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Input-to-State Stability of Networked and Quantized Control Systems ⋆

Wei Ren *, **Junlin Xiong** **

* *Division of Decision and Control Systems, EECS, KTH Royal Institute of Technology, SE-10044, Stockholm, Sweden. (emails: weire@kth.se).*
** *Department of Automation, University of Science and Technology of China, Hefei, 230026, China. (emails: junlin.xiong@gmail.com).*

Abstract: This paper studies input-to-state stability of networked and quantized control systems with external disturbances. Since the external disturbance is considered, the state may escape from the quantization regions, which thus leads to the introduction of the quantization mechanism with zooming-out and zooming-in stages. We first propose a novel hybrid system to incorporate both the open-loop and closed-loop systems. Based on the developed hybrid model, we establish the boundedness of the state in the zooming-out stage, and the convergence of the state in the zooming-in stage. Furthermore, sufficient conditions are established for input-to-state stability in the case of switching between the zooming-in and zooming-out stages. Finally, a numerical example is presented to illustrate the obtained results.

Keywords: Hybrid systems, networked control systems, quantized control, system modelling, input-to-state stability, average dwell-time.

1. INTRODUCTION

With the increasing control applications emerging in many fields like vehicle industry and teleoperation, control of systems via communication channels has been given great attentions in the last few decades; see Zhang et al. (2013); Lunze (2014). In these control applications, system components, such as sensors, actuators, controllers and quantizers, are connected through a communication network. The presence of the network provides many benefits in terms of installation, maintenance, flexible architecture, high reliability and wireless. However, because of the limited capacity of the network, some unavoidable issues are induced, such as time-varying transmission intervals, packet dropouts, quantization effects and communication constraints. See survey papers Zhang et al. (2013); Baillieul and Antsaklis (2007) for a general introduction.

In the literature, there exist two modelling strategies to deal with control systems with the band-limited network (Nešić and Liberzon (2009)). The first one is to transform the sampled information into the digital information by the coding/quantization techniques to match the network constraints. The transmitted digital information is decoded to be the available information for controllers. Control systems based on this modelling strategy are called quantized control systems (QCSs); see Brockett and Liberzon (2000); Liberzon (2003a). The second one is to choose part of the sampled information with finite words to be transmitted via the network, and thus the network constraints are avoided. In this direction, time-scheduling protocols are needed to decide which part to be transmitted. Control systems based on this strategy are called networked control systems (NCSs); see Walsh et al. (2002, 2001). Observing from these two modelling strategies, we can find their similarities in many aspects (Nešić and Liberzon (2009)). For instance, both QCSs and NCSs can be modelled as hybrid systems (Heemels et al. (2010); Liberzon (2003a)) or discrete-time systems (Donkers et al. (2011); Liberzon and Nešić (2007)); the quantizer and network can be seen as information processing devices and only provide part of information (Liberzon (2003b); Nešić and Teel (2004)). Based on these similarities, both QCSs and NCSs can be combined and such control systems are called networked and quantized control systems (NQCSs); see van Loon et al. (2013); Ren and Xiong (2018b); Tian et al. (2007); Nešić and Liberzon (2009). However, the differences between QCSs and NCSs lead to additional assumptions. For instance, the encoding part is open-loop in QCSs (Liberzon (2003a); Liberzon and Nešić (2007)), whereas the system is always closed-loop in NCSs (Nešić and Liberzon (2009); Heemels et al. (2009)). Hence, the absence of the external disturbance (Ren and Xiong (2018b); van Loon et al. (2013)) or the assumption to constrain the system in the quantization regions (Heemels et al. (2009); Nešić and Liberzon (2009)) are required, which lead to conservatism of the obtained results. In this paper, we follows this direction and study this topic further.

In this paper, we focus on input-to-state stability (ISS) of networked and quantized control systems with external disturbance. Since the external disturbance is addressed here, which may result in escaping of the state from the quantization regions, it is necessary to address the quantization mechanism with two stages: zooming-out and zooming-in stages, which thus leads to reconsideration of the system model. As a result, our first contribution is to propose a novel hybrid model for NQCSs with external disturbance based on the formalism of Goebel et al. (2012). Since the general assumption, *which requires the system state to be always in the quantization regions* (Nešić and Liberzon (2009)), is not needed here, the main challenge is to model the data transmissions on the one hand and the switches between zooming-out and the zooming-

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in stages on the other hand. To this end, we describe in detail the quantization mechanism with two stages, and then introduce two clock variables to represent respectively the time elapsed since the last data transmission and the time elapsed since the last stage switch. Both the open-loop and closed-loop cases are included, and thus a high fidelity model is developed to be amenable to controller design and stability analysis. With the developed hybrid model, our second contribution is to derive sufficient conditions to guarantee ISS of the whole system. To this end, we derive the boundedness of the state in the zooming-out stage, the convergence of the state in zooming-in stage, and further study ISS of the whole system in terms of average dwell-time. Finally, a numerical example is presented to illustrate the obtained results.

2. PRELIMINARIES

Basic definitions and notations are introduced in this section. $\mathbb{R} := (-\infty, +\infty)$; $\mathbb{R}_t^+ := [t, +\infty)$ with $t \in \mathbb{R}$; $\mathbb{R}^+ := (0, +\infty)$; $\mathbb{N} := \{0, 1, 2, \ldots\}$; $\mathbb{N}^+ := \{1, 2, \ldots\}$. $|\cdot|$ stands for the Euclidean norm. Given two vectors $x, y \in \mathbb{R}^n$, $(x, y) := (x^\top, y^\top)^\top$ for simplicity of notation, and $\langle x, y\rangle$ denotes the usual inner product. I represents the identity matrix with the appropriate dimension; diag$\{\cdot\}$ to denote the block diagonal matrix. The symbols $\wedge$ and $\vee$ denote separately 'and' and 'or' in logic. Id denotes the identity function. Given a piecewise continuous function $f : \mathbb{R}^+ \to \mathbb{R}$, $f(t^+) := \lim_{s \searrow t} f(s)$; $\|f\|_{[a,b]} :=$ ess. $\sup_{t\in[a,b]} |f(t)|$ with given interval $[a, b] \subseteq \mathbb{R}$; and $\|f\| := \|f\|_{[a,\infty)}$ if $b \to \infty$. A function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathscr{K}$ if it is continuous, $\alpha(0) = 0$, and strictly increasing; it is of class $\mathscr{K}_\infty$ if it is of class $\mathscr{K}$ and unbounded. A function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathscr{K}\mathscr{L}$ if $\beta(s, t)$ is of class $\mathscr{K}$ for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to zero as $t \to 0$ for each fixed $s \geq 0$. A function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathscr{K}\mathscr{L}\mathscr{L}$ if $\beta(r, s, t)$ is of class $\mathscr{K}\mathscr{L}$ for each fixed $s \geq 0$ and of class $\mathscr{K}\mathscr{L}$ for each fixed $t \geq 0$.

Consider the nonlinear hybrid system (Cai and Teel (2009)):

$$\begin{cases} \dot{x} = F(x, w), & (x, w) \in C, \\ x^+ = G(x, w), & (x, w) \in D, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $w \in \mathbb{R}^m$ is the external input, $y \in \mathbb{R}^p$ is the system output, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the flow map, $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the jump map, $C \subset \mathbb{R}^n \times \mathbb{R}^m$ is the flow set, and $D \subset \mathbb{R}^n \times \mathbb{R}^m$ is the jump set. For the hybrid system (1), the following basic assumptions are given in Goebel et al. (2012): the sets $C, D \subset \mathbb{R}^n \times \mathbb{R}^m$ are closed; F is continuous on C; G is continuous on D.

Definition 1. (Cai and Teel (2009)). The system (1) is *input-to-state stable (ISS)*, if there exist $\beta \in \mathscr{K}\mathscr{L}\mathscr{L}$ and $\gamma \in \mathscr{K}$ such that $|x(t, j)| \leq \beta(|x(0, 0)|, t, j) + \gamma(\|w\|)$ for all $(t, j) \in \operatorname{dom} x$ and all the solution to (1).

3. SYSTEM MODEL AND PROBLEM FORMULATION

Consider the plant of the following form

$$\dot{x}_\mathrm{p} = A_\mathrm{p} x_\mathrm{p} + B_\mathrm{p} u + E_\mathrm{p} w, \quad y = C_\mathrm{p} x_\mathrm{p}, \tag{2}$$

where $x_\mathrm{p} \in \mathbb{R}^{n_\mathrm{p}}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control output, $w \in \mathbb{R}^{n_w}$ is the unknown disturbance, and $y \in \mathbb{R}^{n_y}$ is the plant output. The controller designed for the plant (3) is given by

$$\dot{x}_\mathrm{c} = A_\mathrm{c} x_\mathrm{c} + B_\mathrm{c} y + E_\mathrm{c} w, \quad u = C_\mathrm{c} x_\mathrm{c} + D_\mathrm{c} y, \tag{3}$$

where $x_\mathrm{c} \in \mathbb{R}^{n_\mathrm{c}}$ is the controller state. In (3)-(4), $w \in \mathbb{R}^{n_w}$ is assumed to be Lebesgue measurable and locally bounded. Our objective is to implement the controller (3) over the quantizer and network, and to show that under reasonable assumptions, the assumed stability property of the system (2)-(3) will be preserved for NQCSs.

At the transmission times, denoted by $t_i \in \mathbb{R}_0^+$, $i \in \mathbb{N}$, the outputs of the plant (2) and the controller (3) are sampled, quantized and transmitted via the network. According to the band-limited network and the spatial locations of the sensors and actuators, the sensors and actuators are partitioned into $\ell \in \mathbb{N}^+$ nodes connecting the network. Correspondingly, the transmitted information is also divided into ℓ parts. At $t_i \in \mathbb{R}_0^+$, one and only one node is allowed to access to the network to transmit its data packet, and which node is granted is determined by the time-scheduling protocols; see Donkers et al. (2011); Nešić and Liberzon (2009); Nešić and Teel (2004). The transmission time sequence $\{t_i \in \mathbb{R}_0^+ : i \in \mathbb{N}\}$ is strictly increasing, and thus the transmission intervals are defined as $h_i := t_{i+1} - t_i$, $i \in \mathbb{N}$. The following assumption is made for the transmission intervals; see also Nešić and Liberzon (2009); Nešić and Teel (2004).

Assumption 1. There exist $h_\mathrm{mati} > \varepsilon > 0$ such that $\varepsilon \leq h_i \leq h_\mathrm{mati}$ holds for all $i \in \mathbb{N}$.

3.1 Quantization

After sampling via the sensors, the sampled measurements $y(t_i)$ and $u(t_i)$ are quantized to match the bandwidth limitation of the network. A quantizer is a piecewise constant function $q : \mathbb{R}^n \to \mathscr{Q}$, where $\mathscr{Q}$ is a finite subset of $\mathbb{R}^n$; see Liberzon (2003a). That is, the quantizer q divides $\mathbb{R}^n$ into a finite number of the quantization regions of the form $\{z \in \mathbb{R}^n : q(z) = \wp \in \mathscr{Q}\}$. To well define the quantizer, the following assumption is made; see also Ren and Xiong (2018b,a); Liberzon (2003a); Liberzon and Nešić (2007).

Assumption 2. Each node has a quantizer q_j with $j \in \mathfrak{L} := \{1, \ldots, \ell\}$. For each $j \in \mathfrak{L}$, there exist $M_j > \Delta_j > 0$ and $\Delta_{0j} > 0$ such that for all $z_j \in \mathbb{R}^{n_j}$,

$$|z_j| \leq M_j \Rightarrow |q_j(z_j) - z_j| \leq \Delta_j, \tag{4}$$

$$|z_j| > M_j \Rightarrow |q_j(z_j)| > M_j - \Delta_j, \tag{5}$$

$$|z_j| \leq \Delta_{0j} \Rightarrow q_j(z_j) \equiv 0. \tag{6}$$

The dynamic quantizer applied in this paper is given by

$$q(\mu, z) := \mu q(z/\mu), \quad \mu > 0, \tag{7}$$

where $\mu \in \mathbb{R}^+$ is called the quantization parameter. All the quantization parameters are combined as $\mu := (\mu_1, \ldots, \mu_\ell) \in \mathbb{R}^\ell$, and the overall quantizer is given by

$$q(\mu, z) := (q_1(\mu_1, z_1), \ldots, q_\ell(\mu_\ell, z_\ell)).$$

The quantization parameter $\mu \in \mathbb{R}^\ell$ evolves according to a hybrid dynamics, which will be discussed later. The quantized measurements are defined as $\bar{y} := q(\mu, y) \in \mathbb{R}^{n_y}$ and $\bar{u} := q(\mu, u) \in \mathbb{R}^{n_u}$; the quantization errors are defined as $\varepsilon_y := \bar{y} - y \in \mathbb{R}^{n_y}$ and $\varepsilon_u := \bar{u} - u \in \mathbb{R}^{n_u}$.

Since the external disturbance are studied and the dynamic quantizer (7) is applied, the quantization mechanism is introduced with two stages: zooming-out and zooming-in stages. Since the initial state is unknown *a priori* and the state may escape from the quantization regions due to the effects of the external disturbance, the system is in the zooming-out stage, and μ increases such that the quantization regions are expanded to cover the state. Meanwhile, the quantization error is getting large with the increase of μ. Once the state is captured by the quantization regions, the system switches into the zooming-in stage, and μ decreases such that the quantization regions are contracted to drive the state to be bounded and even convergent. Because of these two stages in the quantization mechanism, the applied control strategy also has two stages: the controller is not activated in the zooming-out stage until the state is captured in the quantization regions, whereas activated in the zooming-in stage such that the control input is transmitted via the network.

In the following, we introduce a logical variable $c \in \{0,1\}$ to distinguish the zooming-out and zooming-in stages, and a timer $\tau_2 \in \mathbb{R}^+$ to keep track of these two stages. In particular, $c = 0$ means that the system is in the zooming-out stage, whereas $c = 1$ means that the system is in the zooming-in stage. The dynamics of the variable (τ_2, c) is given by

$$\begin{aligned} (\dot{\tau}_2, \dot{c}) &= (1,0), \quad (\tau_2,c) \in C^{\text{stg}}, \\ (\tau_2^+, c^+) &= (0, 1-c), \quad (\tau_2,c) \in D^{\text{stg}}, \end{aligned} \tag{8}$$

with

$$\begin{aligned} C^{\text{stg}} &:= ([0,T_1(x)] \times \{0\}) \cup ([0,T_2(x)] \times \{1\}), \\ D^{\text{stg}} &:= (\{T_1(x)\} \times \{0\}) \cup (\{T_2(x)\} \times \{1\}), \end{aligned} \tag{9}$$

where $x := (x_\text{p}, x_\text{c}) \in \mathbb{R}^{n_x}$ is the augmented state with $n_x = n_\text{p} + n_\text{c}$. In (9), $T_1(x) := \inf\{t \in \mathbb{R}^+ : (\forall j \in \mathfrak{L}, (|y_j(t)| \wedge |u_j(t)|) \leq M_j\mu_j(t)) \wedge (\exists j \in \mathfrak{L}, (|y_j(0)| \vee |u_j(0)|) > M_j\mu_j(0))\}$ is the time instant from which the quantizer switches into the zooming-out stage from the zooming-in stage, and $T_2(x) := \inf\{t \in \mathbb{R}^+ : (\exists j \in \mathfrak{L}, (|y_j(t)| \vee |u_j(t)|) \leq M_j\mu_j(t)) \wedge (\forall j \in \mathfrak{L}, (|y_j(0)| \vee |u_j(0)|) \leq M_j\mu_j(0))\}$ is the time instant from which the quantizer switches into the zooming-in stage from the zooming-out stage. Both $T_1(x)$ and $T_2(x)$ are time-varying and state-dependent. Hence, the dynamics (8) with (9) is state-dependent and event-triggered, and the switching between the zooming-out and zooming-in stages is event-triggered. The event-triggered conditions are presented in $T_1(x)$ and $T_2(x)$ to verify whether the outputs of the plant and controller are in the quantization regions. Because of the strictly increasing transmission time sequence and Assumption 1, the Zeno phenomena are ruled out, which implies the positiveness of $T_1(x)$ and $T_2(x)$.

3.2 Zooming-out Stage

In the zooming-out stage, the system is open-loop, the quantization parameter increases and the controller is not activated. The following assumption is made first for the increase of the quantization parameter.

Assumption 3. In the zooming-out stage, the quantization parameter μ increases periodically, and the period is $\delta \in \mathbb{R}^+$ satisfying $\delta \in (0, h_\text{mati}]$.

Since the system is open-loop in the zooming-out stage, the transmission times do not affect the increase of the quantization parameter μ, which thus implies that Assumption 3 is reasonable. It has to be noted that the increase of μ is also periodic in the previous works; see Liberzon (2014, 2003a); Liberzon and Hespanha (2005). The period $\delta \in (0, h_\text{mati}]$ can be set arbitrarily, and the smaller δ is, the larger the increase frequency of μ is. We can set δ as the inter-sampling period as in Liberzon and Hespanha (2005) for the sake of convenience.

Next, we introduce a timer $\tau_1 \in \mathbb{R}$ to keep track of the transmission interval (which refers to the update interval of μ in the zooming-out stage). Hence, the open-loop system in the zooming-out stage is denoted by $\mathscr{S}_1$ and modelled below.

$$\left.\begin{aligned} \dot{x} &= \mathscr{A}_1 x + \mathscr{E}_1 w \\ \dot{\mu} &= 0 \\ \dot{\tau}_1 &= 0 \end{aligned}\right\} \quad (\tau_1, \tau_2, c) \in C_1, \tag{10a}$$

$$\left.\begin{aligned} x^+ &= x \\ \mu^+ &= \Omega_\text{out}\mu \\ \tau_1^+ &= 0 \end{aligned}\right\} \quad (\tau_1, \tau_2, c) \in D_1^\text{jump}, \tag{10b}$$

with

$$\begin{aligned} \mathscr{A}_1 &:= \begin{bmatrix} A_\text{p} & 0 \\ 0 & A_\text{c} \end{bmatrix}, \quad \mathscr{E}_1 := \begin{bmatrix} E_\text{p} \\ E_\text{c} \end{bmatrix}, \\ C_1 &:= [0,\delta] \times [0, T_1(x)] \times \{0\}, \\ D_1^\text{jump} &:= \{\delta\} \times [0, T_1(x)] \times \{0\}, \end{aligned} \tag{11}$$

where $\Omega_\text{out} = \text{diag}\{\Omega_{\text{out},1}, \ldots, \Omega_{\text{out},\ell}\} \in \mathbb{R}^\ell$ with the constants $\Omega_{\text{out},j} > 1$ given for all $j \in \mathfrak{L}$.

The dynamics in (10a)-(10b) implies that the timer τ_1 is reset to 0 when it reaches δ during the zooming-out stage, i.e., when $c = 0$ and $\tau_2 \in [0, T_1(x)]$. It is obvious that the switching times for the solutions of the hybrid system (10) are consistent with Assumption 3. In Subsection 5.1, we will establish the existence of the time instants switching from the zooming-out stage to the zooming-in stage, which shows the boundedness of $T_1(x)$. After the zooming-out stage, the state is captured by the quantization regions and thus the system switches into the zooming-in stage.

3.3 Zooming-in Stage

In the zooming-in stage, the controller is activated and the system is closed-loop such that the quantized measurements $\bar{y}$ and $\bar{u}$ are transmitted via the network. Here, the quantization measurements are assumed to be time-invariant in the transmission intervals and updated at the transmission times. The received measurements, denoted by $\hat{y}$ and $\hat{u}$, are assumed to be time-invariant in the transmission intervals $[t_i, t_{i+1}]$, $i \in \mathbb{N}$; see for instance van Loon et al. (2013); Heemels et al. (2010). At the transmission times t_i, $i \in \mathbb{N}$, the updates of $\hat{y}$ and $\hat{u}$ are given as follows.

$$\begin{aligned} \hat{y}(t_i^+) &= \bar{y}(t_i) + h_y(i, e_y(t_i), \varepsilon_y(t_i)), \\ \hat{u}(t_i^+) &= \bar{u}(t_i) + h_u(i, e_u(t_i), \varepsilon_u(t_i)), \end{aligned} \tag{12}$$

where h_y and h_u are the update functions, and depend on the time-scheduling protocols. As a result, the update of the network-induced errors are given by

$$\begin{aligned} e_y(t_i^+) &= \hat{y}(t_i^+) - y(t_i^+) = \varepsilon_y(t_i) + h_y(i, e_y(t_i), \varepsilon_y(t_i)) \\ &=: \mathbf{h}_y(i, x_\text{p}(t_i), e_y(t_i), \mu(t_i)), \end{aligned} \tag{13}$$

where the second "=" holds due to (12). Similarly,

$$e_u(t_i^+) =: \mathbf{h}_u(i, x_\text{c}(t_i), e_u(t_i), \mu(t_i)). \tag{14}$$

Since the functions $\mathbf{h}_y$ and $\mathbf{h}_u$ model the time-scheduling mechanism, which determines the transmissions between the controller and the plant. Following the terminology of Nešić and Teel (2004), the following is referred to as the transmission protocol

$$e^+ = \mathbf{h}(i, x, e, \mu), \tag{15}$$

where $e := (e_y, e_u) \in \mathbb{R}^{n_e}$ with $n_e := n_y + n_u$, and $\mathbf{h}(i,x,e,\mu) := (\mathbf{h}_y(i, x_\text{p}, e_y, \mu), \mathbf{h}_u(i, x_\text{c}, e_u, \mu))$. According to the number of the nodes, e is partitioned as $e = (e_1, \ldots, e_\ell)$. The transmission protocol (15) is to determine the node to transmit its data packet at the transmission times. That is, at each transmission instant t_i, if the node $j \in \mathfrak{L}$ gets access to the network, then the corresponding component of e_j is updated whereas other components of e is kept.

The dynamics of μ in the zooming-in stage is given by

$$\begin{aligned} \dot{\mu} &= 0, \quad t \in [t_i, t_{i+1}], \\ \mu(t_i^+) &= \Psi\Omega_{\text{in}}\mu(t_i) + (I-\Psi)\mu(t_i), \end{aligned} \tag{16}$$

where $\Omega_{\text{in}} = \text{diag}\{\Omega_{\text{in},1}, \ldots, \Omega_{\text{in},\ell}\} \in \mathbb{R}^\ell$ with the constants $\Omega_{\text{in},j} \in (0,1)$ given for all $j \in \mathfrak{L}$. Note that the dynamic of μ in the zooming-in stage is similar to those in Heemels et al. (2009); Nešić and Liberzon (2009).

Combining all the variables and their relationships in (2)-(3), (12)-(16), the closed-loop system in the zooming-in stage is denoted by $\mathscr{S}_2$ and modelled as follows.

$$\left.\begin{aligned} \dot{x} &= \mathscr{A}_2 x + \mathscr{B}_2 e + \mathscr{E}_2 w \\ \dot{e} &= \mathscr{A}_3 x + \mathscr{B}_3 e + \mathscr{E}_3 w \\ \dot{\mu} &= 0, \quad \dot{\tau}_1 = 0 \end{aligned}\right\} \quad (\tau_1, \tau_2, c) \in C_2, \tag{17a}$$

$$\left.\begin{aligned} x^+ &= x \\ e^+ &= \mathbf{h}(i, x, e, \mu) \\ \mu^+ &= \Psi\Omega_{\text{in}}\mu + (I-\Psi)\mu \\ \tau_1^+ &= 0 \end{aligned}\right\} \quad (\tau_1, \tau_2, c) \in D_2^{\text{jump}}, \tag{17b}$$

with

$$\begin{aligned} C_2 &:= [0, h_{\text{mati}}] \times [0, T_2(x)] \times \{1\}, \\ D_2^{\text{jump}} &:= [\varepsilon, h_{\text{mati}}] \times [0, T_2(x)] \times \{1\}, \end{aligned} \tag{18}$$

where

$$\mathscr{A}_2 := \begin{bmatrix} A_\text{p} + B_\text{p} D_\text{c} C_\text{p} & B_\text{p} C_\text{c} \\ B_\text{c} C_\text{p} & A_\text{c} \end{bmatrix}, \quad \mathscr{B}_2 := \begin{bmatrix} B_\text{p} D_\text{c} & B_\text{p} \\ B_\text{c} & 0 \end{bmatrix},$$
$$\mathscr{E}_2 := \mathscr{E}_1, \ \mathscr{A}_3 := -\mathscr{C}\mathscr{A}_2, \quad \mathscr{B}_3 := -\mathscr{C}\mathscr{B}_2, \ \mathscr{E}_3 := -\mathscr{C}\mathscr{E}_2.$$

Different from the periodic increase of the quantization parameter in the zooming-out stage in Subsection 3.2, the transmissions do not occur periodically *a priori* in the zooming-in stage. Hence, both the flow and jump are allowed to occur when $\tau_1 \in [\varepsilon, h_{\text{mati}}]$. If there is a solution to (17a)-(17b) such that $\tau_1 \in [0, \varepsilon)$ and $\tau_2 = T_2(x)$, then it stops to exist, which is not an issue since this case does not occur in practice and the stability in Section 5 still holds.

4. HYBRID MODEL

We combine the system models in both the zooming-out and zooming-in stages, and construct a unified hybrid model as the formalism in Goebel et al. (2012). To begin with, we introduce a counter $\kappa_1 \in \mathbb{N}$ to keep track of the number of the transmissions needed to implement some time-scheduling protocols like the RR protocol, a counter $\kappa_2 \in \mathbb{N}$ to keep track of the number of the switching between the zooming-in and the zooming-out stage, and a timer $\tau \in \mathbb{R}$ to keep track of the length of all the zooming-out stages. Denote $\mathfrak{X} := (x, e, \mu, \kappa_1, \kappa_2, \tau, \tau_1, \tau_2, c) \in \mathscr{R} := \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^\ell \times \mathbb{N}^2 \times \mathbb{R}^3 \times \{0,1\}$, and the NQCS is modelled as

$$\mathscr{H} : \begin{cases} \dot{\mathfrak{X}} = F(\mathfrak{X}, w), & (\mathfrak{X}, w) \in C \times \mathbb{R}^{n_w}, \\ \mathfrak{X}^+ = G(\mathfrak{X}), & (\mathfrak{X}, w) \in D \times \mathbb{R}^{n_w}, \end{cases} \tag{19}$$

where

$$\begin{aligned} C &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^\ell \times \mathbb{N}^2 \times \mathbb{R} \times (C_1 \cup C_2), \\ D &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^\ell \times \mathbb{N}^2 \times \mathbb{R} \times (D_1 \cup D_2), \end{aligned} \tag{20}$$

with C_1 and C_2 defined respectively from (11) and (18), $D_1 := D_1^{\text{in}} \cup D_1^{\text{jump}}$, $D_2 := D_2^{\text{out}} \cup D_2^{\text{jump}}$ with D_1^{jump} and D_2^{jump} defined in (11) and (18) respectively, and

$$\begin{aligned} D_1^{\text{out}} &:= \{0\} \times \{T_1(x)\} \times \{0\}, \\ D_2^{\text{in}} &:= \{0\} \times \{T_2(x)\} \times \{1\}. \end{aligned} \tag{21}$$

The mapping F in (19) is defined as

$$F(\mathfrak{X}, w) := \begin{cases} F_1(\mathfrak{X}, w), & (\tau_1, \tau_2, c) \in C_1, \\ F_2(\mathfrak{X}, w), & (\tau_1, \tau_2, c) \in C_2, \end{cases} \tag{22}$$

where $F_1(\mathfrak{X}, w) := (\mathscr{A}_1 x + \mathscr{E}_1 w, -\mathscr{C}\mathscr{A}_1 x - \mathscr{C}\mathscr{E}_1 w, 0, 0, 0, 1, 1, 1, 0)$ corresponds to the flow (10a) for the zooming-out stage, and $F_2(\mathfrak{X}, w) := (\mathscr{A}_2 x + \mathscr{B}_2 e + \mathscr{E}_2 w, \mathscr{A}_3 x + \mathscr{B}_3 e + \mathscr{E}_3 w, 0, 0, 0, 0, 1, 1, 0)$ corresponds to the flow (17a) for the zooming-in stage. The mapping G in (19) is defined as

$$G(\mathfrak{X}) := \begin{cases} G_1(\mathfrak{X}), & (\tau_1, \tau_2, c) \in D_1^{\text{jump}}, \\ G_2(\mathfrak{X}), & (\tau_1, \tau_2, c) \in D_2^{\text{jump}}, \\ G_3(\mathfrak{X}), & (\tau_1, \tau_2, c) \in D_1^{\text{in}} \cup D_2^{\text{out}}, \end{cases} \tag{23}$$

where $G_1(\mathfrak{X}) := (x, e, \Omega_{\text{out}}\mu, \kappa_1, \kappa_2, \tau, 0, \tau_2, c)$ corresponds to a transmission jump (10b) in the zooming-out stage, $G_2(\mathfrak{X}) := (x, \mathbf{h}_e(\kappa_1, x, e, \mu), \mathbf{h}_\mu(\kappa_1, \mu), \kappa_1 + 1, \kappa_2, \tau, 0, \tau_2, c)$ corresponds to the transmission jump (17b) in the zooming-in stage, and $G_3(\mathfrak{X}) := (x, e, c\Omega_{\text{in}}^{-1}\mu + (1-c)\mu, \kappa_1, \kappa_2 + 1, \tau, 0, 0, 1-c)$ corresponds to the jump caused by the switching between the zooming-out and zooming-in stages.

5. MAIN RESULTS

In this section, we first study the evolution of the system state in the zooming-out stage, and then the convergence of the system state in the zooming-in stage.

5.1 Boundedness in Zooming-out Stage

The boundedness of the system state in the zooming-out stage is investigated in this subsection.

Theorem 1. Consider the system $\mathscr{H}$ in the zooming-out stage, and let $\delta \in (0, \ln \Omega_{\text{out}}^{\min} / \|\mathscr{A}_1\|)$. There exists a finite hybrid time instant $(t_1, j_1) \succeq (t_0, j_0)$ such that for all $(t_0, j_0) \preceq (t, j) \preceq (t_1, j_1)$,

$$|x(t,j)| \leq \beta_x(|x(t_0, j_0)|, t - t_0, j - j_0) + \gamma_x(\|w\|), \tag{24}$$
$$|\mu(t,j)| \leq \beta_\mu(|x(t_0, j_0)|, t - t_0, j - j_0) + \gamma_\mu(\|w\|), \tag{25}$$

where $\beta_x, \beta_\mu \in \mathscr{KLL}$ and $\gamma_x, \gamma_\mu \in \mathscr{K}$. In addition, $x(t_1, j_1)$ is in the quantization regions.

Theorem 1 ensures that the state and the quantization parameter are bounded in each zooming-out stage, and the existence of (t_1, j_1) ensures the finite length of each zooming-out stage, which further implies that $T_1(x)$ is bounded. In addition, the state is in the quantization regions at the end of the zooming-out stages. As a result, there always exists a hybrid time instant such that the system switches from the zooming-out stage into the zooming-in stage.

In quantized control systems (Ren and Xiong (2018a); Liberzon and Nešić (2007)), the existence of the time instant (t_1, j_1) implies that the state will be in the quantization regions from such a time instant despite the effects of the external disturbance. If this is also the case for the NQCS studied in this paper, then we obtain that $T_1(x)$ is finite, $c(t) \equiv 1$ for all $t \in [T_1(x), \infty)$, and the input-to-state stability of the hybrid system $\mathscr{H}$ depends on the convergence of the state in the zooming-in stage, which will be studied in Subsection 5.2. However, due to the effects of the external disturbance and the network-induced errors, the state may escape from the quantization regions such that the system $\mathscr{H}$ switches back to the zooming-out stage.

5.2 Convergence in Zooming-in Stage

We next study the input-to-state stability of the system $\mathscr{H}$ in the zooming-in stage. In order to ensure ISS, some assumptions are provided for the x and (e,μ) subsystems.

Assumption 4. There exist a function $W : \mathbb{N}\times\mathbb{R}^{n_e}\times\mathbb{R}^{\ell}\to\mathbb{R}_0^+$ which is locally Lipschitz in (e,μ) for all $\kappa\in\mathbb{N}$, $\alpha_{1W},\alpha_{2W}\in\mathscr{K}_\infty$ and $\lambda\in[0,1)$ such that for all $(\kappa,e,\mu)\in\mathbb{N}\times\mathbb{R}^{n_e}\times\mathbb{R}^{\ell}$,

$$\alpha_{1W}(|(e,\mu)|)\leq W(\kappa,e,\mu)\leq \alpha_{2W}(|(e,\mu)|),$$
$$W(\kappa+1,\mathbf{h}(\kappa,x,e,\mu),\Omega_{\text{in}}\mu)\leq\lambda W(\kappa,e,\mu);$$

for all $x\in\mathbb{R}^{n_x}$ and almost all $e\in\mathbb{R}^{n_e},\mu\in\mathbb{R}^{\ell}$,

$$\left\langle\frac{\partial W(\kappa,e,\mu)}{\partial e},\mathscr{A}_3x+\mathscr{B}_3e+\mathscr{E}_3w\right\rangle\leq LW(\kappa,e,\mu)+|\tilde{y}|,\quad(26)$$

where $\tilde{y}:\mathbb{R}^{n_x}\times\mathbb{R}^{n_e}\times\mathbb{R}^{n_w}\to\mathbb{R}$ is continuous on (x,e,w).

In Assumption 4, the function W is used to analyze the stability of (e,μ)-subsystem. Assumption 4 is to estimate the jumps of W at the discrete-time instants and the derivative of W in the continuous-time intervals.

Assumption 5. There exist a locally Lipschitz function $V:\mathbb{R}^{n_x}\to\mathbb{R}_0^+$, $\alpha_{1V},\alpha_{2V}\in\mathscr{K}_\infty$, and constants $\rho,\theta,\gamma>0$, such that such that $\alpha_{1V}(|x|)\leq V(x)\leq\alpha_{2V}(|x|)$ holds for all $x\in\mathbb{R}^{n_x}$; and $\langle\nabla V(x),\mathscr{A}_2x+\mathscr{B}_2e+\mathscr{E}_2w\rangle\leq(\gamma^2-\theta)W(\kappa,e,\mu)+\varphi(|w|)-|\tilde{y}|^2-\rho V(x)$ holds for all $(e,\mu)\in\mathbb{R}^{n_e}\times\mathbb{R}^{\ell}$ and almost all $x\in\mathbb{R}^{n_x}$, where $\tilde{y}$ is given in Assumption 4.

The last assumption is on the transmission intervals, and is to bound the MATI in Assumption 1. Consider the following differential equation

$$\dot{\phi}=-2L\phi-\gamma(\phi^2+1),\quad\phi(0)=\lambda^{-1},\quad(27)$$

where $L\geq0$ and $\gamma>0$ are given respectively in Assumptions 4-5. It is noted from Heemels et al. (2010) that the solution to (27) is strictly decreasing as long as $\phi(\tau_1)\geq0$. With the equation (27), the upper bound h_{mati} satisfies $\phi(h_{\text{mati}})=\lambda$; see Jentzen et al. (2010).

Theorem 2. Consider the system $\mathscr{H}$ in the zooming-in stage, and let Assumptions 4-5 hold. If the MATI h_{mati} satisfies $\phi(h_{\text{mati}})=\lambda$, then the state of the system $\mathscr{H}$ is convergent.

6. SWITCHING BETWEEN TWO STAGES

According to Section 5, we can construct the following Lyapunov function for the system $\mathscr{H}$ in (19):

$$\bar{U}(\mathfrak{X}):=(1-c)U_1(\mathfrak{X})+cU_2(\mathfrak{X}),$$

where $U_1(\mathfrak{X})$ is the Lyapunov function for the zooming-out stage and $U_2(\mathfrak{X})$ is the same as the Lyapunov function applied in the zooming-in stage. For the sake of simplicity, we can take the Lyapunov function below

$$\bar{U}(\mathfrak{X}):=V(x)+c\gamma\phi(\tau_1)W^2(\kappa,e,\mu),\quad(28)$$

where $V(x)$ and $W(\kappa,e,\mu)$ are given in the assumptions in Subsection 5.1. Hence, $U_1(\mathfrak{X})=V(x)$ and $U_1(\mathfrak{X})=V(x)+\gamma\phi(\tau_1)W^2(\kappa,e,\mu)$.

For the zooming-out stage, we have from (10) and Theorem 1 that there exist $\mathbf{r}_1\in\mathbb{R}^+$ related to the function f and $\mathbf{r}_2\in\mathscr{K}$ such that

$$\langle\nabla U(\mathfrak{X}),F_1(\mathfrak{X},w)\rangle\leq\mathbf{r}_1U(\mathfrak{X})+\mathbf{r}_2|w|^2,\quad(29)$$

where $\mathbf{r}_1:=\lambda_{\min}^{-1}(\mathscr{P})\lambda_{\max}(\mathscr{A}_1^\top\mathscr{P}+\mathscr{P}\mathscr{A}_1+\mathscr{E}_1^\top\mathscr{E}_1)$ and $\mathbf{r}_2:=\mathscr{P}\mathscr{P}$. For the zooming-in stage, we have from Subsection 5.2 that there exist $\varpi>0$ and $\mathbf{r}_3\in\mathscr{K}$ such that that

$$\langle\nabla U_2(\mathfrak{X}),F_2(\mathfrak{X},w)\rangle\leq-\varpi U_2(\mathfrak{X})+\mathbf{r}_3(|w|).\quad(30)$$

Proposition 3. Consider the system (19), and let Assumptions 1-5 hold. If there exists $\xi>1$ such that

$$\gamma\phi(0)W^2(\kappa,e,\mu)\leq(\xi-1)V(x),\quad(31)$$
$$\tau\leq(\mathbf{r}_1+\varpi)^{-1}\omega t,\quad\omega\in(\varpi/2,\varpi),\quad(32)$$
$$\kappa_2\leq N_0+\frac{2(\omega-\omega_1)t}{\ln\xi},\quad\omega_1\in(2\omega-\varpi,\omega),\quad(33)$$

then the system (19) is ISS.

In Proposition 3, Condition (31) is to ensure that $U_2(\mathfrak{X})\leq\xi U_1(\mathfrak{X})$ at the stage switches, which holds by choosing an appropriate constant $\xi>1$. Condition (32) is to constrain the time length of the zooming-out stage, and Condition (33) is to constrain the number of the switching from the zooming-in stage to the zooming-out stage. In addition, Conditions (32)-(33) can be treated as the average dwell-time (ADT) conditions, and $0.5(\omega-\omega_1)^{-1}\ln\xi$ is the ADT. Conditions (32)-(33) can be satisfied via increasing the quantization parameter as fast as possible to ensure the time length of the zooming-out stage as small as possible. The smaller the constant $\delta>0$ in $\mathscr{H}$ is, the more frequent the increase of μ is, and thus the shorter of the time length of the zooming-out stage is.

7. NUMERICAL EXAMPLE

In this section, we apply our results to stabilize the origin of a batch reactor in Heemels et al. (2010). Consider the plant below

$$\dot{x}_{\text{p}}=A_{\text{p}}x_{\text{p}}+B_{\text{p}}u+C_{\text{p}}w,\quad y=D_{\text{p}}x_{\text{p}},\quad(34)$$

and the designed controller given by

$$\dot{x}_{\text{c}}=A_{\text{c}}x_{\text{c}}+B_{\text{c}}y,\quad u=C_{\text{c}}x_{\text{c}}+D_{\text{c}}y,\quad(35)$$

where $x_{\text{p}}\in\mathbb{R}^4,x_{\text{c}}\in\mathbb{R}^2,y\in\mathbb{R}^2$ and $u\in\mathbb{R}^2$; the matrices in (34)-(35) can be found in Nešić and Teel (2004); Heemels et al. (2010).

As a result, we have that the open-loop model is

$$\dot{x}_{\text{p}}=A_{\text{p}}x_{\text{p}}+C_{\text{p}}w,\quad(36)$$

which means that the system is in the zooming-out stage. Let μ be initialized at E, and $\Omega_{\text{out}}:=\exp(2\delta\|A_{\text{p}}\|)I$, where $\delta>0$ is given in Subsection 3.2. Thus, there exists a time instance such that $|x|\leq M|\mu|$, and such time instance can be set as the initial time for the zooming-in stage.

In the zooming-in stage, the closed-loop model is

$$\begin{aligned}\dot{x}&=A_{11}x+A_{12}e+A_{13}w,\\\dot{e}&=A_{21}x+A_{22}e+A_{23}w,\end{aligned}\quad(37)$$

where the matrices in (37) can be found in Heemels et al. (2010). Here, we choose $\max_j\Delta_j=0.8$ and $\max\Omega_{\text{in},j}=0.6$, $j=\{1,2\}$, which implies that $\sqrt{(\ell-1)/\ell}\geq\max_j\Omega_{\text{in},j}+\varepsilon\max_j\Delta_j/\sqrt{\ell}$ holds for the TOD protocol case and $\sqrt{(\ell-1)/\ell}\geq\max_j\Omega_{\text{in},j}+\varepsilon\sqrt{\ell}\max_j\Delta_j$ holds for the RR protocol case. As a result, Assumption 4-5 hold with $\lambda=\sqrt{2}/2,\rho=M+1,L=15.730,\gamma=21.5275$, and $\tilde{y}=\sqrt{2}A_{21}x$ for the RR protocol case,

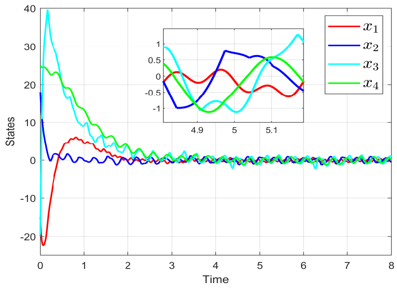

Fig. 1. The state response in the TOD protocol case.

Table 1. Bounds on MATI in Section 5

	MATI	
	RR	TOD
Theorem 1	0.0010	0.0029
Simulations	0.0663	0.0961

and with $\lambda = \sqrt{2}/2, \rho = M+1, L = 15.730, \gamma = 15.9222$, and $\tilde{y} = A_{21}x$ for the TOD protocol case. The MATI bounds are summarized and compared to the bounds estimated via simulations in Table 1.

Given a switching time instant $T_k \in \mathbb{R}^+$, the next switching time from the zooming-out stage to the zooming-in stage is obtained by $T_1(x) = T_{2k+1} = \inf\{t \geq T_{2k} : (|y_l(t)| \wedge |u_l(t)|) \leq M\mu(t), l = 1,2\}$, and the next switching time from the zooming-in stage to the zooming-out stage is obtained by $T_2(x) = T_{2k+2} = \inf\{t \geq T_{2k+1} : (|y_l(t)| \vee |u_l(t)|) > M\mu(t), l = 1,2\}$. Hence, we verify whether the state is in the quantization regions, and then determine the switching time instants and further which stage that the system is in. Under the external disturbance $w(t) = (10\sin(50t), 5\cos(20t))$ and $M = 5$, the state response is shown in Fig. 1, which implies that the system state converges to a region around the origin, which is caused due to the external disturbance, and implies input-to-state stability of the whole system.

8. CONCLUSIONS

In this paper, system modelling and input-to-state stability were studied for networked and quantized control systems with external disturbances. A unified hybrid system framework was developed for networked and quantized control systems. In particular, the quantization mechanism was considered. According to the developed system model, a Lyapunov ISS protocol was proposed and sufficient conditions for input-to-state stability were established. A numerical example was used to illustrate the developed theory.

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