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Metzler matrix-based switching control scheme for linear systems with prescribed performance guarantees ⋆

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Abstract: This paper investigates the issue of control design for linear systems with prescribed performance. The novelty of the proposed solution relies on a switching control scheme that uses the interval theory. Different from the existing prescribed performance control approaches, the error transformation and the logarithmic/tangent function are no longer required in the presented control. Alternatively, a switching scheme inspired by the interval observer technique is introduced to ensure the errors not violating performance bound functions. The proposed control establishes an unified prescribed performance control framework which covers the bounded stability, asymptotic stability and finite-time stability only via selecting the corresponding performance bound functions. Numerical examples are simulated to demonstrate the effectiveness of the proposed approach.

Keywords: Metzler matrix; prescribed performance control; switching; bounded stability; asymptotic stability; finite-time stability

1. INTRODUCTION

Prescribed performance control (PPC) is a recent control approach that aims to ensure the tracking error to satisfy predefined transient and steady-state bounds, see (Bechlioulis et al., 2008; Bu et al., 2018; Li et al., 2018) to cite a few papers. PPC means that the tracking error should be limited to an arbitrarily small residual set, while its convergence rate is not less than a given constant and the maximum overshoot is less than a prescribed value.
This technique is also exploited in (Bechlioulis and Rovithakis, 2011) in a robust setting for cascade systems by applying partial states. Under the framework of backstepping, a PPC method is presented in (Hu Y., 2014; Song H., 2014) and further investigated in (Bu et al., 2018). Other approaches combine PPC with other techniques to solve the adaptive control (Li et al., 2018), fault-tolerant control (Zhang et al., 2017) and unknown control direction issue (Li et al., 2018). Recently, a low-complexity approximation-free universal control scheme was proposed for unknown pure-feedback systems (Bechlioulis et al., 2014) inspired by the PPC. This method extends the PPC to the robust field since there is no requirement for any disturbance observer or fuzzy/neural networks approximation mechanism. Furthermore, some practical applications, such as flexible joint robots subject to actual system nonlinearities (Kostarigka et al., 2013), hypersonic flight under parametric uncertainties (Bu et al., 2016; Guo et al., 2020) and spacecraft (Hu et al., 2018) have also been addressed by the PPC technique.

Actually, the basic thought of the PPC includes two steps: *i*) The error transformation is conducted in the first step to convert the constrained system to a unconstrained one. In this phase, the logarithmic or tangent function is commonly used to define the new error variables based on the performance bound functions. *ii*) The second step is to design a control to stabilize the transformed system. Here, all existing control theories are potentially viable.

However, there exist two major drawbacks in the existing PPC theories. *i*) The first one is related to the system stability. The standard PPC theory only guarantees the bounded stability (Bechlioulis et al., 2008). A PPC framework covering all types of stability such as asymptotic and finite-time stability, has not been yet established. Note that solutions have been proposed to this problem, see (Li et al., 2018) for asymptotic and (Chen, 2019) for

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finite-time stability issues. However, the stability relies on the control method itself, and not at the performance bound function tuning level. *ii*) The second drawback is concerned by the prescribed value that can be set for the steady-state bound. In the existing PPC, this bound cannot be set to zero since it results in the large control burden since the state is very close to the performance bound. Thus, the control parameters tuning is quite sensitive to this steady-state bound.

Motivated by these drawbacks, the aim of this paper is to investigate an unified solution for the PPC theory. Especially, we investigate a PPC framework whose design is based only on tuning the performance bound functions, so that, by tuning them adequately, it is possible to enforce independently bounded, asymptotic or finite-time stability. In other words, the PPC scheme is thought unified and the obtained stability only depends on the selection of the performance bound functions. To proceed, it is proposed to use the interval theory, see for instance (Farina et al., 2000; Raissi et al., 2012; Ellero et al., 2019). The basic thought of the interval theory is to enforce signals to stay in an envelop, whose tight is minimized by means of an optimisation procedure. To some degrees, this philosophy is similar with the PPC at the point of restricting the variables into a certain range, and this is the genesis of the PPC theory we investigate in this paper.

Compared to existing PPC schemes, the presented approach no longer needs the error transformation and logarithmic/tangent function. This provides an alternative solution. More interestingly, the proposed solution allows the steady-state bound to be set to zero, which means that asymptotic stability can be enforced. Moreover, the finite-time performance bound function can be chosen such as the output converges to zero in finite time and always falls into the prescribed performance limitations. However, it is worth to note that the proposed theory is currently only available for a restricted class of linear time invariant system, i.e. systems of relative order equal to 1. Further investigations are necessary to deal with higher relative order systems.

The paper is organized as follows. Section 2 lays down some preliminaries and section 3 states the problem. Main results are given in sections 4 and 5. Section 6 provides simulation examples.

2. PRELIMINARIES

This section is devoted to notations, definitions and lemmas that will be later used in the paper.

$\mathbb{R}$ and $\mathbb{R}^+ := [0, \infty) \subset \mathbb{R}$ are the set of real numbers and the set of nonnegative real numbers, respectively. $\mathbb{R}^n$ is a n-dimensional real space and $\mathbb{R}^{n\times m}$ is the set of real $n \times m$ matrices. The function $\text{sgn}(y)$ denotes the standard sign function for a scalar y, and $\text{sgn}(x) = \text{diag}(\text{sgn}(x_i)) \in \mathbb{R}^{n\times n}$ $(i = 1, 2, \ldots, n)$ for a vector $x \in \mathbb{R}^n$. For a matrix $A \in \mathbb{R}^{n\times m}$, $\geq, \leq, >, <$ refer to component-wise. For vectors $a_1 \in \mathbb{R}^n, \ldots, a_m \in \mathbb{R}^n$, the function $\text{Max}(a_1, \ldots, a_m) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$ is also defined component-wise, *i.e.,* $\text{Max}(a_1, \ldots, a_m) = [\max(a_{11}, \ldots, a_{m1}), \ldots, \max(a_{1n}, \ldots, a_{mn})]^T$.

Definition 1. (Bechlioulis et al., 2008) (Prescribed performance control PPC). Consider a system's realization with state $x(t) \in \mathbb{R}^n$ and control input $u(t) \in \mathbb{R}^m$. Any control $u(t)$ that guarantees $x(t) \in [\underline{\rho}(t), \overline{\rho}(t)], \forall t \geq 0$ is called a prescribed performance control. The functions $\bar{\rho}(t), \underline{\rho}(t) \in \mathbb{R}^n$ are called performance bound functions.

Definition 2. (Polyakov et al., 2014) (bounded, asymptotic and finite-time stability) Define that $\mathbb{B}(r) = \{x \in \mathbb{R}^n : ||x|| < r\}$ is an open ball of the radius $r \in \mathbb{R}^+$ with the origin. The origin of the system (1) is said to be
i) bounded stable (Lyapunov stable), if for $\forall \varepsilon \in \mathbb{R}^+$, $\exists \delta = \delta(t_0, \varepsilon) \in \mathbb{R}^+$ such that for $\forall x(t_0) \in \mathbb{B}(\delta), x(t) \in \mathbb{B}(\varepsilon)$ for $t \geq t_0$.
ii) asymptotically stable, if it is bounded stable and $\lim\limits_{t\to\infty} ||x(t)|| = 0$.
iii) finite-time stable, if it is bounded stable and there exists a function $T : \mathbb{R}^n \to \mathbb{R}$ such that for $t \in [0, T(x))$, $\lim\limits_{t\to T(x)} ||x(t)|| = 0$.

Definition 3. (Farina et al., 2000) (Metzler matrix). A square matrix S is said to be Metzler if all its off-diagonal elements are non negative, i.e. $S_{i,j} \geq 0, 1 \leq i \neq j \leq n$.

Lemma 1. (Farina et al., 2000). Let an autonomous system be given by $\dot{z}(t) = Sz(t) + \theta(t), z \in \mathbb{R}, \theta : \mathbb{R}_+ \to \mathbb{R}^n_+$ where the matrix $S \in \mathbb{R}^{n\times n}$ is Metzler. If $z(t_0) \geq 0$ then $z(t) \geq 0$, $\forall t \geq t_0$. Such systems are called cooperative.

3. PROBLEM STATEMENT

Consider the particular class of linear time-invariant (LTI) systems

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= x \end{aligned} \qquad A, B \in \mathbb{R}^{n\times n} \tag{1}$$

where the state $x \in \mathbb{R}^n$ is assumed measured and $u \in \mathbb{R}^n$ denotes the control input. (A, B) is supposed to be controllable without loss of generality. Obviously, the relative order of (1) equals to one and the control matrix B is a square matrix.

The goal we pursue is to design a (state feedback) control law so that $x(t) \in \mathbb{R}^n$ belongs to an *a priori* given interval $[\underline{\rho}(t), \overline{\rho}(t)], \forall t \geq 0$ with performance bound functions $\underline{\rho}, \overline{\rho} \in \mathbb{R}^n$. In other words, we look for a PPC law according to Definition 1.

The following assumptions are made:

Assumption 1. (Farina et al., 2000) For the linear system (1), $\exists K$ such that the matrix $A + K$ is Metzler.

Assumption 2. The performance bound functions $\bar{\rho}(t)$ and $\underline{\rho}(t)$ can be set as $\bar{\rho}(t) = -\underline{\rho}(t) = \rho(t)$ where $\rho(t)$ is a differentiable, bounded, positive and decreasing function of time. Furthermore, $\rho(t)$ is assumed to be known.

Assumption 3. The control matrix B is nonsingular.

Assumption 1 will enable us to use the theory of cooperative systems, see lemma 1. Assumption 2 is nothing else than a centred formulation of the performance bound function proposed in (Bechlioulis et al., 2008, 2014) which is defined as $\rho(t) = [\rho_1(t), \ldots, \rho_n(t)]^T$ where $\rho_i(i = 1, \ldots, n)$ is given by the following form

$$\rho_i(t) = (\rho_{i,0} - \rho_{i,\infty})e^{-l_i t} + \rho_{i,\infty} \tag{2}$$

where $\rho_{i,0} > \rho_{i,\infty} > 0$, and $l_i > 0$. Assumption 3 is required for the proposed solution to be valid. Assumption 3 seems conservative for the control design. However, the aim of this paper is only to show the basic spirit of the novel PPC framework inspired by the interval theory, and the solution for the system with general form (such as the cascade form) will be discussed in future work.

4. SOLUTION FOR $n = 1$

Let us first consider the case $n = 1$ for a good understanding. Then (1) is re-written according to

$$\dot{x} = ax + bu \tag{3}$$

where the state x, the control u and a, b are scalars. The proposed PPC scheme is given by

$$u = \frac{1}{b}\left(-r\mathrm{sgn}(x) + \begin{cases} -k\bar{e} & x > 0 \\ 0 & x = 0 \\ k\underline{e} & x < 0 \end{cases}\right) \tag{4}$$

where the parameter $k \in \mathbb{R}$ is to be designed and the errors are defined as $\bar{e} = \bar{\rho} - x$ and $\underline{e} = x - \underline{\rho}$. The gain $r \in \mathbb{R}$ is selected to satisfy

$$r \geq a\rho - \dot{\rho} \quad \forall t \tag{5}$$

Besides, it is assumed that the initial condition of the system state satisfies $\underline{\rho}(0) \leq x(0) \leq \bar{\rho}(0)$.

Since symmetric performance bound functions are considered, the fact that $x(t) \geq 0$ $\forall t$ implies that $\underline{e}(t) > 0$ $\forall t$ holds and, vice versa, $x(t) \leq 0$ $\forall t$ ensures that $\bar{e} > 0$ $\forall t$. Thus, without loss of generality, let us assume that $x(0) \geq 0$ is set. According to (4), $u = \frac{1}{b}(-k\bar{e} - r)$ and it follows

$$\dot{\bar{e}} = \dot{\bar{\rho}} - ax + k\bar{e} + r = (a + k)\bar{e} + \dot{\bar{\rho}} - a\bar{\rho} + r \tag{6}$$

or equivalently

$$\dot{\bar{e}} = (a + k)\bar{e} + v \tag{7}$$

with $v = r + \dot{\bar{\rho}} - a\bar{\rho}$. Using assumption 2, it follows $v = r + \dot{\rho} - a\rho$ and by virtue of (5), we have $v(t) \geq 0$ $\forall t \geq 0$. Then, by virtue of lemma 1 and since $\bar{e}(0) \geq 0$ (thanks to the condition $x(0) \leq \bar{\rho}(0)$), it follows $\bar{e}(t) \geq 0$ $\forall t \geq 0$ and any value for k (including zero) is suitable to keep $\bar{e}(t)$ positive. This case is also a special case for scalars with respect to Lemma 1.

The property $\bar{e}(t) \geq 0$ $\forall t \geq 0$ does not guarantee that the system state $x(t)$ do not cross zero. In this case, there exists t_1: $x(t_1) = 0$. Then the control is switched into $u = \frac{1}{b}(k\underline{e} + r)$ for $t_1 + \Delta t$ where $\Delta t > 0$ denotes a small time interval. Differentiating the error $\underline{e}$ leads to

$$\dot{\underline{e}} = ax + k\underline{e} + r - \dot{\underline{\rho}} = (a + k)\underline{e} + a\underline{\rho} - \dot{\underline{\rho}} + r \tag{8}$$

or equivalently with Assumption 2

$$\dot{\underline{e}} = (a + k)\underline{e} + v \tag{9}$$

with $v(t) \geq 0$ $\forall t$ according to (5). Noting that $\underline{e}(t_1 + \Delta t) > 0$, it follows by virtue of Lemma 1 that $\underline{e}(t) \geq 0$ $\forall t \geq t_1 + \Delta t$.

This reasoning leads to the following theorem.

Theorem 1. Consider the system (3) satisfying the initial condition $\underline{\rho}(0) \leq x(0) \leq \bar{\rho}(0)$, the control law (4) and the performance bound function (2). If the gain r satisfies (5), then the (closed-loop) system state is bounded so that $\underline{\rho}(t) \leq x(t) \leq \bar{\rho}(t)$ $\forall t \geq 0$.

Proof. Immediate using the above developments. □

Based on the analysis above, the overall dynamics of $\bar{e}$ and $\underline{e}$ can be written according to

$$\dot{\bar{e}} \in \begin{cases} (a + k)\,\bar{e} + \dot{\bar{\rho}} - a\bar{\rho} + r & x > 0 \\ a\bar{e} - k\underline{e} + \dot{\bar{\rho}} - a\bar{\rho} - r & x < 0 \\ \mathrm{Co}\left\{ \bigcup_{j \in N(t,x)} \left\{ \begin{array}{l} (a + k)\,\bar{e} + \dot{\bar{\rho}} - a\bar{\rho} + r, \\ a\bar{e} - k + \dot{\bar{\rho}} - a\bar{\rho} - r \end{array} \right\} \right\} & x = 0 \end{cases} \tag{10}$$

and

$$\dot{\underline{e}} \in \begin{cases} a\underline{e} - k\bar{e} + a\underline{\rho} - r - \dot{\underline{\rho}} & x > 0 \\ (a + k)\,\underline{e} + r + a\underline{\rho} - \dot{\underline{\rho}} & x < 0 \\ \mathrm{Co}\left\{ \bigcup_{j \in N(t,x)} \left\{ \begin{array}{l} a\underline{e} - k\bar{e} + a\underline{\rho} - r - \dot{\underline{\rho}}, \\ (a + k\underline{e}) + r + a\underline{\rho} - \dot{\underline{\rho}} \end{array} \right\} \right\} & x = 0 \end{cases} \tag{11}$$

where the symbol "Co" represents the convex closure. Therefore, the solution of closed-loop discontinuous system (3) with its control law (4) should be understood in the right-hand side under the Filippov sense (Filippov, 1998).

5. EXTENSION TO $n > 1$

5.1 Main results

Let us now consider the generalization of the developments stated in section 4 to the case $n > 1$. Define the PPC law according to the following switching control scheme

$$u = B^{-1}\left(Kx - K\mathrm{sgn}(x)\rho - \mathrm{sgn}(x)R\right) \tag{12}$$

where the matrix $K \in \mathbb{R}^{n\times n}$ is designed such that $A + K$ is Metzler. The function $R \in \mathbb{R}^n$ represents the control gain to be designed. The initial system state is assumed to satisfy $\underline{\rho}(0) \leq x(0) \leq \bar{\rho}(0)$. Since the state x is a vector, the positiveness/negativeness of $x(0)$ must be discussed in the following way. Without loss of generality, consider the case that some elements of $x(0)$ are negative and others are positive. Let us denote l as the number of non-negative elements of $x(0)$, which can be represented as $x_{s_i}(0)$ ($i = 1, \ldots, l, s_i \in \{1, 2, \ldots, n\}$). Thus, there exist $m - l$ negative elements, say $x_{q_j}(0)$ ($j = 1, \ldots, n - l, q_j \in \{1, 2, \ldots, n\}$). Define that

$$\begin{aligned} x_s = [x_{s_1}, \ldots, x_{s_i}, \ldots, x_{s_l}]^T \in \mathbb{R}^l,\ 0 \leq l \leq n \\ i = 1, \ldots, l, s_i \in \{1, 2, \ldots, n\} \\ x_q = [x_{q_1}, \ldots, x_{q_j}, \ldots, x_{q_{m-l}}]^T \in \mathbb{R}^{n-l} \\ j = 1, \ldots, n - l, q_j \in \{1, 2, \ldots, n\} \end{aligned} \tag{13}$$

satisfying $x_s(0) \geq 0$, $x_q(0) < 0$. Consider the following upper and lower errors associated to x_s and x_q respectively

$$\bar{e}_s = \bar{\rho}_s - x_s,\ \underline{e}_q = x_q - \underline{\rho}_q \tag{14}$$

where $\bar{\rho}_s = \rho_s \in \mathbb{R}^l, \underline{\rho}_q = -\rho_q \in \mathbb{R}^{n-l}$ play similar roles with $\bar{\rho}, \underline{\rho}$, where $\rho_s = [\rho_{s_1}, \ldots, \rho_{s_l}]^T$ and $\rho_q = [\rho_{q_1}, \ldots, \rho_{q_{n-l}}]^T$. Define that $B_s = [B_{s_1}; \ldots; B_{s_l}] \in \mathbb{R}^{l\times n}$ and $B_q = [B_{q_1}; \ldots; B_{q_{n-l}}] \in \mathbb{R}^{l\times(m-l)}$ where $B_{s_i} \in \mathbb{R}^{1\times n}$ and $B_{q_j} \in \mathbb{R}^{1\times n}$ denote the s_ith and q_jth row of the matrix B. Define that $u_s = [u_{s_1}, \ldots, u_{s_l}]^T \in \mathbb{R}^l$ and $u_q = [u_{q_1}, \ldots, u_{q_{n-l}}]^T \in \mathbb{R}^{n-l}$ where $u_{s_i}(i = 1, \ldots, l), u_{q_j}(j = 1, \ldots, n - l)$ denote the corresponding s_ith and q_jth elements of $Bu \in \mathbb{R}^{n\times 1}$. Recall that $\mathrm{sgn}(x) \in \mathbb{R}^{n\times n}$ is a diagonal matrix, and it can be verified that

$$\begin{aligned} x &= (B_sB^{-1})^T x_s + (B_qB^{-1})^T x_q \\ \mathrm{sgn}(x)\rho &= (B_sB^{-1})^T \rho_s - (B_qB^{-1})^T \rho_q \end{aligned} \tag{15}$$

hold, and then one has that

$$\begin{aligned} u_s &= B_sB^{-1}Bu \\ &= B_sB^{-1}Kx - B_sB^{-1}K\mathrm{sgn}(x)\rho - B_sB^{-1}R \\ &= B_sB^{-1}K(B_sB^{-1})^T x_s + B_sB^{-1}K(B_qB^{-1})^T x_q \\ &\quad - B_sB^{-1}K(B_sB^{-1})^T\rho_s + B_sB^{-1}K(B_qB^{-1})^T\rho_q \\ &\quad - B_sB^{-1}R \\ &= -B_sB^{-1}K(B_sB^{-1})^T\bar{e}_s + B_sB^{-1}K(B_qB^{-1})^T\underline{e}_q \\ &\quad - B_sB^{-1}R \end{aligned} \tag{16}$$

and

$$\begin{aligned} u_q &= B_qB^{-1}Bu \\ &= B_qB^{-1}Kx - B_qB^{-1}K\mathrm{sgn}(x)\rho + B_qB^{-1}R \\ &= B_qB^{-1}K(B_sB^{-1})^T x_s + B_qB^{-1}K(B_qB^{-1})^T x_q \\ &\quad - B_qB^{-1}K(B_sB^{-1})^T\rho_s + B_qB^{-1}K(B_qB^{-1})^T\rho_q \\ &\quad + B_sB^{-1}R \\ &= -B_qB^{-1}K(B_sB^{-1})^T\bar{e}_s + B_qB^{-1}K(B_qB^{-1})^T\underline{e}_q \\ &\quad + B_qB^{-1}R \end{aligned} \tag{17}$$

Note that $\dot{\bar{e}}_s = \dot{\bar{\rho}}_s - \dot{x}_s$ and $\dot{\underline{e}}_q = \dot{x}_q - \dot{\underline{\rho}}_q$, and combining (16) and (17) yields

$$\begin{aligned} \dot{\bar{e}}_s &= \dot{\rho}_s - B_sB^{-1}(Ax + Bu) \\ &= \dot{\rho}_s - B_sB^{-1}Ax - u_s \\ &= B_sB^{-1}(A+K)(B_sB^{-1})^T\bar{e}_s - B_sB^{-1}A(B_sB^{-1})^T\rho_s \\ &\quad - B_sB^{-1}(A+K)(B_qB^{-1})^T x_q - B_sB^{-1}K(B_qB^{-1})^T\rho_q \\ &\quad + \dot{\rho}_s + B_sB^{-1}R \end{aligned} \tag{18}$$

and

$$\begin{aligned} \dot{\underline{e}}_q &= B_qB^{-1}(Ax + Bu) + \dot{\rho}_q \\ &= B_qB^{-1}Ax + u_q + \dot{\rho}_q \\ &= B_qB^{-1}(A+K)(B_sB^{-1})^T x_s - B_qB^{-1}K(B_sB^{-1})^T\rho_s \\ &\quad + B_qB^{-1}(A+K)(B_qB^{-1})^T\underline{e}_q - B_qB^{-1}A(B_qB^{-1})^T\rho_q \\ &\quad + \dot{\rho}_q + B_qB^{-1}R \end{aligned} \tag{19}$$

Further, (18) and (19) can be rewritten as the following compact dynamic system

$$\begin{aligned} \begin{bmatrix} \dot{\bar{e}}_s \\ \dot{\underline{e}}_q \end{bmatrix} &= \\ &\begin{bmatrix} B_sB^{-1}(A+K)(B_sB^{-1})^T & 0 \\ 0 & B_qB^{-1}(A+K)(B_qB^{-1})^T \end{bmatrix} \\ &\cdot \begin{bmatrix} \bar{e}_s \\ \underline{e}_q \end{bmatrix} + \begin{bmatrix} -B_sB^{-1}(A+K)(B_qB^{-1})^T x_q \\ B_qB^{-1}(A+K)(B_sB^{-1})^T x_s \end{bmatrix} + \begin{bmatrix} V_s \\ V_q \end{bmatrix} \end{aligned} \tag{20}$$

where $V_s = B_sB^{-1}(R + \dot{\rho} - A\rho - (K-A)(B_qB^{-1})^T\rho_q)$ and $V_q = B_qB^{-1}(R + \dot{\rho} - A\rho - (K-A)(B_sB^{-1})^T\rho_s)$. The function $R \in \mathbb{R}^n$ should be selected to satisfy the following requirement

$$\begin{aligned} R \geq\ & A\rho - \dot{\rho} \\ &+ \mathrm{Max}((K-A)(B_qB^{-1})^T\rho_q, \\ &\qquad (K-A)(B_sB^{-1})^T\rho_s, 0_{n\times 1}) \quad \forall t \end{aligned} \tag{21}$$

which leads to the following property.

Lemma 2. For the system (20), if the matrix $A + K$ is Metzler and the condition (21) holds, then it can be verified that:

$$\begin{aligned} -B_sB^{-1}(A+K)(B_qB^{-1})^T x_q(t) + V_s(t) \geq 0 \quad \forall t \geq 0 \\ B_qB^{-1}(A+K)(B_sB^{-1})^T x_s(t) + V_q(t) \geq 0 \quad \forall t \geq 0 \end{aligned} \tag{22}$$

and the matrix

$$\begin{bmatrix} B_sB^{-1}(A+K)(B_sB^{-1})^T & 0 \\ 0 & B_qB^{-1}(A+K)(B_qB^{-1})^T \end{bmatrix} \tag{23}$$

is Metzler.

Proof. Let $\bar{A} = A + K = [\bar{a}_{ij}]$ $(i, j = 1, 2, \ldots, n)$, then the following relationships are satisfied

$$\begin{aligned} &B_sB^{-1}(A+K)(B_sB^{-1})^T \\ &= [\bar{a}_{ij}]_{l\times l}(i = s_1, \ldots, s_l; j = s_1, \ldots, s_l) \\ &B_qB^{-1}(A+K)(B_qB^{-1})^T \\ &= [\bar{a}_{ij}]_{(n-l)\times(n-l)}(i = q_1, \ldots, q_{n-l}; q_1, \ldots, q_{n-l}) \\ &B_sB^{-1}(A+K)(B_qB^{-1})^T \\ &= [\bar{a}_{ij}]_{l\times(n-l)}(i = s_1, \ldots, s_l; j = q_1, \ldots, q_{n-l}) \\ &B_qB^{-1}(A+K)(B_sB^{-1})^T \\ &= [\bar{a}_{ij}]_{(n-l)\times l}(i = q_1, \ldots, q_{n-l}; j = s_1, \ldots, s_l) \end{aligned} \tag{24}$$

Since that $A + K$ is Metzler and $\bar{a}_{ij} \geq 0$ $(i \neq j)$, it can be obtained that the matrices $B_sB^{-1}(A+K)(B_sB^{-1})^T$ and $B_qB^{-1}(A+K)(B_qB^{-1})^T$ are Metzler and $B_sB^{-1}(A+K)(B_qB^{-1})^T \geq 0$ and $B_qB^{-1}(A+K)(B_sB^{-1})^T \geq 0$. Thus, (23) must be hold.

Note that $x_s(t)$ includes the non-negative elements, and thus $x_s(t) \geq 0$ holds for $\forall t \geq 0$. Also, $x_q(t) < 0, \forall t \geq 0$. Furthermore, the condition (21) implies that $V_s(t) \geq 0$ and $V_q(t) \geq 0$ $\forall t \geq 0$. Therefore, (22) is guaranteed, and this completes the proof. □

Remark 1. The matrix B_sB^{-1} acts an role of selecting the corresponding s_i row of a vector. For example, if $n = 3$ and $s_1 = 1, s_2 = 3$, then $B_sB^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ holds. Similarly, the q_i row of a vector is selected with the help of B_qB^{-1}.

By virtue of Lemma 1 and Assumption 1, it is demonstrated that $\bar{e}_s(t) \geq 0, \underline{e}_q(t) \geq 0$ $\forall t \geq 0$, since $\bar{e}_s(0) \geq 0$ and $\underline{e}_q(0) \geq 0$. The following theorem then yields.

Theorem 2. Consider the system (1) satisfying the initial condition $\underline{\rho}(0) \leq x(0) \leq \bar{\rho}(0)$, the control law (12) and the performance bound functions (2). If the gain R satisfies (21), then the (closed-loop) system state is bounded so that $\underline{\rho}(t) \leq x(t) \leq \bar{\rho}(t)$ $\forall t \geq 0$.

Proof. Consider the ith element x_i of x. If the initial value $x_i(0)$ satisfies $x_i(0) \geq 0$, then $x_i(0) \in x_s(0)$, and thus $e_s(t) \geq 0$ $\forall t \geq 0$ holds, which implies that $\bar{e}_i \geq 0$ for this ith element. In the same time period, $x_i \geq 0 \geq \underline{\rho}_i$ must be satisfied, and thus $\underline{\rho}_i \leq x_i \leq \bar{\rho}_i$ holds.

If there exists a moment t_1 when $x_i(t_1) < 0$, then the non-negative or positive elements have changed, and thus the symbols s and q can be changed into s' and q'. Correspondingly, (20) will be reformulated as a $[\bar{e}_{s'}^T, \underline{e}_{q'}^T]^T$ system with the similar form. In this new dynamics, the moment t_1 is taken as the new initial point. The

conclusions in Lemma 2 also hold and $\bar{e}_{s'}(t) \geq 0, \underline{e}_{q'}(t) \geq 0$ for $\forall t \geq t_1$. Note that $x_i(t_1) \in \underline{x}_{q'}(t_1)$, and thus $\underline{e}_i \geq 0$. Combining that $\bar{\rho}_i \geq 0 \geq x_i$, and it follows that $\underline{\rho}_i \leq x_i \leq \bar{\rho}_i$. By using the similar statements, we can extend the conclusion to other time interval where the sign of $x_i(t)$ changes. Therefore, it can be concluded that the state $x_i(t)$ will keep inside the upper bound $\bar{\rho}_i(t)$ and lower bound $\underline{\rho}_i(t)$ even the switching happens.
The same conclusion can be obtained when $x_i(0) < 0$. Thus, $\underline{\rho}_i(t) \leq x_i(t) \leq \bar{\rho}_i(t)$ is always satisfied. Similar analysis can be conducted into the other elements of the state x. Therefore, it can be concluded that the state $\underline{\rho}_i(t) \leq x_i(t) \leq \bar{\rho}_i(t), \forall t \geq 0$. Note that the performance bound functions $\bar{\rho}(t)$ and $\underline{\rho}(t)$ are bounded, and thus the closed-loop system is ultimately bounded. This completes the proof. □

Remark 2. Compared with existing PPC approaches (Bechlioulis et al., 2008, 2014; Zhang et al., 2017; Li et al., 2018), the presented control no longer needs the error transformation and logarithmic/tangent function, and provides an alternative switching scheme to ensure the performance restricted in a certain limitation. More interestingly, the proposed control allows the final bound ρ_∞ to be set to zero which cannot be admitted in existing PPC approaches due to the logarithmic or tangent function.

5.2 Discussion on the unified PPC framework

The proposed control provides a new way to address the PPC issue through the following corollary.

Corollary 1. Consider the system (1) satisfying the initial condition $\underline{\rho}(0) \leq x(0) \leq \bar{\rho}(0)$, the control (12) and the performance bound functions (2). If R satisfies (21), then $\underline{\rho}(t) \leq x(t) \leq \bar{\rho}(t)$ is always satisfied and (see definition 2)

(1) arbitrary $\rho(t)$ allows to guarantee bounded stability of the closed-loop;
(2) If the final value $\rho_{i,\infty} = 0$ $\forall i$ are set in the performance bound function (2), then asymptotic stability is achieved for the controlled system;
(3) If the performance bound function $\rho(t)$ is chosen as a finite-time and at least first-order differentiable function, then finite time stability is guaranteed for the closed loop.

Proof. Immediate considering the property $\underline{\rho}(t) \leq x(t) \leq \bar{\rho}(t)$ $\forall t \geq 0$. □

As stated in Corollary 1, the different system stability can be obtained only by using different performance bound functions, and thus the proposed control method can establish a framework to achieve the bounded, asymptotic or finite-time stability. For instance, the finite-time performance bound function can be selected as

$$\dot{\rho}(t) = -c\,\mathrm{sgn}(\rho(t)),\ c > 0 \tag{25}$$

6. SIMULATION STUDY

Consider the simple linear system

$$\dot{x} = x + u \tag{26}$$

with $x(0) = 2$, and the following performance bound functions

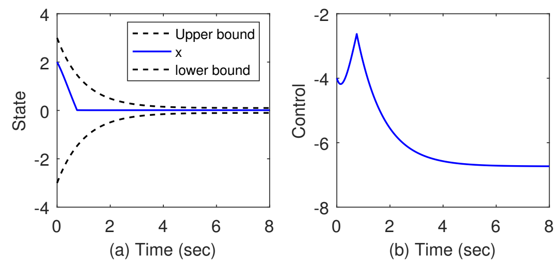

Fig. 1. Responses of the state and control in Case 1

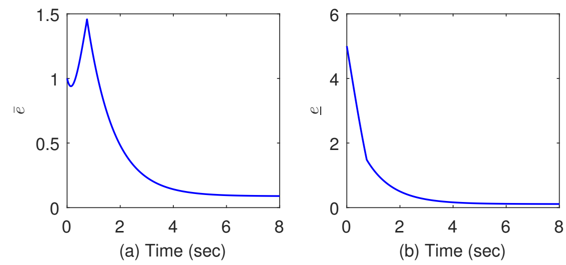

Fig. 2. Responses of $\bar{e}, \underline{e}$ in Case 1

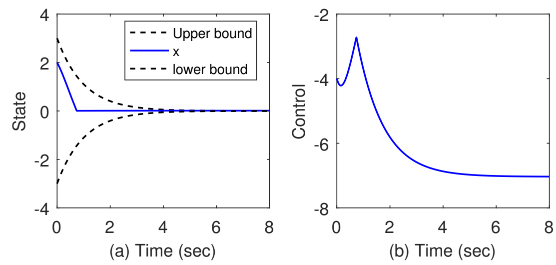

Fig. 3. Responses of the state and control in Case 2

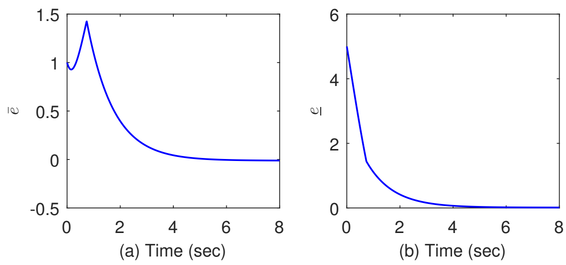

Fig. 4. Responses of $\bar{e}, \underline{e}$ in Case 2

- Case 1: (2) with $\rho_0 = 3, l = 1,\ \rho_\infty = 0.1$
- Case 2: (2) with $\rho_0 = 3, l = 1,\ \rho_\infty = 0$
- Case 3: (25) with $c = 1$ and $\rho_0 = 3$

Case 1 is to demonstrate the bounded stability, case 2 is for asymptotic stability and case 3 is concerned by finite-time stability.

The control parameters are set as $r = 7, k = -3$. In fact, $\dot{\rho} = -l(\rho_0 - \rho_\infty)e^{-lt} = -2.9e^{-t}$, and thus it can be verified that the condition (21) holds.

As shown in Fig. 1, Fig. 3 and Fig. 5, the state x is restricted between the prescribed performance functions, and thus simulation results verify the proposed control method in all three cases. Note that Fig. 4(a) and Fig. 6(a) reveal that $\bar{e}(t) < 0$ for some t, and thus x_1 escapes the

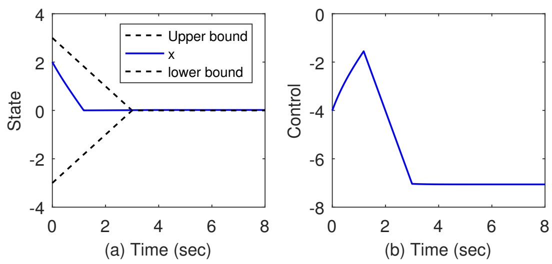

Fig. 5. Responses of the state and control in Case 3

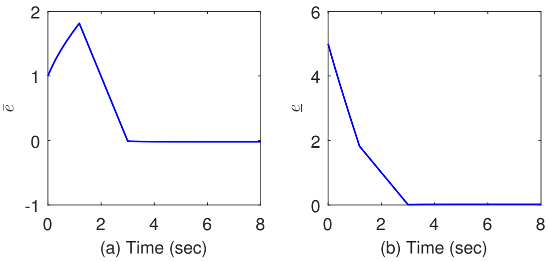

Fig. 6. Responses of $\bar{e}, \underline{e}$ in Case 3

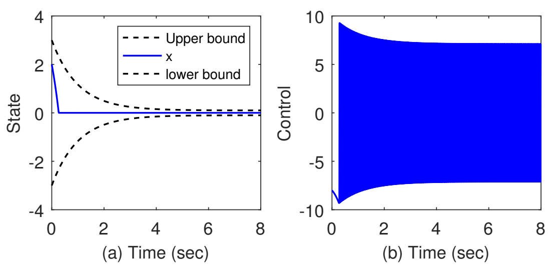

Fig. 7. Responses of the state and control under $k = 1$

zero, see Fig. 3(a) and Fig. 5(a). It can be explained that the absolute zero-tracking cannot be achieved in the numerical simulation via the switching scheme. However, this phenomenon exactly demonstrates the advantage of the proposed control approach. The escape from the required performance bound does not destroy the control system, but the existing PPC system will be terminated once the escape occurs due to the logarithmic/tangent function.

Finally, we would like to discuss the selection of k. For $k = -3$, the transition matrix of the closed-loop is Hurwitz, which is not the case for $k = 1$. However, the state x can still be driven to zero using the proposed control scheme, and thus, the bounded stability can be ensured, see Fig. 7. This means that the state is forced to the region between the upper bound and lower bound, thus, switching occurs around zero which causes a chattering in the control u.

7. CONCLUSION

This paper proposes a switching control scheme to guarantee prescribed performance for a particular case of linear time invariant system. The main feature consists of a unified prescribed performance control framework which covers the bounded stability, asymptotic stability and finite-time stability only via choosing the corresponding performance bound functions. Simulation results verify the effectiveness of the proposed method. Future works include the extension to the industrial system with general form.

REFERENCES

Bechlioulis, C. P., and Rovithakis, G. A. (2008). Robust adaptive control of feedback linearizable MIMO nonlinear systems with prescribed performance. *IEEE Transactions on Automatic Control*, 53(9), 2090–2099.

Bechlioulis, C.P. and Rovithakis, G.A. (2011). Robust partial-state feedback prescribed performance control of cascade systems with unknown nonlinearities. *IEEE Transactions on Automatic Control*, 56(9), 2224–2230.

Bechlioulis, C. P., and Rovithakis, G. A. (2014). A low-complexity global approximation-free control scheme with prescribed performance for unknown pure feedback systems. *Automatica*, 50(4), 1217–1226.

Bu, X., He, G., and Wei, D. (2018). A new prescribed performance control approach for uncertain nonlinear dynamic systems via back-stepping. *Journal of the Franklin Institute*, 355(17), 8510 – 8536.

Bu X., Wu X., Huang J., and Wei D. (2016). A guaranteed transient performance-based adaptive neural control scheme with low-complexity computation for flexible air-breathing hypersonic vehicles. *Nonlinear Dynamics*, 84(4), 2175–2194.

Guo Z., Ma Q., Guo J., Zhao B., and Zhou J. (2020). Performance-involved coupling effect-triggered scheme for robust attitude control of HRV. *IEEE/ASME Transactions on Mechatronics*, doi: 10.1109/TMECH.2020.2973708.

Chen, C. C. (2019). A unified approach to finite-time stabilization of high-order nonlinear systems with and without an output constraint. *International Journal of Robust and Nonlinear Control*, 29, 393–407.

Ellero, N., Gucik-Derigny, D., and Henry, D. (2019). An unknown input interval observer for LPV systems under L_2-gain and L_∞-gain criteria. *Automatica*, 103, 294–301.

Farina, L., and Rinaldi, S. (2000). Positive linear systems: theory and applications. John Wiley & Sons.

Filippov, A. F. (1988). *Differential Equations with Discontinuous Righthand Sides*, Dordrecht, The Netherlands: Kluwer Academic.

Hu Y., Li H., and Huang. H. (2014). Prescribed performance-based backstepping design for synchronization of cross-strict feedback hyperchaotic systems with uncertainties. *Nonlinear Dynamics*, 76, 103–113.

Hu, Q. L., Shao, X. D., and Guo, L. (2018). Adaptive fault-tolerant attitude tracking control of spacecraft with prescribed performance. *IEEE/ASME Transactions on Mechtronics*, 23(1), 331–341.

Kostarigka, A.K., Doulgeri, Z., and Rovithakis, G.A. (2013). Prescribed performance tracking for flexible joint robots with unknown dynamics and variable elasticity. *Automatica*, 49(5), 1137 – 1147.

Li, F. Z., and Liu, Y. G. (2018). Control design with prescribed performance for nonlinear systems with unknown control directions and nonparametric uncertainties. *IEEE Transactions on Automatic Control*, 63(10), 3573–3580.

Liu, Y. H., and Li., H. Y. (2018). Adaptive asymptotic tracking using barrier functions. *Automatica*, 98, 239–246.

Polyakov, A., and Fridman, L. (2014). Stability notions and Lyapunov functions for sliding mode control systems. *Journal of the Franklin Institute*, 351, 1831–1865.

Raissi, T., Efimov, D. and Zolghadri, A. (2012) Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57(1), 260–265.

Song H., Zhang T., and Zhang G.. (2014). Robust dynamic surface control of nonlinear systems with prescribed performance. *Nonlinear Dynamics*, 76, 599–608.

Torres-Gonzalez, V., Sanchez, T., Fridman, L. M., and Moreno, J. A. (2017). Design of continuous twisting algorithm. *Automatica*, 80, 119–126.

Zhang, J. X., and Yang, G. H. (2017). Prescribed performance fault-tolerant control of uncertain nonlinear systems with unknown control directions. *IEEE Transactions on Automatic Control*, 62(12), 6529–6535.