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Stability Analysis for A Class of Linear Hyperbolic System of Balance Laws with Sampled-data Control ⋆

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Abstract: The stability for a class of linear hyperbolic systems with distributed sampled-data controllers is discussed in this paper. The original sampled-data system is firstly transformed into a new equivalent system by modelling the sampling induced error as a reset integrator operator. Then we construct an appropriate Lyapunov function and obtain sufficient conditions for the Rε - stability of the system based on linear matrix inequalities (LMIs). A numerical example illustrates our results: when the sampling interval is within the allowable range, the solution of the system converges from the domain of attraction to a positive invariant set.

Keywords: Linear hyperbolic system, sampled-data control, LMIs methods, Lyapunov function, Rε-stability.

1. INTRODUCTION

Nowadays, control systems are usually implemented on digital platforms. The properties of digital implementations are studied in the domain of sampled-data systems (Åström and Wittenmark (2013); Chen and Francis (2012)). For finite-dimensional systems (linear or nonlinear), the stability and control design problems have been extensively studied (see the references in Hetel et al. (2017); Laila et al. (2006); Monaco and Normand-Cyrot (2001, 2007)). Here we are interested in the sampled-data control problem of partial differential equations (PDEs) which involves an infinite dimensional state space. More precisely we focused on sampled-data control of systems governed by hyperbolic PDEs since many typical examples of processes can be described by this kind of system Bastin and Coron (2016), such as the transmission of electrical energy, the flow of liquid in a pipe, the light propagation in optical fibers, the road traffic, etc. Different from the research method of finite dimensional systems, fewer results are available for hyperbolic PDEs.

For the case of idealized sample-and-hold process, Logemann et al. (2003, 2005) provided sampled-data control design methods which stabilized an infinite dimensional system for sufficiently small sampling periods. E. Fridman and co-authors have addressed several problems related to the sampled-data control of parabolic PDEs using a time-delay method (Kang and Fridman (2018); Selivanov and Fridman (2016, 2017)). For hyperbolic PDEs, event-triggered sampled-data control with controller on the boundaries was considered in Espitia et al. (2016, 2017). In Karafyllis and Krstic (2017), the application results of Zero-Order-Hold boundary feedback control in one-dimensional linear hyperbolic systems with non-local terms on bounded domains were given. The boundary feedback control of a 2×2 hyperbolic system was implemented by backstepping method in Davó et al. (2018), and the global asymptotic stability was realized. As we can see from the literature review, the analysis of sampled-data controller for hyperbolic PDEs is a wide open research field and there is still the place for important contributions.

In this paper, we address the stability problem for sampled-data hyperbolic PDEs. Differently from the existing works where boundary control has been considered, here we study distributed sampled-data controllers. The problem is studied from an Input-Output point of view, extending the approaches for finite dimensional systems (Fujioka (2009); Kao and Rantzer (2007); Mirkin (2007); Omran et al. (2014)). The main idea is to represent the sampling induced error as a perturbation for a continuous time hyperbolic equation. By converting the original sampled-data system into an interconnection of a continuous-time PDEs and a reset-integral operator, constructive local stability results are derived.

The layout of this paper is as follows. Section 2 introduces the systems we will study and the problem we have to deal with. In Section 3, we give the equivalent transformation form of the system including sampling error and the concrete stability analysis process. A numerical example in Section 4 shows that our method is feasible. Finally, the summary and prospect of the paper are presented in Section 5.

Notations: $\mathbb{N}$ is the set of nonnegative integers from 0 to infinity, $\mathbb{R}_+$ is the set of nonnegative reals, $\mathbb{R}^n$ is used to denote the set of n-dimensional Euclidean space

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with the norm $|\cdot|$. $L^2(0,L)$ stands for the Hilbert space of square integrable scalar functions on $(0,L)$ with the corresponding norm $\|\cdot\|_{L^2(0,L)}$, defined by $\| e \|_{L^2(0,L)} = \sqrt{\int_0^L |e(x)|^2 dx}$. The associated norm to Sobolev space $H^1(0,L)$ is defined as $\| e \|_{H^1(0,L)} = \sqrt{\int_0^L \left(|e(x)|^2 + |e_x(x)|^2\right) dx}$. The set of functions $\varphi : [0,L] \to \mathbb{R}^n$ such that $\int_0^L |\varphi(x)|^2 dx < \infty$ is denoted by $L^2([0,L];\mathbb{R}^n)$ and given a functional $V : H^1([0,L];\mathbb{R}^n) \to \mathbb{R}_+$ such that $\mathfrak{L}_{V\leq C} = \left\{y \in H^1([0,L];\mathbb{R}^n) : V(y) \leq C\right\}$. The notation $M \leq 0$ denotes that M is symmetric and negative semidefinite. The symmetric elements are denoted by $*$ in the symmetric matrix. The identity matrix is denoted by I and $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ are the minimum and maximum eigenvalues of the matrix Q. $\mathcal{C}^0$ is the continuous function and $\mathcal{C}^1$ is the continuously differentiable function.

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

2.1 System Description

We consider the sampled-data controlled hyperbolic system (1) given below

$$\begin{cases} \partial_t y(t,x) + \Lambda \partial_x y(t,x) + \Upsilon y(t,x) + u(t,x) = 0, & \text{(1a)} \\ u(t,x) = K y(t_k, x), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & \text{(1b)} \\ y(t,0) = y(t,L) = 0, \forall t \geq 0, & \text{(1c)} \\ y(0,x) = y_0(x), \forall x \in [0,L], & \text{(1d)} \end{cases}$$

where $y : [0,+\infty) \times [0,L] \to \mathbb{R}^n$, $\Lambda = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$ with $\lambda_1, \lambda_2, ..., \lambda_n > 0$, K and Υ are real $n \times n$ constants matrices. The sequence is defined as $\{t_k\}_{k\in\mathbb{N}}$ where

$$t_0 = 0, t_{k+1} - t_k \in (0, \bar{h}], \tag{2}$$

and $\bar{h} > 0$.

To deal with the problem under consideration, we need the following compatibility condition:

Condition 1. The initial condition $y_0(x)$ for $\forall x \in [0,L]$, satisfies

$$y_0(0) = y_0(L) = 0, \partial_x y_0(0) = \partial_x y_0(L) = 0. \tag{3}$$

Remark 1. Let us discuss the notion of solution used in the present work. The system (1)-(2) can be rewritten as a first order system

$$\begin{cases} \frac{dy(t)}{dt} = \mathfrak{A} y(t) + f(y(t_k)), t \in [t_k, t_{k+1}), k \in \mathbb{N}, \\ y(0) = y_0, \end{cases}$$

where $f(y(t_k)) = -ky(t_k)$, and the operator $\mathfrak{A}$ is defined by

$$\mathfrak{A} y = -\Lambda \partial_x y(t,x) - \Upsilon y(t,x), \tag{4}$$

with domain

$$D(\mathfrak{A}) = \left\{ y \in H^1(0,L;\mathbb{R}^n) \,\middle|\, \begin{array}{c} y(0) = y(L) = 0, \\ y_x(0) = y_x(L) = 0. \end{array} \right\} \tag{5}$$

The operator $\mathfrak{A}$ generates a stable C_0 semigroup (see the proof of theorem A.1. in Bastin and Coron (2016)). Moreover, we note that $f_k : H^1(0,L) \to H^1(0,L)$ is continuously differentiable for $t \in [t_k, t_{k+1})$. If $y_0 \in D(\mathfrak{A})$, then according to Theorem 6.1.5 of Pazy (1983), there exists a classical solution for each $t \in [t_k, t_{k+1}), k \in \mathbb{N}$. Therefore, we can construct a solution by choosing the last value of the previous sampling interval as the initial condition of the following sampling interval such that it is continuous at each sampling instant.

2.2 Problem Formulation

In our work, we prove Rε-stability, which is defined in Definition 1 below.

Definition 1. Rε-stability Polyakov (2008)
Consider positive scalars R and ε, such that $0 < \varepsilon < R$, and a candidate Lyapunov function $V : H^1([0,L];\mathbb{R}^n) \to \mathbb{R}_+$. If for all solutions of system (1) with $y_0(x) \in \mathfrak{L}_{V\leq R}$, the trajectory of the state $y(t,x)$ converges to $\mathfrak{L}_{V\leq\varepsilon}$ as t goes to infinity and the set $\mathfrak{L}_{V\leq\varepsilon}$ is positive invariant, then, system (1) is said to be $R\varepsilon$-stable from $\mathfrak{L}_{V\leq R}$ to $\mathfrak{L}_{V\leq\varepsilon}$.

In this work, our main goal is to guarantee the $R\varepsilon$-stability of the closed loop system (1)-(2) based on an Input-Output approach.

3. MAIN RESULT

This section is divided into two parts. First, we represent the sampled-data system as an equivalent continuous hyperbolic PDE where the sampling induced error appears in the input, as a disturbance. Secondly, based on the provided model, constructive Rε-stability criteria are provided.

3.1 System Remodelling

System (1) can be equivalently re-expressed as

$$\begin{cases} \partial_t y(t,x) + \Lambda \partial_x y(t,x) + (\Upsilon + K) y(t,x) \\ \qquad + K\omega(t,x) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & \text{(6a)} \\ y(t,0) = y(t,L) = 0, \forall t \geq 0, & \text{(6b)} \\ y(0,x) = y_0(x), \forall x \in [0,L]. & \text{(6c)} \end{cases}$$

with the sampling induced error

$$\omega(t,x) = y(t_k, x) - y(t,x). \tag{7}$$

Define the function φ as

$$\varphi(t,x) = \frac{\partial y(t,x)}{\partial t}, \forall t \geq 0, x \in [0,L]. \tag{8}$$

Note that

$$\omega(t,x) = -\int_{t_k}^{t} \frac{\partial y(\theta,x)}{\partial \theta} d\theta = -\int_{t_k}^{t} \varphi(\theta,x) d\theta,$$
$$\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, x \in [0,L]. \tag{9}$$

As a result, the closed-loop system can be regarded as the interconnection of two systems G and Ψ shown in Figure 1, where the operator $G : L^2(0,L) \to L^2(0,L)$ is defined by

$$G : \begin{cases} \partial_t y(t,x) = -\Lambda \partial_x y(t,x) - (K + \Upsilon) y(t,x) - K\omega(t,x), \\ y(t,0) = y(t,L) = 0, \forall t \geq 0, \\ y(0,x) = y_0(x), \forall x \in [0,L], \\ \varphi(t,x) = -\Lambda \partial_x y(t,x) - (K + \Upsilon) y(t,x) - K\omega(t,x) \\ \qquad = \partial_t y(t,x), \end{cases} \tag{10}$$

and the operator $\Psi : L^2(0,L) \to L^2(0,L)$ is defined by

$$\Psi : \begin{cases} \omega(t,x) = (\Psi y)(t,x) = -\int_{t_k}^{t} \varphi(\theta,x) d\theta, \\ \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, x \in [0,L]. \end{cases} \tag{11}$$

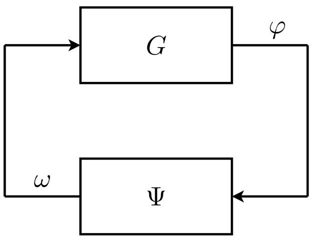

Fig. 1. Alternative representation of the closed-loop system.

3.2 Stability Analysis

In the following, we introduce our main result.

Theorem 1. Consider systems (10)-(11) with (2)(or equivalently (1) with (2)) and the initial condition satisfying (3):

(i) Let $\underline{\lambda} = \min_{i \in \{1,\dots,n\}} \lambda_i$. Assume that there exist $\mu_1, \gamma_1, \mu_2, \gamma_2 > 0$, and symmetric positive matrices $Q_1 \in \mathbb{R}^{n\times n}$, $Q_2 \in \mathbb{R}^{n\times n}$ satisfying

$$M(0) \leq 0, \quad M(L) \leq 0, \tag{12}$$

with $M(x)$ defined for all $x \in [0, L]$ as

$$M(x) = \begin{bmatrix} \Omega & -e^{-2\mu_1 x} Q_1 K & 0 & 0 \\ * & -\gamma_1 I & 0 & 0 \\ * & * & E & -e^{-2\mu_2 x} Q_2 K \\ * & * & * & -\gamma_2 I \end{bmatrix}, \tag{13}$$

where

$$\Omega = -e^{-2\mu_1 x}\left[(K+\Upsilon)^T Q_1 + Q_1(K+\Upsilon)\right],$$
$$\mathrm{E} = -e^{-2\mu_2 x}\left[\Upsilon^T Q_2 + Q_2\Upsilon + \beta Q_2\right]. \tag{14}$$

(ii) If $\exists \varepsilon \in \mathbb{R}_+$; $R \in \mathbb{R}_+$ s.t. $0 < \varepsilon < R$ and

$$\gamma_1 3\bar{h}\left(|\Lambda|^2\Phi + \left(|\Upsilon|^2 + |K|^2\right)\Psi\right) + \gamma_2\Phi < min(2\nu_1, 2\nu_2 - \beta)\varepsilon, \tag{15}$$

with

$$\Phi = \frac{R}{\lambda_{\min}(Q_2)\, e^{-2\mu_2 L}}, \Psi = \frac{R}{\lambda_{\min}(Q_1)\, e^{-2\mu_1 L}},$$

where $\nu_1 = \mu_1\underline{\lambda}$, $\nu_2 = \mu_2\underline{\lambda}$, $0 < \beta < 2\nu_2$.

Then the considered system (1) is Rε-stable from $\mathfrak{L}_{V\leq R}$ to $\mathfrak{L}_{V\leq\varepsilon}$ for any sampling sequence satisfying (2). Moreover, a Lyapunov function is defined as

$$V(y) = V_1(y) + V_2(y), \tag{16}$$

with

$$V_1(y) = \int_0^L y^T e^{-2\mu_1 x} Q_1 y dx, \tag{17}$$

$$V_2(y) = \int_0^L y_x^T e^{-2\mu_2 x} Q_2 y_x dx. \tag{18}$$

Proof. Consider the Lyapunov function (16)-(18). It can be bounded as follows:

$$\Theta \left\|y(t,\cdot)\right\|^2_{H^1([0,L]\,;\,\mathbb{R}^n)} \leq V(y(t,\cdot)) \leq \Xi \left\|y(t,\cdot)\right\|^2_{H^1([0,L]\,;\,\mathbb{R}^n)}, \tag{19}$$

where
$\Theta = min(\lambda_{\min}(Q_1), \lambda_{\min}(Q_2))e^{-2\mu L}$, $\mu = max(\mu_1, \mu_2)$,
$\Xi = max(\lambda_{\max}(Q_1), \lambda_{\max}(Q_2))$.

Step 1: In this step we study the continuity of the function V defined in (16).

(1) Since $y(t,x)$ is continuous with respect to t for all $t \in [t_k,\ t_{k+1}), k \in \mathbb{N}$, and continuous at sampling instants by construction (see Remark 1) , then V_1 is continuous for all $t \geq 0$.
(2) From system (1), we can get

$$y_x(t,x) = \Lambda^{-1}\left(-y_t(t,x) - \Upsilon y(t,x) - K y(t_k, x)\right), \tag{20}$$

for all $t \in [t_k, t_{k+1}),\ k \in \mathbb{N}$. Since all the terms on the right of the equation (20) are continuous in t on (t_k, t_{k+1}), $\forall k \in \mathbb{N}$, then $y_x(t,x)$ and thus V_2 are also continuous in t for all $(t_k, t_{k+1}), k \in \mathbb{N}$.

Now we consider the time interval $[t_k, t_{k+1})$, for some $k \in \mathbb{N}$ and an initial condition $y_k(x)$. The solution of (1) is defined as $y(t,x)$ on the time interval $[t_k, t_{k+1})$, and is such that y and y_x are both $\mathcal{C}^0$ in $t \in [t_k, t_{k+1})$.

Next, we prolong the solution to $\mathcal{C}^1$ in t on $[t_k, t_{k+1}]$. We denote $z(t,x)$ the solution on $[t_k, t_{k+1}]$ with initial condition $y_k(x)$. $z(t,x)$ and $z_x(t,x)$ are $\mathcal{C}^0$ in t on $[t_k, t_{k+1}]$. We got the following properties on $[t_k, t_{k+1})$

$$\begin{cases} y(t,x) = z(t,x), \\ y_x(t,x) = z_x(t,x). \end{cases} \tag{21}$$

Then the left limit can be calculated as

$$\lim_{t\to t_{k+1}^-} y_x(t,x) = \lim_{t\to t_{k+1}^-} z_x(t,x) = z_x(t_{k+1}, x). \tag{22}$$

For the next time interval $[t_{k+1}, t_{k+2})$, we set the initial condition $y_{k+1}(x) = z(t_{k+1}, x)$. Then the solution $y(t,x)$ of system (1) on $[t_{k+1}, t_{k+2})$ satisfies $y_x(t,x)$ is $\mathcal{C}^0$ in t on $[t_{k+1}, t_{k+2})$. Therefore, we have the right limit property

$$\lim_{t\to t_{k+1}^+} y_x(t,x) = y_x(t_{k+1}, x) = z_x(t_{k+1}, x). \tag{23}$$

According to (22) and (23), we can see that by construction, $y_x(t,x)$ is continuous in t at time instant t_{k+1}. Similarly, we can show that the function $y_x(t,x)$ is continuous at all sampling instants, which shows both the continuity of $y_x(t,x)$ with respect to time for all $t \geq 0$ and the continuity of V_2.

Remark 2. Here V_1 is used in order to bound y, and V_2 is used to deal with the term y_x that appears in the derivative of V_1.

Step 2: In this step we study the time derivative of the function of $V(y)$ defined in (16). We first compute the time derivative of $V_1(y)$ along the solutions to (10)-(11),$\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$,

$$\begin{aligned} \dot{V}_1(y) &= \int_0^L \left(\partial_t y^T e^{-2\mu_1 x} Q_1 y + y^T e^{-2\mu_1 x} Q_1 \partial_t y\right) dx \\ &= \int_0^L \Big((-\Lambda\partial_x y - (K+\Upsilon)y - K\omega)^T e^{-2\mu_1 x} Q_1 y \\ &\quad + y^T e^{-2\mu_1 x} Q_1 \left(-\Lambda\partial_x y - (K+\Upsilon)y - K\omega\right)\Big) dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^L -\partial_x \left[y^T \Lambda e^{-2\mu_1 x} Q_1 y\right] dx \\
&\quad + \int_0^L \Big(-y^T (K+\Upsilon)^T e^{-2\mu_1 x} Q_1 y \\
&\quad - y^T e^{-2\mu_1 x} Q_1 (K+\Upsilon) y - y^T e^{-2\mu_1 x} Q_1 K \omega \\
&\quad -2\mu_1 y^T \Lambda e^{-2\mu_1 x} Q_1 y - \omega^T K^T e^{-2\mu_1 x} Q_1 y \Big) dx \\
&= - \left[y^T \Lambda e^{-2\mu_1 x} Q_1 y\right]_0^L \\
&\quad + \int_0^L \Big(-y^T \Big((K+\Upsilon)^T e^{-2\mu_1 x} Q_1 \\
&\quad + e^{-2\mu_1 x} Q_1 (K+\Upsilon) \Big) y \\
&\quad - \omega^T K^T e^{-2\mu_1 x} Q_1 y - y^T e^{-2\mu_1 x} Q_1 K \omega \Big) dx \\
&\quad - 2\mu_1 \int_0^L y^T \Lambda e^{-2\mu_1 x} Q_1 y dx. \qquad (24)
\end{aligned}
$$

In order to get the time derivative of y_x in V_2, we refer to the original system (1). Since $y : [0, +\infty) \times [0, L] \to \mathbb{R}^n$ has consecutive partial derivatives in $[0, +\infty) \times [0, L]$, according to Schwartz's theorem (James (1966)) we can obtain

$$
\begin{aligned}
\partial_{xt} y(t,x) &= \partial_{tx} y(t,x) \\
&= -\Lambda \partial_{xx} y(t,x) - \Upsilon \partial_x y(t,x) - K \partial_x y(t_k, x), \\
&\forall t \in (t_k, t_{k+1}), k \in \mathbb{N}. \qquad (25)
\end{aligned}
$$

For the next calculation of the time derivative of V_2, we use Lemma 1 in the appendix. According to (25) and Lemma 1, we have

$$
\begin{cases}
\partial_x y(t,0) = \partial_x y(t,L) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & (26a) \\
y_0(0) = y_0(L) = 0, \quad \partial_x y_0(0) = \partial_x y_0(L) = 0. & (26b)
\end{cases}
$$

Similarly to the computation of $\dot{V}_1$, the time derivative of $V_2(y)$ along the solutions to (25)-(26), $\forall t \in (t_k, t_{k+1}), k \in \mathbb{N}$ is shown as follows

$$
\begin{aligned}
\dot{V}_2(y) &= - \left[\partial_x y^T \Lambda e^{-2\mu_2 x} Q_2 \partial_x y\right]_0^L \\
&\quad + \int_0^L \Big(-\partial_x y^T \left(\Upsilon^T e^{-2\mu_2 x} Q_2 + e^{-2\mu_2 x} Q_2 \Upsilon\right) \partial_x y \\
&\quad - \partial_x y^T (t_k, \cdot) K^T e^{-2\mu_2 x} Q_2 \partial_x y \\
&\quad - \partial_x y^T e^{-2\mu_2 x} Q_2 K \partial_x y^T (t_k, \cdot) \Big) dx \\
&\quad - 2\mu_2 \int_0^L \partial_x y^T \Lambda e^{-2\mu_2 x} Q_2 \partial_x y dx. \qquad (27)
\end{aligned}
$$

Adding $\gamma_1 \|\omega(s,\cdot)\|^2_{L^2([0,L]\,;\mathbb{R}^n)} - \gamma_1 \|\omega(s,\cdot)\|^2_{L^2([0,L]\,;\mathbb{R}^n)}$ to (24) and

$$
\gamma_2 \|\partial_x y(t_k,\cdot)\|^2_{L^2([0,L]\,;\mathbb{R}^n)} - \gamma_2 \|\partial_x y(t_k,\cdot)\|^2_{L^2([0,L]\,;\mathbb{R}^n)},
$$

$$
\beta \int_0^L y_x^T e^{-2\mu_2 x} Q_2 y_x dx - \beta \int_0^L y_x^T e^{-2\mu_2 x} Q_2 y_x dx
$$

to (27) for some $\gamma_1 > 0, \gamma_2 > 0, \beta > 0$, and using boundary condition (6b) and (26b) we have

$$
\begin{aligned}
\dot{V}(y) &= \dot{V}_1(y) + \dot{V}_2(y) \\
&\leq -2\nu_1 V_1(y) - (2\nu_2 - \beta) V_2(y) \\
&\quad + \int_0^L \eta^T M(x) \eta dx + \gamma_1 \|\omega(s,\cdot)\|^2_{L^2([0,L]\,;\mathbb{R}^n)} \\
&\quad + \gamma_2 \|\partial_x y(t_k,\cdot)\|^2_{L^2([0,L]\,;\mathbb{R}^n)}. \qquad (28)
\end{aligned}
$$

with

$$
\begin{aligned}
&\nu_1 = \mu_1 \underline{\lambda}, \nu_2 = \mu_2 \underline{\lambda}, \\
&\eta = [y^T \ \omega^T \ (\partial_x y)^T \ (\partial_x y)^T (t_k, \cdot)]^T \qquad (29)
\end{aligned}
$$

and $M(x)$ defined in (13).

Since $M(x)$ satisfies LMIs (12), by convexity we have $\eta^T M(x) \eta \leq 0$, $\forall x \in [0, L]$ and thus

$$
\int_0^L \eta^T M(x) \eta dx \leq 0, \qquad (30)
$$

we deduce from (28)

$$
\begin{aligned}
\dot{V}(y) \leq & -2\nu_1 V_1(y) - (2\nu_2 - \beta) V_2(y) \\
& + \gamma_1 \|\omega(s,\cdot)\|^2_{L^2([0,L]\,;\mathbb{R}^n)} \\
& + \gamma_2 \|\partial_x y(t_k,\cdot)\|^2_{L^2([0,L]\,;\mathbb{R}^n)}. \qquad (31)
\end{aligned}
$$

Step 3: The negative properties of function $\dot{V}(y)$ for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ will be discussed in this step. Consider some $k \in \mathbb{N}$ and let us first assume that $y(t,\cdot) \in \mathfrak{L}_{V \leq R} \setminus \mathfrak{L}_{V \leq \varepsilon}$ and $\forall \theta \in [t_k, t]$, $y(\theta,\cdot) \in \mathfrak{L}_{V \leq R}$.

$$
\begin{aligned}
&\int_0^L y^T e^{-2\mu_1 x} Q_1 y dx \leq R, \forall y(\theta,\cdot) \in \mathfrak{L}_{V \leq R}, \\
&\int_0^L \partial_x y^T e^{-2\mu_2 x} Q_2 \partial_x y dx \leq R, \forall \partial_x y(\theta,\cdot) \in \mathfrak{L}_{V \leq R}. \qquad (32)
\end{aligned}
$$

Then the following inequalities are further derived

$$
\|y(\theta,\cdot)\|^2_{L^2([0,L];\mathbb{R}^n)} \leq \frac{R}{\lambda_{\min}(Q_1) e^{-2\mu_1 L}}, \forall \theta \in [t_k, t], \qquad (33)
$$

$$
\|\partial_x y(\theta,\cdot)\|^2_{L^2([0,L];\mathbb{R}^n)} \leq \frac{R}{\lambda_{\min}(Q_2) e^{-2\mu_2 L}}, \forall \theta \in [t_k, t]. \qquad (34)
$$

Recalling $y(t,\cdot) \in \mathfrak{L}_{V \leq R} \setminus \mathfrak{L}_{V \leq \varepsilon}$ and $\forall \theta \in [t_k, t]$, $y(\theta,\cdot) \in \mathfrak{L}_{V \leq R}$, for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$. The bound of $\|\omega(s,\cdot)\|^2_{L^2([0,L];\mathbb{R}^n)}$ can be calculated by (9)

$$
\begin{aligned}
&\|\omega(s,\cdot)\|^2_{L^2([0,L]\,;\mathbb{R}^n)} \\
&= \int_0^L |\omega(s,x)|^2 dx = \int_0^L \left| \int_{t_k}^t \frac{\partial y(\theta,x)}{\partial \theta} d\theta \right|^2 dx \\
&\leq 3\bar{h} \Bigg(|\Lambda|^2 \frac{R}{\lambda_{\min}(Q_2) e^{-2\mu_2 L}} \\
&\quad + \left(|\Upsilon|^2 + |K|^2 \right) \frac{R}{\lambda_{\min}(Q_1) e^{-2\mu_1 L}} \Bigg) \\
&= \varpi. \qquad (35)
\end{aligned}
$$

In addition, since $y(t,\cdot), \partial_x y(t,\cdot) \notin \mathfrak{L}_{V \leq \varepsilon}$, we have

$$
\begin{aligned}
& -2\nu_1 V_1(y) - (2\nu_2 - \beta) V_2(y) \\
& \leq -min(2\nu_1, 2\nu_2 - \beta)(V_1(y) + V_2(y)) \\
& < -min(2\nu_1, 2\nu_2 - \beta)\varepsilon. \qquad (36)
\end{aligned}
$$

Therefore, instituting (35) and (36) into (31), we have that for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$,

$$
\begin{aligned}
\dot{V}(y) < & - \ min(2\nu_1, 2\nu_2 - \beta)\varepsilon + \gamma_1 \varpi \\
& + \gamma_2 \frac{R}{\lambda_{\min}(Q_2) e^{-2\mu_2 L}}, \qquad (37)
\end{aligned}
$$

and thus from assumption (15), we have shown that if $y(t,\cdot) \in \mathfrak{L}_{V \leq R} \setminus \mathfrak{L}_{V \leq \varepsilon}$ and $\forall \theta \in [t_k, t]$, $y(\theta,\cdot) \in \mathfrak{L}_{V \leq R}$ for $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$

$$
\dot{V}(y) < 0, \qquad (38)
$$

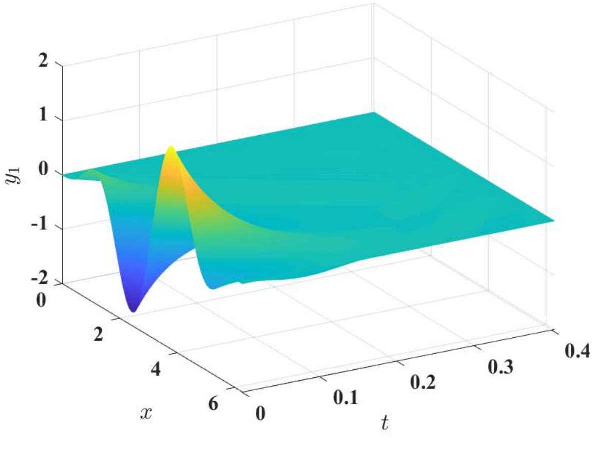

Fig. 2. Response of state y_1.

which means that since V is continuous, that y will remain in $\mathfrak{L}_{V\leq R}$ during the whole sampling interval $[t_k, t_{k+1}]$, and by recursion, we can see that it will always remains there. As a consequence, $\mathfrak{L}_{V\leq R}$ is positively invariant. Furthermore, since $\dot{V} < 0$ wherever $y \notin \mathfrak{L}_{V\leq\varepsilon}$, that means that $\mathfrak{L}_{V\leq\varepsilon}$ is attractive, which ends the proof of Rε-stability. ■

4. NUMERICAL SIMULATION

In this section, we present a numerical example to illustrate the stability we proposed in section 3. Consider system (1)

$$\begin{cases} \partial_t y\,(t,x) + \Lambda \partial_x y\,(t,x) + \Upsilon y\,(t,x) + u\,(t,x) = 0, \\ u\,(t,x) = Ky\,(t_k,x)\,, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, \\ y(t,0) = y(t,L) = 0, \forall t \geq 0, \\ y(0,x) = y_0(x), \forall x \in [0,L]\,, \end{cases}$$

where

$$\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & 12 \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} 20 & 15 \\ 20 & 25 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix},$$

$$L = 2\pi, \quad y_0\,(x) = \begin{bmatrix} (1-\cos x)\sin 2x \\ 3(1-\cos x)\sin x \end{bmatrix}.$$

The parameters in Theorem 1 are selected as:

$$\bar{h} = 0.0001, \quad \beta = 0.01, \quad \mu_1 = 0.1, \quad \mu_2 = 0.08,$$

$$\gamma_1 = \gamma_2 = 0.01, \quad R = 10, \quad \varepsilon = 3.3842,$$

$$Q_1 = \begin{bmatrix} 0.1232 & -0.0432 \\ -0.0432 & 0.1106 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.1299 & -0.0368 \\ -0.0368 & 0.1319 \end{bmatrix},$$

which satisfy the conditions (12), (13), and (15). The results of numerical simulations are presented in Figs. 2-4. Figs. 2-3 show the states converge to the vicinity of the origin with the controller. The initial conditions satisfying the compatibility condition (3) are given in Figs. 4-5. The time-evolution of Lyapunov function V is shown in Fig. 6. One can see that the Lyapunov function V deceases when $\varepsilon < V(y) < R$.

5. CONCLUSION

The main work of this paper is to construct a sampling controller for distributed control of linear hyperbolic balance laws. The closed-loop system is reformulated from an Input-Output point of view. In addition, we prove the local

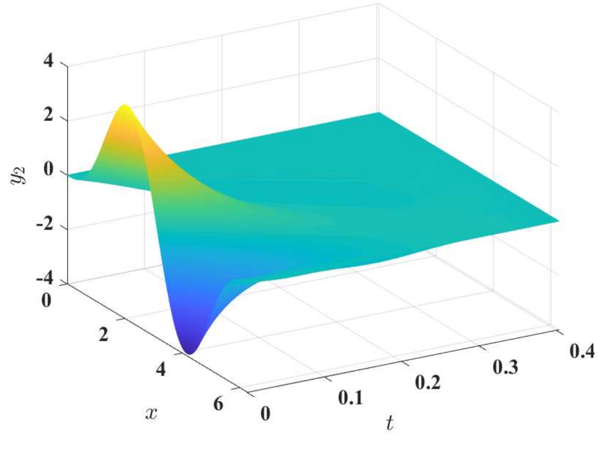

Fig. 3. Response of state y_2.

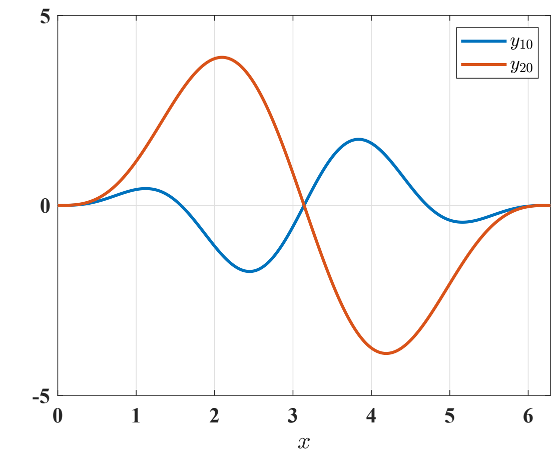

Fig. 4. Initial data of y_1 and y_2.

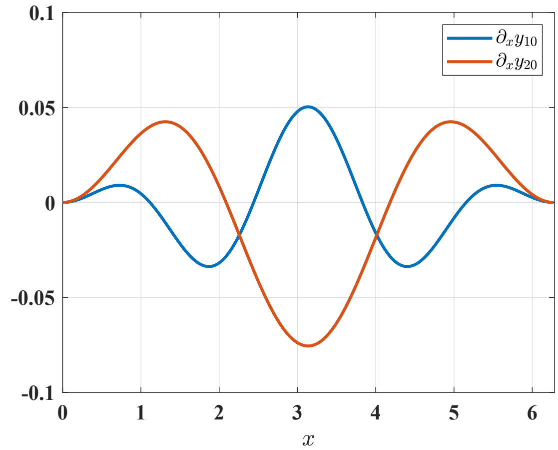

Fig. 5. Initial data of $\partial_x y_1$ and $\partial_x y_2$.

stability of the system by means of the Lyapunov method. This is somewhat conservative because the sampling interval needs to be small enough. In the future, we will expand the sampling interval by virtue of dissipativity theory, and sample the controller in space not only in time, so as to propose a more general stability theory.

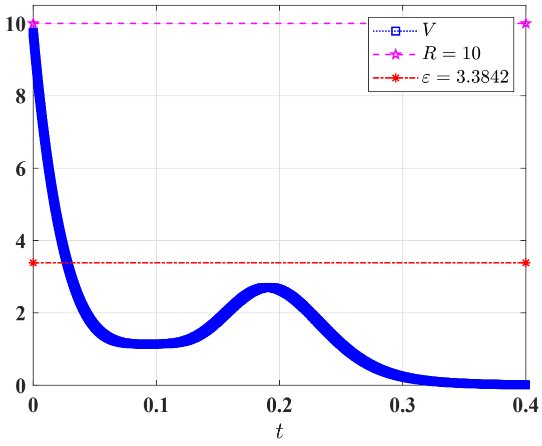

Fig. 6. Time-evolution of function V.

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Appendix A. PROOF OF LEMMA 1

Lemma 1. Consider the system (1)-(2) with initial condition y_0 satisfying Condition 1. Then $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$, $\partial_x y(t,0) = \partial_x y(t,L) = 0$.

Proof. We recall system (1)

$$\begin{cases} \partial_t y\,(t,x) + \Lambda \partial_x y\,(t,x) + \Upsilon y\,(t,x) + u\,(t,x) = 0, & \text{(A.1a)} \\ u\,(t,x) = Ky\,(t_k,x)\,, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & \text{(A.1b)} \\ y(t,0) = y(t,L) = 0, \forall t \geq 0, & \text{(A.1c)} \\ y(0,x) = y_0(x), \forall x \in [0,L]\,, & \text{(A.1d)} \end{cases}$$

The time derivative of the boundary condition leads to

$$\partial_t y\,(t,0) = \partial_t y\,(t,L) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \quad \text{(A.2)}$$

Combining (A.1a) with (A.2), we obtain

$$\begin{cases} 0 = \partial_t y\,(t,0) = -\Lambda \partial_x y\,(t,0) - \Upsilon y\,(t,0) - Ky\,(t_k,0)\,, \\ 0 = \partial_t y\,(t,L) = -\Lambda \partial_x y\,(t,L) - \Upsilon y\,(t,L) - Ky\,(t_k,L)\,. \end{cases}$$

Since $y(t,0) = y(t,L) = 0, \forall t \geq 0$, we have

$$\partial_x y(t,0) = \partial_x y(t,L) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \quad \text{(A.3)}$$

■