;\le\; \varepsilon_x.$$

Proof. The statement can be deduced from a similar application of integration by parts as in the proof of Lemma 11 for the integral in (A.2c). Note that suitable antiderivatives of both U^ω and W^ω contribute a factor $\frac{1}{\omega}$. This factor decays sufficiently fast in order to compensate the powers of ω from the observer gains as well as the impact of a transient peak of the observer state. For this reason the integral over R^ω tends uniformly to 0 for $\omega \to \infty$ as long as $\tilde{x}$ runs in $\tilde{X}$. □

Now we are ready to prove Theorem 1. Note that (15) can be written as

$$\dot{\bar{x}} \;=\; F(t,\bar{x},0). \tag{A.3}$$

Assume that $\bar{X}_0 \subseteq \bar{X} \subseteq \mathbb{R}^3$ are compact sets such that, for every $t_0 \in \mathbb{R}$ and every maximal solution $\bar{x}$ of (15) with $\bar{x}(t_0) \in \bar{X}_0$, we have $\bar{x}(t) \in \bar{X}$ for every $t \geq t_0$. Let $\tilde{Z}_0 \subseteq \mathbb{R}^2$ be a compact set and fix arbitrary $\varepsilon > 0$ and $T > 0$. Let $\tilde{X}$ be the closed ε-neighborhood of $\bar{X}$ in $\mathbb{R}^3$. For the compact set $\tilde{X}$, there exist constants $b, L_x, L_z > 0$ as in Lemma 10. After possibly increasing L_x or T, we may assume that $\mathrm{e}^{-L_x T} < 1$. Define $\varepsilon_x := \frac{\varepsilon}{3}\mathrm{e}^{-L_x T}$ and $\varepsilon_z := \frac{\varepsilon}{6 L_z T}\mathrm{e}^{-L_x T}$. Then, there exists some sufficiently large $\omega_0 > 0$ such that, for every $\omega \geq \omega_0$, the estimates in Lemmas 11 and 12 are satisfied. By Remark 2, after possibly increasing ω_0, we have $\|\mathrm{e}^{(t-t_0)A^\omega}\tilde{z}_0\| \leq \varepsilon_z$ for every $\omega \geq \omega_0$, every $(t - t_0) \geq \tau^\omega := \omega^{-1/8}$, and every $\tilde{z}_0 \in \tilde{Z}_0$. Again, after possibly increasing ω_0, we can ensure that $\tau^\omega \leq \min\{T, \frac{\varepsilon}{6b}\mathrm{e}^{-L_x T}\}$ for every $\omega \geq \omega_0$.

Fix arbitrary $\omega \geq \omega_0$, $t_0 \in \mathbb{R}$, $\tilde{x}_0 \in \bar{X}_0$, and $\tilde{z}_0 \in \tilde{Z}_0$. Let $(\tilde{x}, \tilde{z})\colon I \to \mathbb{R}^3 \times \mathbb{R}^2$ be the maximal solution of (A.1) with $\tilde{x}(t_0) = \tilde{x}_0$ and $\tilde{z}(t_0) = \tilde{z}_0$. Let $\bar{x}$ be the maximal solution of (A.3) with $\bar{x}(t_0) = \tilde{x}_0$ and $\tilde{z}(t_0) \in \tilde{Z}_0$. Then, we have

$$\begin{aligned} \tilde{x}(t_1) &= \tilde{x}_0 + \int_{t_0}^{t_1} \big(F(t, \tilde{x}(t), \tilde{z}(t)) + R^\omega(t, \tilde{x}(t), \tilde{z}(t))\big)\mathrm{d}t, \\ \bar{x}(t_1) &= \tilde{x}_0 + \int_{t_0}^{t_1} F(t, \bar{x}(t), 0)\,\mathrm{d}t, \end{aligned}$$

and therefore

$$\bar{x}(t_1) - \tilde{x}(t_1) = -\int_{t_0}^{t_1} R^\omega(t, \tilde{x}(t), \tilde{z}(t))\,\mathrm{d}t \tag{A.4a}$$

$$+\int_{t_0}^{t_1} \big(F(t, \bar{x}(t), 0) - F(t, \tilde{x}(t), \tilde{z}(t))\big)\,\mathrm{d}t \tag{A.4b}$$

for every $t \in I$. Now, for every $t_1 \in I \cap [t_0, t_0 + \tau^\omega]$, the following implication holds: if $\tilde{x}(t) \in \tilde{X}$ for every $t \in [t_0, t_1]$, then we obtain from Lemma 11 that (16c) holds for every $t \in [t_0, t_1]$, and, by applying the estimates in Lemmas 10 and 12 to (A.4) that

$$\|\bar{x}(t_1) - \tilde{x}(t_1)\| \leq \varepsilon_x + 2\,\tau^\omega\, b < 2\varepsilon/3$$

for every $t \in [t_0, t_1]$. It follows that $\tilde{x}$ exists on $[t_0, t_0 + \tau^\omega]$ and that $\|\bar{x}(t) - \tilde{x}(t)\| < \varepsilon$ for every $t \in [t_0, t_0 + \tau^\omega]$. It is left to prove that the same holds on the interval $[t_0 + \tau^\omega, t_0 + T]$. For this purpose, we split up the integral in (A.4a) into integrals from t_0 to $t_0 + \tau^\omega$ and from $t_0 + \tau^\omega$ to $t_0 + T$. We already know from the preceding considerations that contribution of the integral from t_0 to $t_0 + \tau^\omega$ is $\leq 2\,\tau^\omega\, b$. For the integral from $t_0 + \tau^\omega$ to $t_0 + T$, we obtain from Lemma 10 that

$$\|F(t, \bar{x}(t), 0) - F(t, \tilde{x}(t), \tilde{z}(t))\| \leq L_z\|\tilde{z}(t)\| + L_x\|\bar{x}(t) - \tilde{x}(t)\|$$

as long as $\tilde{x}(t) \in \tilde{X}$. Thus, for every $t_1 \in I \cap [t_0 + \tau^\omega, t_0 + T]$, the following implication holds: if $\tilde{x}(t) \in \tilde{X}$ for every $t \in [t_0, t_1]$, then we obtain from Lemma 11 that (16c) holds for every $t \in [t_0, t_1]$, and from (A.4) that

$$\begin{aligned} \|\bar{x}(t_1) - \tilde{x}(t_1)\| \leq{} & \varepsilon_x + 2\,\tau^\omega\, b + 2\,T\, L_z\, \varepsilon_z \\ & + L_x \int_{t_0}^{t_1} \|\bar{x}(t) - \tilde{x}(t)\|\,\mathrm{d}t \end{aligned}$$

for every $t \in [t_0, t_1]$. Since $\varepsilon_x + 2\,\tau^\omega\, b + 2\,T\, L_z\, \varepsilon_z < \varepsilon\,\mathrm{e}^{-L_x T}$, we conclude from the Gronwall inequality that $\tilde{x}$ exists on $[t_0 + \tau^\omega, t_0 + T]$ and that $\|\bar{x}(t) - \tilde{x}(t)\| < \varepsilon$ for every $t \in [t_0 + \tau^\omega, t_0 + T]$. Consequently, (16c) and $\|\bar{x}(t) - \tilde{x}(t)\| < \varepsilon$ are satisfied for every $t \in [t_0, t_0 + T]$.