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Homogeneous Output-Feedback Control

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Abstract: New types of homogeneous and homogeneous-in-bilimit filtering observers and differentiators are proposed and applied for the global robust homogeneous asymptotic output-feedback stabilization of disturbed integrator chains. The type of convergence (finite-time, fixed-time to any ball or just asymptotic) is determined by the chosen system homogeneity degree (HD). Output-feedback sliding-mode control is an important particular case. Stabilization accuracy is calculated in the presence of possibly *unbounded* noises having a bounded multiple integral. Successful stabilization is demonstrated for very large noises and different HDs.

Keywords: Nonlinear output-feedback control, filtering, sliding mode control, robustness.

1. INTRODUCTION

The classical output-regulation problem is often reduced to the stabilization of an integrator chain (Brunowsky system) $\sigma^{(n)} = h + gu$, $g \neq 0$, with uncertain functions h and g depending on the time and the state (Isidori, 1989). It is natural to replace the unknown functions h, g with some *known* sets producing a separate differential inclusion to be stabilized. The length of the chain dictates the number of the output derivatives to be estimated for the output-feedback control.

The control design method is often based on the homogeneous extension (Bernuau et al., 2013). A well-known case is the high-order sliding-mode (SM) control (SMC), since it only requires the boundedness of the functions h, g (Bartolini et al., 2003; Cruz-Zavala and Moreno, 2016a; Davila et al., 2009; Floquet et al., 2003; Harmouche et al., 2017; Koch et al., 2020; Levant, 2003, 2005, 2017). Integrator chains are stabilized by homogeneous controllers in many papers, in particular by Hong (2002).

The problem is solved *in finite time* (FT) for the negative homogeneity degrees (HDs), or *asymptotically* for non-negative HDs (Levant et al., 2016). The convergence time to any ball around the origin is uniformly bounded for positive HDs (*fixed time (FxT) convergence*) (Andrieu et al., 2008; Angulo et al., 2013; Polyakov, 2012). SMC corresponds to the system HD $-1/n$, provided the HD $\deg \sigma = 1$ is chosen.

Being very sensitive to sampling times and very large already at moderate distances from the origin, FxT control is often not feasible. Nevertheless, it becomes the best choice in the control of explosive systems capable of finite-time escape to infinity. In the homogeneity theory it corresponds to the positive system HD.

It is well-known that linear control and SMC allow effective output-feedback control design (Atassi and Khalil, 2000; Levant, 2003). Such methods are unknown for the system HDs larger than $-1/n$ and not approximating 0. Standard SM-based differentiators correspond to semiglobal output-feedback controllers and destroy the closed system accuracy. This paper is to close that gap. For this end we apply homogeneous differentiators (Perruquetti et al., 2008; Cruz-Zavala and Moreno, 2016b) also for positive HDs. We prove that in spite of being exact only on polynomials, they become effective observers in the homogeneous feedback.

Noisy sampling make observation difficult Khalil and Priess (2016). We equip the proposed observers with the filtering extensions (Levant and Yu, 2018; Levant and Livne, 2019) preserving *robust exact* observation and providing for good estimation even in the presence of unbounded noises, having a small multiple integral.

We propose a new type of differentiators called hybrid filtering differentiators, extending the fast-converging hybrid differentiators (Levant and Livne, 2018) to the general negative homogeneity degrees and equipping them with the filtering capabilities. Their accuracy is calculated in the output-feedback SMC case.

Our simulation demonstrates successful stabilization in the presence of **very large** measurement noises.

Notation. A binary operation $\diamond$ of two sets is defined as $A \diamond B = \{a \diamond b | \ a \in A, b \in B\}$; $a \diamond B = \{a\} \diamond B$. A function of a set is the set of function values on this set; $||x||_h$ is a homogeneous norm; $\lfloor a \rceil^b = |a|^b \operatorname{sign} a$, $\lfloor a \rceil^0 = \operatorname{sign} a$.

2. HOMOGENEOUS STABILIZATION PROBLEM

Recall that solutions of the differential inclusion (DI)

$$\dot{x} \in F(x),\ F(x) \subset T_x\mathbb{R}^{n_x}, \qquad (1)$$

are defined as locally absolutely continuous functions $x(t)$, satisfying the DI for almost all t. Here $T_x\mathbb{R}^{n_x}$ denotes the tangent space to $\mathbb{R}^{n_x}$ at $x \in \mathbb{R}^{n_x}$.

We call the DI (1) *Filippov DI*, if the vector-set field $F(x) \subset T_x\mathbb{R}^{n_x}$ is non-empty, compact and convex for any x, and F is an upper-semicontinuous set function. The latter means that the maximal distance of the points of $F(x)$ from the set $F(y)$ tends to zero, as $x \to y$.

Filippov DIs feature existence, extendability, etc. of solutions, but not their uniqueness (Filippov, 1988). The Filippov definition (Filippov, 1988) replaces a discontinuous vector field $f(x)$ with a Filippov DI.

2.1 Coordinate homogeneity basics

Introduce the homogeneous weights $m_1, ..., m_{n_x} > 0$ of the coordinates $x_1, ..., x_{n_x}$ in $\mathbb{R}^{n_x}$, $\deg x_i = m_i$, and the dilation (Bacciotti and Rosier, 2005)

$$d_\kappa : (x_1, x_2, ..., x_{n_x}) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_{n_x}} x_{n_x}),$$

where $\kappa \geq 0$. Recall that a function $g : \mathbb{R}^{n_x} \to \mathbb{R}^m$ is said to have the homogeneity degree (HD) (weight) $q \in \mathbb{R}$, $\deg g = q$, if the identity $g(x) = \kappa^{-q} g(d_\kappa x)$ holds for any $x \in \mathbb{R}^{n_x}$ and $\kappa > 0$.

Consider the combined time-coordinate transformation

$$(t, x) \mapsto (\kappa^{-q} t, d_\kappa x), \quad \kappa > 0, \tag{2}$$

where the number $-q \in \mathbb{R}$ might naturally be considered as the weight of t. The DI $\dot{x} \in F(x), x \in \mathbb{R}^{n_x}$ and the vector-set field $F(x) \subset T_x\mathbb{R}^{n_x}$ are called homogeneous of the HD q, if the identity $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$ holds for any x and $\kappa > 0$. It implies that the DI is invariant with respect to (2), i.e. $\dot{x} \in F(x) \Leftrightarrow \frac{d(d_\kappa x)}{d(\kappa^{-q} t)} \in F(d_\kappa x)$.

A system of differential equations (DEs) $\dot{x}_i = f_i(x)$, $i = 1, ..., n_x$, is a particular case of DI, when the set $F(x)$ contains only one vector $f(x)$ and is reduced to the classical definition $\deg \dot{x}_i = \deg x_i - \deg t = m_i + q = \deg f_i$. If f is discontinuous, the DE is equivalent to the corresponding homogeneous Filippov DI (1).

Note that the weights/degrees $-q, m_1, ..., m_{n_x}$ are defined up to proportionality.

Any continuous positive-definite function of the HD 1 is called a homogeneous norm. We denote it $||x||_h$. In particular, denote $||x||_{h\infty} = \max_i |x_i|^{1/m_i}$. The quotient of any two homogeneous norms is uniformly bounded and separated from zero for $x \neq 0$.

It is proved by Levant (2005); Levant and Livne (2016); Levant et al. (2016) that if $q > 0$ then asymptotic stability (AS) implies FxT convergence to any ball around 0, AS is exponential for $q = 0$, and if $q < 0$ then AS implies FT stability. In that case, in the presence of a maximal delay $\tau \geq 0$ and noises of the magnitudes $\varepsilon_i \geq 0$, $i = 1, 2, ..., n_x$, all extendable-in-time solutions of the disturbed DI

$$\dot{x} \in F(x(t - \tau[0, 1]) + [-\varepsilon_1, \varepsilon_1] \times ... \times [-\varepsilon_{n_x}, \varepsilon_{n_x}])$$

starting from some time satisfy the inequalities $|x_i| \leq \mu_i \rho^{m_i}$ for some $\mu_i > 0$, $\rho = \max[||\varepsilon||_{h\infty}, \tau^{-1/q}]$.

2.2 Stabilization problem

Let $\sigma \in \mathbb{R}$, introduce the homogeneous weights $\deg \sigma^{(i)} = 1 + iq$, $i = 0, 1, ..., n$. Denote $\overrightarrow{\sigma}_k = (\sigma, \dot{\sigma}, ..., \sigma^{(k)})$, $k \in \mathbb{N} \cup \{0\}$. Consider the DI

$$\begin{array}{l} \sigma^{(n)} \in [-C, C] ||\overrightarrow{\sigma}_{n-1}||_h^{1+nq} + [K_m, K_M] u, \\ C \geq 0, \; 0 < K_m \leq K_M, \end{array} \tag{3}$$

where $u \in \mathbb{R}$ is the control, $|| \cdot ||_h$ is some homogeneous norm. The problem is to globally asymptotically stabilize the system only using the real-time measurements of σ.

A large number of homogeneous stabilizers are known, which solve the problem using the full knowledge of $\overrightarrow{\sigma}_{n-1}(t)$. The problem has a well-known output-feedback HOSM stabilization solution for $q = -1/n$. It cannot be solved applying the standard differentiators with constant (Levant (2003)) or variable (Levant and Livne (2018)) gains for $q \neq -1/n$, since $||\overrightarrow{\sigma}_{n-1}||_h$ is not available and can also feature higher than exponential growth. Thus, the problem remains challenging for $q \neq 0, -1/n$.

3. HOMOGENEOUS STABILIZATION

3.1 Filtering homogeneous observation

Introduce the weights $\deg w_i = 1 - (n_f + 1 - i)q$, $i = 1, ..., n_f$, $\deg z_i = 1 + iq$, $i = 0, 1, ..., n_d$. The homogeneous filtering differentiator of the differentiation order $n_d \geq 0$ and the filtering order $n_f \geq 0$, $-1/(n_d + 1) \leq q < 1/n_f$, with the input $f(t)$ and the parameter $L > 0$ has the form

$$\begin{array}{l} \dot{w}_1 = -\tilde{\lambda}_{n_d+n_f} L^{\frac{|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1-(n_f-1)q}{1-n_f q}} + w_2, \\ \dot{w}_2 = -\tilde{\lambda}_{n_d+n_f-1} L^{\frac{2|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1-(n_f-2)q}{1-n_f q}} + w_3, \\ \quad \cdots \\ \dot{w}_{n_f-1} = -\tilde{\lambda}_{n_d+2} L^{\frac{(n_f-1)|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1-q}{1-n_f q}} + w_{n_f}, \\ \dot{w}_{n_f} = -\tilde{\lambda}_{n_d+1} L^{\frac{n_f|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1}{1-n_f q}} + z_0 - f(t), \end{array} \tag{4}$$

$$\begin{array}{l} \dot{z}_0 = -\tilde{\lambda}_{n_d} L^{\frac{(1+n_f)|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1+q}{1-n_f q}} + z_1, \\ \dot{z}_1 = -\tilde{\lambda}_{n_d-1} L^{\frac{(2+n_f)|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1+2q}{1-n_f q}} + z_2, \\ \quad \cdots \\ \dot{z}_{n_d} = -\tilde{\lambda}_0 L^{\frac{(n_d+1+n_f)|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1+(n_d+1)q}{1-n_f q}}. \end{array} \tag{5}$$

Here $\tilde{\lambda}_j > 0$, $j = 0, ..., n_d + n_f$. Formally define that for $n_f = 0$ equations (4) disappear, and w_1 is replaced with $z_0 - f(t)$ in (5).

In the case $q = -1/(n_d + 1)$ the filtering differentiator (Levant and Livne, 2019) is obtained which is based on SMs. The "continuous differentiator" (Perruquetti et al., 2008; Sanchez et al., 2018) is obtained for $n_f = 0$, $0 > q > -1/(n_d + 1)$. In the case $q = 0$ a linear filter is obtained. Note that system (4), (5) is only homogeneous for $f(t) \equiv 0$.

Theorem 1. Fix any $\gamma_L \geq 0$, and denote $\zeta_i = z_i - f^{(i)}$. Then there exist such $\tilde{\lambda}_0, ..., \tilde{\lambda}_{n_d+n_f} > 0$ that filter (4), (5) provides for the asymptotically exact estimations z_i of $f^{(i)}(t)$ provided $L \geq 1$ and

$$|f^{(n_d+1)}(t)| \leq \gamma_L L^{\frac{|q|}{1+n_d q}} ||(\zeta_0, ..., \zeta_{n_d})||_{h\infty}^{1+(n_d+1)q} \tag{6}$$

*holds. The convergence is **in FT** for $q < 0$, **in FxT** to any ball of errors ζ_i for $q > 0$ and **exponential** for $q = 0$.*

In the case $\gamma_L = 0$ condition (6) means exactness on polynomials of the degrees not exceeding n_d.

It is difficult to find $\tilde{\lambda}_0, ..., \tilde{\lambda}_{n_d+n_f} > 0$ for each $n_d, n_f \geq 0$ and q, $-1/(n_d + 1) \leq q < 1/n_f$. The task is facilitated by the identical recursive form of the filter

$$\begin{aligned}
\dot w_1 &= -\lambda_{n_d+n_f} L^{\frac{|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1-(n_f-1)q}{1-n_f q}} + w_2,\\
\dot w_2 &= -\lambda_{n_d+n_f-1} L^{\frac{|q|}{1-(n_f-1)q}} \lfloor w_2 - \dot w_1 \rceil^{\frac{1-(n_f-2)q}{1-(n_f-1)q}} + w_3,\\
&\dots\\
\dot w_{n_f-1} &= -\lambda_{n_d+2} L^{\frac{|q|}{1-2q}} \lfloor w_{n_f-1} - \dot w_{n_f-2} \rceil^{\frac{1-q}{1-2q}} + w_{n_f},\\
\dot w_{n_f} &= -\lambda_{n_d+1} L^{\frac{|q|}{1-q}} \lfloor w_{n_f} - \dot w_{n_f-1} \rceil^{\frac{1}{1-q}} + z_0 - f(t),
\end{aligned} \tag{7}$$

$$\begin{aligned}
\dot z_0 &= -\lambda_{n_d} L^{\frac{|q|}{1}} \lfloor z_0 - f(t) - \dot w_{n_f} \rceil^{\frac{1+q}{1}} + z_1,\\
\dot z_1 &= -\lambda_{n_d-1} L^{\frac{|q|}{1+q}} \lfloor z_1 - \dot z_0 \rceil^{\frac{1+2q}{1+q}} + z_2,\\
&\dots\\
\dot z_{n_d} &= -\lambda_0 L^{\frac{|q|}{1+n_d q}} \lfloor z_{n_d} - \dot z_{n_d-1} \rceil^{\frac{1+(n_d+1)q}{1+n_d q}}.
\end{aligned} \tag{8}$$

If $n_f = 0$ one substitutes $\dot w_{n_f} = 0$ in (8), and if also $n_d = 0$ one takes the last equation with $\dot z_{n_d-1} = 0$. It is easy to see that

$$\begin{aligned}
&\tilde\lambda_k = \lambda_k, \text{ where } k = n_d + n_f;\\
&\tilde\lambda_{k-i} = \lambda_{k-i} \cdot \tilde\lambda_{k-i+1}^{\frac{1-(n_f-i)q}{1-(n_f-i+1)q}}, \ i = 1, 2, ..., n_f,\\
&\tilde\lambda_{k-i} = \lambda_{k-i} \cdot \tilde\lambda_{n_d-i+1}^{\frac{1+(n_d-i+1)q}{1+(n_d-i)q}}, \ i = n_d, n_d+1, ..., k.
\end{aligned}$$

If $q \neq 0$ one can rewrite (6) as

$$\begin{aligned}
&d = \begin{cases} 1/|q| + (n_d+1)\operatorname{sign} q, & q \neq 0,\\ \infty, & q = 0, \end{cases} \quad L \geq 1,\\
&|f^{(n_d+1)}(t)| \leq \gamma_L L^{\frac{1}{d-\operatorname{sign} q}} \max_{i=0,\dots,n_d} |\zeta_i|^{\frac{d}{d-(n_d+1-i)\operatorname{sign} q}},
\end{aligned} \tag{9}$$

which corresponds to the proportional change of weights with the HD becoming ± 1, the weight of $\dot z_{n_d}$ becoming d. If $q = 0$ we formally take $d = \infty$, $\frac{1}{d-\operatorname{sign} q} = 0$, $\frac{d}{d-(n_d+1-i)\operatorname{sign} q} = 1$, and, correspondingly, L disappears.

Denote $s_q = \operatorname{sign} q$. Then the recursive form (7), (8) turns into

$$\begin{aligned}
\dot w_1 &= -\lambda_{n_d+n_f} L^{\frac{1}{d-(n_d+n_f+1)s_q}} \lfloor w_1 \rceil^{\frac{d-(n_d+n_f)s_q}{d-(n_d+n_f+1)s_q}} + w_2,\\
\dot w_2 &= -\lambda_{n_d+n_f-1} L^{\frac{1}{d-(n_d+n_f)s_q}} \lfloor w_2 - \dot w_1 \rceil^{\frac{d-(n_d+n_f-1)s_q}{d-(n_d+n_f)s_q}}\\
&\qquad + w_3,\\
&\dots\\
\dot w_{n_f} &= -\lambda_{n_d+1} L^{\frac{1}{d-(n_d+2)s_q}} \lfloor w_{n_f} - \dot w_{n_f-1} \rceil^{\frac{d-(n_d+1)s_q}{d-(n_d+2)s_q}}\\
&\qquad + z_0 - f(t),
\end{aligned} \tag{10}$$

$$\begin{aligned}
\dot z_0 &= -\lambda_{n_d} L^{\frac{1}{d-(n_d+1)s_q}} \lfloor z_0 - f(t) - \dot w_{n_f} \rceil^{\frac{d-n_d s_q}{d-(n_d+1)s_q}}\\
&\qquad + z_1,\\
\dot z_1 &= -\lambda_{n_d-1} L^{\frac{1}{d-n_d s_q}} \lfloor z_1 - \dot z_0 \rceil^{\frac{d-(n_d-1)s_q}{d-n_d s_q}} + z_2,\\
&\dots\\
\dot z_{n_d} &= -\lambda_0 L^{\frac{1}{d-s_q}} \lfloor z_{n_d} - \dot z_{n_d-1} \rceil^{\frac{d}{d-s_q}}, \ s_q = \operatorname{sign} q.
\end{aligned} \tag{11}$$

The following theorem establishes recursive choice of parameters λ_i, $i = 0, 1, \dots$.

Theorem 2. Let $q \geq -1$, $\lambda_0 > \gamma_L$ *and* $d \geq 0$. *Then there exists a universal positive sequence* $\overrightarrow{\lambda} = \{\lambda_0, \lambda_1, ...\}$ *which is* ***infinite*** *for* $q \leq 0$ *and* ***finite*** *for* $q > 0$. *It is valid for any* $n_f, n_d \geq 0$, $d > (n_d + n_f + 1)s_q$, *any input* $f(t)$ *satisfying* (9) *(and, equivalently,* (6)*) filter* (10), (11) *(and* (4), (5)*), and provides for the asymptotic convergence of the estimations* z_i *to the* ***exact*** *derivatives* $f^{(i)}(t)$. *The convergence is in* ***FT*** *for* $q < 0$. *Parameters* λ_i *are chosen one-by-one sufficiently large starting from* λ_1.

Note that $d = 0$ implies the SMC case $q = -1/(n_d+1)$. In that case the valid choice $\overrightarrow{\lambda} = \{1.1, 1, 5, 2, 3, 5, 7, 9, 12...\}$ is well-known for $\gamma_L = 1$ and is sufficient for $n_d + n_f \leq 7$ (Levant and Livne, 2018, 2019). The standard linear technique by Atassi and Khalil (2000) is probably more adequate in the case $q = 0$.

Introduce the short notation for (4), (5):

$$\begin{aligned}
\dot w &= \Omega_{n_d,n_f,q}(w, z_0 - f, L, \overrightarrow{\lambda}_{n_d+n_f}),\\
\dot z &= D_{n_d,n_f,q}(w_1, z, L, \overrightarrow{\lambda}_{n_d+n_f}),
\end{aligned} \tag{12}$$

where $\overrightarrow{\lambda}_{n_d+n_f} = \{\lambda_0, ..., \lambda_{n_d+n_f}\}$ appear in (10), (11).

3.2 Output feedback stabilization

In practice condition (6) is not natural for $q \neq -1/(n_d+1)$, and, generally speaking, differentiators (4)(5) require $f^{(n_d+1)} \equiv 0$, i.e. only differentiate polynomials of the order n_d and less. In this section we show that they still can be used as observers in a homogeneous feedback.

Theorem 3. Let $q \neq 0$, $\gamma_L \geq 0$, *and assume that the locally-bounded homogeneous control* $u = U(\overrightarrow{\sigma}_{n-1})$, $\deg U = 1 + nq$, *asymptotically stabilizes system* (3). *Then the output-feedback control*

$$\begin{aligned}
u &= U(z),\\
\dot w &= \Omega_{n-1,n_f,q}(w, z_0 - \sigma, L, \overrightarrow{\lambda}_{n+n_f-1}),\\
\dot z &= D_{n-1,n_f,q}(w_1, z, L, \overrightarrow{\lambda}_{n+n_f-1})
\end{aligned} \tag{13}$$

asymptotically stabilizes system (3) *for any* $n_f \geq 0$ *and sufficiently large* L. *The convergence is* ***in FT*** *for* $q < 0$, *and* ***in FxT*** *to any ball around the origin for* $q > 0$.

A similar result is obtained for $q = 0$ by tuning the observer eigenvalues as in (Atassi and Khalil, 2000).

3.3 Hybrid filtering differentiation

Here we propose the *new type of differentiator* combining the accelerated FT convergence with the filtering features. Its only practical form is the recursive form

$$\begin{aligned}
\dot w_1 &= -\lambda_{n_d+n_f} L^{\frac{|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1-(n_f-1)q}{1-n_f q}}\\
&\qquad - \mu_{n_d+n_f} M w_1 + w_2,\\
&\dots\\
\dot w_{n_f-1} &= -\lambda_{n_d+2} L^{\frac{|q|}{1-2q}} \lfloor w_{n_f-1} - \dot w_{n_f-2} \rceil^{\frac{1-q}{1-2q}}\\
&\qquad - \mu_{n_d+2} M (w_{n_f-1} - \dot w_{n_f-2}) + w_{n_f},\\
\dot w_{n_f} &= -\lambda_{n_d+1} L^{\frac{|q|}{1-q}} \lfloor w_{n_f} - \dot w_{n_f-1} \rceil^{\frac{1}{1-q}}\\
&\qquad - \mu_{n_d+1} M (w_{n_f} - \dot w_{n_f-1}) + z_0 - f(t),
\end{aligned} \tag{14}$$

$$\begin{aligned}
\dot z_0 &= -\lambda_{n_d} L^{\frac{|q|}{1}} \lfloor z_0 - f(t) - \dot w_{n_f} \rceil^{\frac{1+q}{1}}\\
&\qquad - \mu_{n_d} M (z_0 - f(t) - \dot w_{n_f}) + z_1,\\
\dot z_1 &= -\lambda_{n_d-1} L^{\frac{|q|}{1+q}} \lfloor z_1 - \dot z_0 \rceil^{\frac{1+2q}{1+q}}\\
&\qquad - \mu_{n_d-1} M (z_1 - \dot z_0) + z_2,\\
&\dots\\
\dot z_{n_d} &= -\lambda_0 L^{\frac{|q|}{1+n_d q}} \lfloor z_{n_d} - \dot z_{n_d-1} \rceil^{\frac{1+(n_d+1)q}{1+n_d q}}\\
&\qquad - \mu_0 M (z_{n_d} - \dot z_{n_d-1}).
\end{aligned} \tag{15}$$

It probably is practical only for $q < 0$ and converges even for variable $L(t) > 0$ under the condition $|\dot L|/L \leq M$. In the case $n_f = 0$, $q = -1/(n_d+1)$ one obtains the hybrid differentiator by Levant and Livne (2018). If also $L = 0$ and $M >> 1$ one gets the classical high-gain observer

(Atassi and Khalil, 2000). $M = 0$ produces the filtering differentiator (7), (8).

In the following only the SMC case $q = -1/(n_d + 1)$, $L = const$ is considered. The filter error dynamics are homogeneous in bilimit (Andrieu et al., 2008) with the HD q at zero and the HD 0 at infinity. Parameters $\overrightarrow{\lambda} = \{1.1, 1.5, 2, 3, 5, 7, 10, 12, \ ...\}$ and $\overrightarrow{\mu} = \{2, 3, 4, 7, 9, 13, 19, 23, ...\}$ are well-checked for $\gamma_L = 1$ (Levant and Livne, 2018) and are sufficient for $n_d + n_f \leq 7$.

3.4 Performance in the presence of noise

Recall the simplest notions from the general theory of the filtering homogeneous differentiation (Levant and Livne, 2019). Because of the lack of space the discretization issues (Levant and Livne, 2019) are not considered.

A signal $\nu(t)$, $\nu : [0, \infty) \to \mathbb{R}$, is called *globally filterable*, or a *signal of the (global) filtering order* $k \geq 0$, if it is a locally integrable Lebesgue-measurable function, and there exists a globally bounded Caratheodory solution $\xi(t)$, $\xi : [0, \infty) \to \mathbb{R}$, of the equation $\xi^{(k)} = \nu$. Correspondingly $\xi^{(k-1)}(t)$ is locally absolutely-continuous if $k > 0$, and $\nu(t)$ is of the filtering order $k = 0$, if ν is essentially bounded. Any number exceeding $\sup |\xi(t)|$ is called a *kth-order (global) integral magnitude of* ν.

Let the output $\sigma(t)$ of system (3)) be sampled with the noise $\eta(t) = \eta_0(t) + \eta_1(t) + ... + \eta_{n_f}(t)$, where each η_k, $k = 0, ..., n_f$, is a signal of the global filtering order k and the kth-order integral magnitude $\varepsilon_k \geq 0$. Hence, components $\eta_1, ..., \eta_{n_f}$ are possibly *unbounded*, $|\eta_0| \leq \varepsilon_0$.

Theorem 4. Let $L = const$, $q \neq 0$, $-1/n \leq q < 1/n_f$. *Consider proposed output-feedback controls (with the hybrid filtering in the case* $q = -1/n$*). Then all solutions stabilize in the set* $||\overrightarrow{\sigma}_{n-1}||_{h\infty} \leq \nu\rho$, *for some* $\nu > 0$ *and*

$$\rho = \max[\varepsilon_0{}^{\frac{1}{1}}, \varepsilon_1{}^{\frac{1}{1-q}}, ..., \varepsilon_{n_f}{}^{\frac{1}{1-n_f q}}]. \tag{16}$$

Here ν *only depends on the system parameters* C, K_m, K_M, $\overrightarrow{\lambda}$, $\overrightarrow{\mu}$ *and* $\sup U(\overrightarrow{\sigma}_{n-1})/||\overrightarrow{\sigma}_{n-1}||_{h\infty}^{1+nq}$. *In the case* $q = -1/n$, $M > 0$ *the asymptotics hold only for small enough* ρ, *the system practical stability is always preserved.*

4. PROOF SKETCHES

Proof of Theorem 1. The form (4),(5) is obtained from the form with $L = 1$ by means of the substitution $z_i := Lz_i$ for $q < 0$ and $z_i := z_i/L$ for $q > 0$. Due to the robustness of the AS of homogeneous systems (Levant et al., 2016) it is enough to prove the existence of parameters for $\gamma_L = 0$. Correspondingly the task is reduced to the case $n_f = 0$ (Levant and Livne, 2019) and $f(t) \equiv 0$.

After a suitable recursive change of coefficients the reduced error dynamics get the form

$$\begin{aligned} \dot{z}_0 &= c_0(z_1 - \lfloor z_0 \rceil^{1+q}), \\ \dot{z}_1 &= c_1(z_2 - \lfloor z_0 \rceil^{1+2q}), \\ &... \\ \dot{z}_{n_d} &= -c_{n_d} \lfloor z_0 \rceil^{1+(n_d+1)q}. \end{aligned} \tag{17}$$

A proper choice of $c_i > 0$ is to asymptotically stabilize (17). Fix any $a > 3\max[1, 1 + (n_d + 1)q]$. The following Lyapunov function (LF) candidate is inspired by Hong (2002); Cruz-Zavala and Moreno (2019):

$$\begin{aligned} V = &\int_{\lfloor z_1 \rceil^{\frac{1}{1+q}}}^{z_0} (\lfloor s \rceil^{\frac{a-1}{1}} - \lfloor z_1 \rceil^{\frac{a-1}{1+q}}) ds + ... + \\ &\int_{\lfloor z_{n_d} \rceil^{\frac{1+(n_d-2)q}{1+(n_d-1)q}}}^{z_{n_d-1}} (\lfloor s \rceil^{\frac{a-1-(n_d-2)q}{1+(n_d-2)q}} - \lfloor z_{n_d} \rceil^{\frac{a-1-(n_d-1)q}{1+(n_d-1)q}}) ds \\ &+ \int_0^{z_{n_d}} \lfloor s \rceil^{\frac{a-1-(n_d-1)q}{1+(n_d-1)q}} ds. \end{aligned} \tag{18}$$

It is easily checked that V is positive definite. Its derivative has the form

$$\begin{aligned} \dot{V} = &-[c_0 W_{\dot{z}_0 \neq 0}(z) + c_1 H_{\dot{z}_0 = 0}(z)] - ... \\ &- [c_i W_{\dot{z}_{i-1}=0, \dot{z}_i \neq 0}(z) + c_{i+1} H_{\dot{z}_{i-1}, \dot{z}_i = 0}(z)] - ... \\ &- c_{n_d} W_{\dot{z}_{n_d-1}=0, \dot{z}_{n_d} \neq 0}(z), \end{aligned}$$

where all components are continuous homogeneous functions, $\deg \dot{V} = a + q > 0$, $W_{\dot{z}_0 \neq 0}$ is positive whenever $\dot{z}_0 \neq 0$, $W_{\dot{z}_{i-1}=0, \dot{z}_i \neq 0}$ is positive whenever $\dot{z}_{i-1} = 0$ and $\dot{z}_i \neq 0$, $i = 1, ..., n_d$. The function $H_{\dot{z}_{i-1}, \dot{z}_i = 0}$ vanishes whenever $\dot{z}_{i-1} = \dot{z}_i = 0$, and $H_{\dot{z}_0 = 0} = 0$ if $\dot{z}_0 = 0$.

Fix any $c_{n_d} > 0$, then choose c_i sufficiently large to provide for the negative-definiteness of $\dot{V}$ on the set $\dot{z}_0 = ... = \dot{z}_{i-1} = 0$, $i = n_d - 1, ..., 1$. At last c_0 is chosen sufficiently large providing for the global negative-definiteness of $\dot{V}$. □

Proof of Theorem 2. The proof is by induction and similar to that for the SM case $q = -1/(n_d + 1)$ (Levant, 2003; Levant and Livne, 2018). □

Proof of Theorem 3. Define $\xi_i = z_i - \sigma^{(i)}$, $\deg \xi_i = \deg z_i = \deg \sigma^{(i)} = 1 + iq$, $\deg w_i = 1 - (n_f - i + 1)q$. The closed loop system can be rewritten as the homogeneous DI of the HD q,

$$\begin{aligned} \sigma^{(n)} &\in [-C, C]||\overrightarrow{\sigma}_{n-1}||_h^{1+nq} \\ &\qquad + [K_m, K_M]U(\overrightarrow{\sigma}_{n-1} + \xi), \\ \dot{w} &= \Omega_{n-1, n_f, q}(w, \xi_0, L, \overrightarrow{\lambda}_{n+n_f-1}), \\ \dot{\xi} &\in D_{n-1, n_f, q}(w_1, \xi, L, \overrightarrow{\lambda}_{n+n_f-1}). \end{aligned} \tag{19}$$

Under the exact measurements of $\overrightarrow{\sigma}_{n-1}$ get

$$\begin{aligned} |\sigma^{(n)}| \leq [-C, C]||\overrightarrow{\sigma}_{n-1}||_h^{1+nq} + [K_m, K_M]U(\overrightarrow{\sigma}_{n-1})| \\ \leq L_0||\overrightarrow{\sigma}_{n-1}||_{h\infty}^{1+nq})^T \end{aligned}$$

for some $L_0 > 0$. The statement of Theorem 1 is equivalent to the AS of the homogeneous DI

$$\begin{aligned} \dot{w} = &\ \Omega_{n-1, n_f, q}(w, \xi_0, L, \overrightarrow{\lambda}_{n+n_f-1}), \\ \dot{\xi} \in &\ D_{n-1, n_f, q}(w_1, \xi, L, \overrightarrow{\lambda}_{n+n_f-1}) \\ &+ (0, ..., 0, \gamma_L L^{\frac{|q|}{1+(n-1)q}} ||\xi||_{h\infty}^{1+nq})^T. \end{aligned} \tag{20}$$

So choose $L > 1$ such that $\gamma_L L^{\frac{|q|}{1+(n-1)q}} > L_0$, and let $V_u(\overrightarrow{\sigma}_{n-1})$ and $V_f(w, \xi)$ be the homogeneous C^1 LFs for (3) closed by $u = U(\overrightarrow{\sigma}_{n-1})$ and (20) respectively, $\deg V_u = \deg V_f = a > -q$. Such functions always exist (Bernuau et al., 2013).

Note that $\dot{V}_u(\overrightarrow{\sigma}_{n-1}) = \{\nabla V_u \dot{\overrightarrow{\sigma}}_{n-1}\}$ and similarly $\dot{V}_f(w, \xi)$ are compact numeric sets. Concequently $\sup \dot{V}_u(\overrightarrow{\sigma}_{n-1}) \leq -W_u(\overrightarrow{\sigma}_{n-1})$, $\sup \dot{V}_f(w, \xi) \leq -W_f(w, \xi)$ where W_u, W_f are positive-definite functions of their arguments, $\deg W_u = \deg W_f = a + q > 0$.

Search for the LF in the form $V(\vec{\sigma}_{n-1}, w, \xi) = V_u(\vec{\sigma}_{n-1}) + \mu V_f(w, \xi)$, $\mu > 0$. Then

$$\dot{V} = \dot{V}_u(\vec{\sigma}_{n-1}) + \frac{\partial}{\partial \sigma^{(n-1)}} V_u[K_m, K_M](U(\vec{\sigma}_{n-1}+\xi) - U(\vec{\sigma}_{n-1})) + \mu \dot{V}_f(w, \xi).$$

$$\sup \dot{V} \leq -W_u(\vec{\sigma}_{n-1}) + W_1(\vec{\sigma}_{n-1}, \xi) - \mu W_f(w, \xi),$$

where the continuous homogeneous function $W_1(\vec{\sigma}_{n-1}, \xi)$ vanishes for $\xi = 0$. As follows now from the standard Lemma by Andrieu et al. (2008), the right hand side is negative definite for μ large enough. □

Proof of Theorem 4. The proof is a combination of the proofs by Levant and Livne (2019, 2018). □

5. SIMULATION

Let $n = 3$. Consider the system

$$\dddot{\sigma} = \cos(12t)|\sigma|^{1+3q} + |\ddot{\sigma}|^{\frac{1+3q}{1+2q}} \operatorname{sign}(\dot{\sigma}) + [2 + \cos(t - \dot{\sigma})]u. \quad (21)$$

Correspondingly the system satisfies the DI

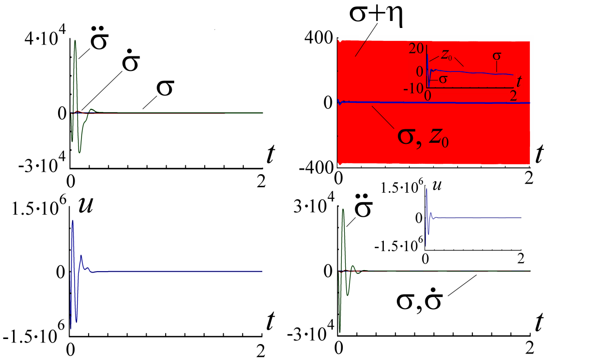
$$\dddot{\sigma} \in [-2, 2]||\vec{\sigma}_2||_h^{1+3q} + [1, 3]u,$$
$$||\vec{\sigma}_2||_h = |\sigma| + |\dot{\sigma}|^{\frac{1}{1+q}} + |\ddot{\sigma}|^{\frac{1}{1+2q}}.$$

The homogeneous stabilizing control (Levant, 2017) is

$$u = -5||\vec{\sigma}_2||_h^{\frac{1}{2}+3q}[\lfloor\ddot{\sigma}\rceil^{\frac{1}{2(1+2q)}} + 2\lfloor\dot{\sigma}\rceil^{\frac{1}{2(1+q)}} + \lfloor\sigma\rceil^{\frac{1}{2}}]. \quad (22)$$

It does not allow developing a Lyapunov function in the standard way (Cruz-Zavala and Moreno, 2016a).

The cases $q = 0.1, -0.1, -1/3$ are considered. The authors have taken $\gamma_L = 1$ and found parameters $\vec{\lambda}$ one by one by simulation for corresponding d, using the recursive form (10), (11). The integration is performed by the Euler method with the integration step $\tau = 10^{-5}, 10^{-6}$.

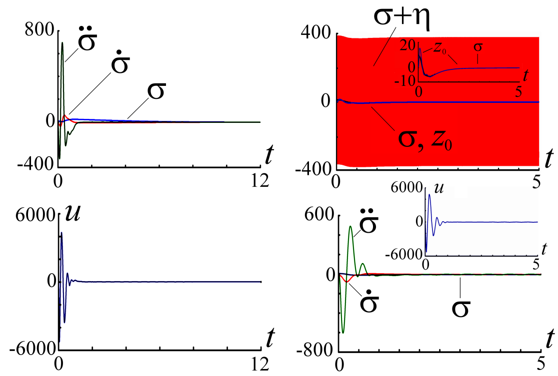


a. Exact sampling b. Noisy sampling

Fig. 1. The case $q = +0.1$. a: Asymptotic stability is got for the exact sampling. b: The system remains practically exact in spite of the very large sampling noise η.

1. Let $q = 0.1$, and $n_d = 2, n_f = 3$, $d = 13$, $\vec{\sigma}_2(0) = (10, -10, 10)^T$, $z(0) = 0$, $\tau = 10^{-6}$. The parameters $\vec{\lambda} = \{1.1, 2, 3, 8.5, 12, 900\}$, $L = 6 \cdot 10^{15}$ are found. Actually L is not large, since (10), (11) involve $L^{1/12}, ..., L^{1/7}$.

In the absence of noises the accuracy is described by the component-wise inequality $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (3.5 \cdot 10^{-6}, 1.5 \cdot 10^{-5}, 8 \cdot 10^{-4})$ for $t \geq 30$ (Fig. 1a). Introduce the sampling noise $\eta(t) = 100\cos(10000t) - 200\cos(50000t) + 100\sin(70000t)$. In spite of it the system converges into the region $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (0.4, 5, 70)$ (Fig. 1b).

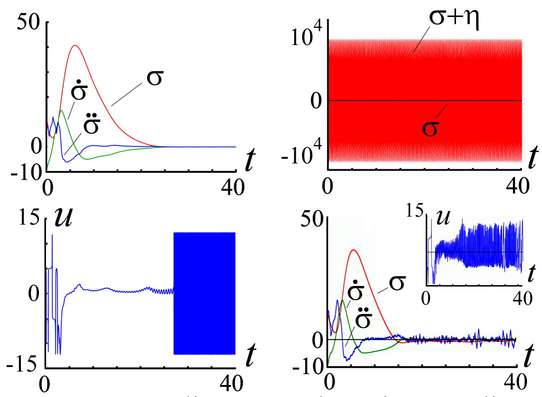


a. Exact sampling b. Noisy sampling

Fig. 2. The case $q = -0.1$. a: FT stability is got for the exact sampling. b: The system remains practically exact in spite of the very large sampling noise η

2. Let $q = -0.1$, and $n_d = 2, n_f = 5$, then $d = 7$, $\vec{\sigma}_2(0) = (10, -10, 10)^T$, $z(0) = 0$, $\tau = 10^{-6}$. The parameters $\vec{\lambda} = \{1.1, 1.4, 2.4, 5, 6, 12, 25, 35\}$, $L = 10^{12}$ are found. Also here actually L is not large.

The system stabilizes at $t = 10$. In the absence of noises the accuracy is described by the component-wise inequality $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (10^{-34}, 10^{-30}, 10^{-26})$ for $t \geq 10$ (Fig. 2a). In the presence of the same noise the system converges into the region $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (0.034, 0.87, 4)$ (Fig. 2b).



a: Exact sampling b: Noisy sampling

Fig. 3. The SMC case $q = -1/3$, hybrid filtering differentiator is applied, $z(0) = (-100, 100, -100)$. a: FT stability and very fast convergence are got for the exact sampling. b: The system remains accurate in spite of the very large sampling noise η.

3. Let $q = -1/3$ which corresponds to SMC. Apply the filtering hybrid differentiator with $n_d = 2$, $n_f = 5$, $M = 0.2$, $d = 0$, for $L = 35[\max\{2 - 0.1t, 1\} + 0.01\cos(1111t)]$, $z(0) = (-100, 100, -100)^T$. Let $\vec{\sigma}_2(0) = (10, -10, 10)^T$, $\tau = 10^{-5}$. Note that L is variable and contains a noise.

The differentiator converges at $t = 3$ in spite of large initial error $z(0)$. The system stabilizes at about $t = 27$. In the absence of noises the accuracy $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (4.0 \cdot 10^{-8}, 2.2 \cdot 10^{-5}, 2.7 \cdot 10^{-2})$ is kept for $t \geq 30$ (Fig. 3a).

Introduce the **unbounded** noise $\eta(t) = 10^4 \cos(10^7 t) + 0.1 \sin(2 \cdot 10^6 t) \lfloor \cos(2 \cdot 10^6 t) \rceil^{-0.5} + \eta_G(t)$, where $\eta_G(t) \in N(0, 0.2^2)$ is a Gaussian noise. In spite of it the system converges into the region $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (0.89, 0.4, 2.9)$ (Fig. 3b). Note the very high noise frequencies $10^7, 2 \cdot 10^6$ in comparison to the sampling step $\tau = 10^{-5}$.

6. CONCLUSION

The robust homogeneous output-feedback stabilization of disturbed integrator chains has been obtained for any homogeneous degree. A corresponding separation principle is formulated.

The proposed new homogeneous observers are exact, feature strong filtering properties, and ensure system robustness for any homogeneity degree in the presence of large and even unbounded, small-in-average noises.

A new type of SM-based differentiator is proposed, called hybrid filtering differentiator, which combines exactness with the filtering capabilities and global fast convergence for variable Lipschitz parameter L.

The proposed extremely robust output-feedback stabilizers of positive homogeneity degree can probably be useful in controlling systems capable of fixed-time escape, in particular, nuclear and thermonuclear reactors.

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