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The ISS property for a feedback connection of an ODE with a parabolic PDE ⋆

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Abstract: We consider a linear non-autonomous ODE with an input signal, which is a solution of a linear non-autonomous parabolic PDE for which the solution of the ODE enters as an input. Moreover there are external disturbances entering through the boundary conditions of the parabolic equation. In this work we derive ISS estimates for this feedback connection under certain linear matrix inequality conditions. The derivation of results is based on the Lyapunov method.

Keywords: Stability, robustness, linear ODE-PDE interconnections, parabolic PDEs, Lyapunov method, input-to-state stability.

1. INTRODUCTION

Systems of interconnected ordinary and partial differential equations appear in mathematical modeling of different physical, biological, chemical and other processes as well as in dynamics of discrete mechanical systems including elastic and fluid objects. For example such a coupling appears in the modeling thermo-elasticity or thermo-plasticity processes with phase transformations, see Dashkovskiy and Narimanyan (2007) and Dachkovski and Böhm (2004), respectively, where the temperature distribution as well as elastic deformations are described by partial differential equations (PDE) and the evolution of solid-solid phase transformations is given in terms of ordinary differential equations (ODE). Stability properties of linear and nonlinear coupled ordinary and partial differential equations of the parabolic type were studied in many papers, see for example Bartyšev (1980); Kedyk (1991); Martynyuk and Slyn'ko (2008); Kang and Fridman (2017); Ahmed-Ali et al. (2017); Karafyllis and Krstic (2019), see also Mironchenko and Prieur (2020). To investigate different dynamical properties of a coupled system of a parabolic PDE and ODE Bartyšev (1980) applies the method of vector Lyapunov function, see Lakshmikantham et al. (1991). In the work of Kedyk (1991) stability problems of the monotone dynamic extensions of coupled ODE with PDE of the parabolic type were investigated. Also here the method of vector Lyapunov function is applied and hence in these works the asymptotic stability for each subsystem is assumed. Similar assumptions on the dynamics of subsystems are applied in the derivations of the small-gain type developed in the ISS framework for interconnected systems Karafyllis and Krstic (2019). Martynyuk and Slyn'ko (2008) propose a Lyapnov function for a coupled linear ODE-PDE parabolic system that allows to study stability in case when the ODE is unstable. The results in Martynyuk and Slyn'ko (2008); Karafyllis and Krstic (2019); Bartyšev (1980) were developed for autonomous ODE-PDE couplings. Some special cases of non-autonomous ODE-PDE systems were considered in Kedyk (1991). The derivation of Lyapunov functions in the general non-autonomous case of a coupled ODE-PDE parabolic system is still an open problem. We also note that in most of works devoted to the investigations of the ISS-like properties for distributed parameter systems the case of autonomous systems only is considered. The ISS framework of non-autonomous infinite dimensional systems needs to be developed yet.

In this work we investigate the ISS properties of a linear periodic system of the coupled ODE-PDE parabolic system with input signals which enter at the boundary. Using the ideas from Slyn'ko (2019) we provide a time dependent Lyapunov function and derive conditions guaranteeing the ISS property including explicit estimations for solutions.

Notation. For a continuous function $r \in C([0,l];\mathbb{R})$, $r(z) > 0$ by $L^2_r(0,1)$ we denote the Banach weighted space of Lebesgue measurable and square integrable functions with weight r and norm given by $\|f\|_{L^2_r(0,1)} = (\int_0^1 r(z)|f(s)|^2\,ds)^{1/2}$. For a Banach space $\mathcal{X}$ and the subset $M \subset \mathbb{R}$ by $C(M,\mathcal{X})$ we denote the space of continuous functions $\mathbb{R} \to \mathcal{X}$ and by $C^1(\mathbb{R},\mathcal{X})$ the space of continuously differential functions with values in $\mathcal{X}$. $L^\infty(\mathbb{R}_+)$ denotes the space of essentially bounded Lebesgue measurable functions with the norm $\|f\|_\infty = ess\sup_{t\geq 0}|f(t)|$. In many definitions and estimates we will use the classes of comparison functions $\mathcal{K},\mathcal{L},\mathcal{KL}$ as usually in the ISS framework, see Sontag (1989) and Liu et al. (2017).

$\mathbb{R}^n$ denotes the n-dimensional space with the euclidean norm $\|\cdot\|$, $\mathbb{R}^{n\times m}$ is the linear space of $n\times m$-matrices, which forms a Banach algebra in case $m = n$. Let the norm in $\mathbb{R}^{n\times n}$ be induced by the norm in $\mathbb{R}^n$, that is $\|A\| = \sup_{\|x\|=1}\|Ax\| = \lambda^{1/2}_{\max}(A^{\mathrm{T}}A)$. In case of the norm $\|x\|_1 =$

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$\sum_{l=1}^{n}|x_l|$ for $x=(x_1,\ldots,x_n)^{\mathrm{T}}\in\mathbb{R}^n$, the noinduced norm in $\mathbb{R}^{n\times n}$ is given by $\|A\|_1=\sup_{\|x\|_1=1}\|Ax\|_1$. By $\mathbb{S}^n$ we denote the set of square symmetric matrices of the size n. The partial order in $\mathbb{S}^n$ is given as follows: for $P,Q\in\mathbb{S}^n$ we say that $P\succ Q$ meaning that $P-Q$ is positive definite. For $A\in\mathbb{S}^n$ by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ we denote the minimal and maximal eigenvalue of A. For $A\in\mathbb{R}^{n\times n}$ by $r_\sigma(A)$ we denote the spectral radius of A.

2. PRELIMINARIES

The following notation and notions that we will use throughout the paper are borrowed from Magnus (1954). The commutator of matrices $A,B\in\mathbb{R}^{n\times n}$ given by

$$[A,B]=AB-BA$$

defines the Lie-Algebra structure in $\mathbb{R}^{n\times n}$.

For matrix variables X, Y and Z let $F(X,Y)$ be some formal series of the variables X, Y and for $\lambda\in\mathbb{R}$ the polarization identity

$$F(X+\lambda Z,Y)=F(X,Y)+\lambda F_1(X,Y,Z)+\lambda^2F_2(X,Y,Z)+...,$$

where $F_1(X,Y,Z)$ and $F_2(X,Y,Z)$ are certain formal series of the variables X, Y, Z defines the Hausdorff derivative $\Big(Z\frac{\partial}{\partial X}\Big)F(X,Y):=F_1(X,Y,Z)$.

The Lie-polynomials of matrix variables X,Y are defined recursively as follows (see Magnus (1954) for more details on Lie-elements)

$$\{Y,X^0\}=Y,\quad \{Y,X^{l+1}\}=[\{Y,X^l\},X],\quad l\in\mathbb{Z}_+.$$

The following Hausdorff equalities will play an important role in the sequel

$$\begin{aligned}\Big(Y\frac{\partial}{\partial X}\Big)e^X&=e^X\Big(Y+\sum_{k=1}^{\infty}\frac{1}{(k+1)!}\{Y,X^k\}\Big),\\ \Big(Y\frac{\partial}{\partial X}\Big)e^X&=\Big(Y+\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{Y,X^k\}\Big)e^X.\end{aligned}\tag{1}$$

As well we will need the chain rule for the differentiation of compositions: Let $p(X)$ be a series of the matrix variable X and $A:(a,b)\to\mathbb{R}^{n\times n}$ be a mapping differentiable at the point $t_0\in(a,b)$ such that $B(t):=P(A(t))$ is well defined in a neighborhood of t_0, then

$$\frac{dB}{dt}\Big|_{t=t_0}:=\Big(Y\frac{\partial}{\partial X}\Big)P(X),\quad Y:=\frac{dA}{dt}\Big|_{t=t_0},X=A(t_0).\tag{2}$$

3. PROBLEM STATEMENT

Let us consider the following system of differential equations

$$\begin{aligned}u_t(z,t)&=\mathcal{A}u(z,t)+B(z,t)x(t),\\ \dot{x}(t)&=C(t)x(t)+\int_0^l D(z,t)u(z,t)\,dz,\end{aligned}\tag{3}$$

with initial

$$\begin{aligned}x(0)=x_0\in\mathbb{R}^n,\ u(z,0)=\varphi(z),\ \varphi(0)=0\\ \varphi\in L_r^2([0,l]),\quad z\in(0,l),\end{aligned}\tag{4}$$

and boundary conditions

$$\begin{aligned}u(0,t)=d_1(t),\quad u(l,t)=d_2(t),\\ d_i\in C^1(\mathbb{R})\cap L^\infty(\mathbb{R}),\quad t\in(0,+\infty),\end{aligned}\tag{5}$$

where $\mathcal{A}$ is the linear Sturm-Liouville operator given by

$$\begin{aligned}\mathcal{A}f&=\frac{1}{r(z)}\frac{d}{dz}\Big(r(z)\frac{df}{dz}\Big)+\frac{q(z)}{r(z)}f,\\ D(\mathcal{A})&=\{f\ :f\in C^2((0,l);\mathbb{R}),\quad f(0)=f(l)=0\},\end{aligned}\tag{6}$$

where $r\in C([0,l],\mathbb{R}_+)$, $r(z)>0$, $q\in C([0,l],\mathbb{R})$, $B\in C([0,l]\times\mathbb{R}_+,\mathbb{R}^{1\times n})$ and $\|B(\cdot,t)\|_{L_r^2[0,l]}\in L^\infty$.

The mapping $C:\mathbb{R}\to\mathbb{R}^{n\times n}$ is piece-wise continuous and periodic with period $\theta>0$ and assumed to be such that the linear ODE

$$\dot{\xi}=C(t)\xi\tag{7}$$

is asymptotically stable. Moreover it is assumed that for any $N\in\mathbb{N}$ there exist positive constants a_m, b_m, c_m, $m=0,\ldots,N-1$ such that for $t\in(mh,(m+1)h]$ and $h=\frac{\theta}{N}$ the following holds

$$\sup_{t\in(mh,(m+1)h]}\|C(t)\|\le a_m,\quad \Big\|\int_{mh}^{t}C(s)\,ds\Big\|\le c_m(t-mh),$$

$$\Big\|\Big[C(t),\int_{mh}^{t}C(\tau)\,d\tau\Big]\Big\|\le b_m(t-mh).$$

The function $D\in C([0,l]\times\mathbb{R}_+,\mathbb{R}^{n\times1})$ is assumed to be uniformly bounded, that is $\|D(\cdot,t)\|_{L_r^2[0,l]}\in L^\infty$.

In this work we consider only such initial conditions and disturbances that the problem (3)-(5) has classical solutions $(u,x)\in C^{2,1}([0,l]\times\mathbb{R}_+,\mathbb{R})\times C^1(\mathbb{R}_+,\mathbb{R}^n)$. The questions of existence and uniqueness of such solutions were studied in Slyn'ko (2006); Karafyllis and Krstic (2019).

Definition The linear system (3)-(5) is called ISS, if there exist functions $\beta_i\in\mathcal{KL}$, $\gamma_i\in\mathcal{K}$, $i=1,2$ such that for any initial state in (4) and any input functions $d_1,d_2\in C^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ the corresponding solution satisfies

$$\begin{aligned}\|u(\cdot,t,\varphi_0,x_0)\|_{L_r^2[0,l]}&\le\beta_1(\|\varphi_0\|_{L_r^2[0,l]},\|x_0\|,t)+\gamma_1(d_\infty),\\ \|x(t,\varphi_0,x_0)\|&\le\beta_2(\|\varphi_0\|_{L_r^2[0,l]},\|x_0\|,t)+\gamma_2(d_\infty),\end{aligned}\tag{8}$$

where $d_\infty=\max(\|d_1\|_{L^\infty},\|d_2\|_{L^\infty})$.

4. AUXILIARY RESULTS

Lemma 1. Let $(u(t,z),x(t))$ be a pair of functions satisfying (3), then

$$u(z,t)=w(z,t)+v(z,t),\tag{9}$$

where $(v(z,t),x(t))$ is the solution to the system

$$\begin{aligned}v_t(z,t)&=\mathcal{A}v(z,t)+B(z,t)x(t),\\ \dot{x}(t)&=C(t)x(t)+\int_0^l D(z,t)v(z,t)\,dz+\int_0^l D(z,t)w(z,t)\,dz\end{aligned}\tag{10}$$

with initial conditions

$$x(0)=x_0,\quad v(z,0)=\varphi(z),\quad \varphi(0)=0,\tag{11}$$

and boundary conditions

$$v(0,t)=0,\quad v(l,t)=0,\tag{12}$$

and function $w(z,t)$ satisfies the following estimation

$$\sup_{(z,t)\in[0,l]\times(0,\infty)}|w(z,t)|\le\max\{\|d_1\|_\infty,\|d_2\|_\infty\}=d_\infty.\tag{13}$$

Proof. Let us substitute $u(z,t) = w(z,t) + v(z,t)$ to the equations (3), then we get

$$v_t(z,t) = \mathcal{A}v(z,t) + B(z,t)x(t) + \mathcal{A}w(z,t) - w_t(z,t),$$
$$\dot{x}(t) = C(t)x(t) + \int_0^l D(z,t)v(z,t)\,dz + \int_0^l D(z,t)w(z,t)\,dz. \tag{14}$$

Let function $w(z,t)$ satisfy the following equations:

$$w_t(z,t) - \mathcal{A}w(z,t) = 0,$$
$$w(0,t) = d_1(t), \quad w(l,t) = d_2(t), \quad w(z,0) = 0.$$

Then function $v(z,t)$ needs to satisfy the following initial and boundary conditions

$$v(0,t) = 0, \quad v(l,t) = 0, \quad v(z,0) = \varphi(z).$$

By the maximum principle applied to the function $w(z,t)$ we obtain the desired estimation (13). Then from (14) it follows that the function $v(z,t)$ satisfies the system of linear differential equations (10) subject to (11) and (12), which completes the proof of the lemma.

Next, we are going to derive estimates for the solution to (10) by means of the direct Lyapunov method. The Lyapunov function will be constructed by the discretization method as follows: Let

$$V(t, v(\cdot,t), x) = p(t)V_1(v(\cdot,t)) + V_2(t,x), \tag{15}$$

with

$$V_1(v(\cdot,t)) = \int_0^l r(z)v^2(z,t)\,dz = \|v(\cdot,t)\|^2_{L^2_r[0,l]}, \tag{16}$$
$$V_2(t,x) = x^{\mathrm{T}} P(t)x$$

where $p : \mathbb{R} \to \mathbb{R}_+$ and $P : \mathbb{R} \to \mathbb{S}^n$ are left continuous periodic functions with period θ.

Due to the periodicity it is enough to define the function $P(t)$ on the interval $(0,\theta]$. Let N be a natural number and $h = \frac{\theta}{N}$ be the discretization step. First we define the following matrices

$$\widehat{C}_m(t) = \int_{mh}^{t} C(s)\,ds,$$
$$\widetilde{C}_m = \frac{1}{h}\widehat{C}_m((m+1)h) = \frac{1}{h}\int_{mh}^{(m+1)h} C(s)\,ds,$$
$$\Phi = e^{h\widetilde{C}_{N-1}} \dots e^{h\widetilde{C}_0}.$$

By the asymptotic stability of the linear periodic system (7) we can choose the number N such that $r_\sigma(\Phi) < 1$. Hence we can choose a positive definite matrix P_0 such that the following inequality is satisfied

$$\Phi^{\mathrm{T}} P_0 \Phi - P_0 \prec 0. \tag{17}$$

Further we define the following positive definite matrices

$$P_{m+1} = e^{-h\widetilde{C}_m^{\mathrm{T}}} P_m e^{-h\widetilde{C}_m}, \quad m = 0, \dots, N-1$$

and finally the function $P(t)$ is defined by

$$P(t) = e^{-\widehat{C}_m^{\mathrm{T}}(t)} P_m e^{-\widehat{C}_m(t)}, \quad t \in (mh, (m+1)h].$$

Note that from $P_0 \succ 0$ follows that $P(t) \succ 0$ for $t \geq 0$.

Now let us estimate the derivative of $V_1(v(\cdot,t))$ along solutions of the system (10) for $t \in (k\theta{+}mh, k\theta{+}(m{+}1)h]$, $k \in \mathbb{Z}_+$, $m = 0, \dots, N-1$.

Let us denote for short $r_{\min} = \min_{z\in[0,l]} r(z)$, $r_{\max} = \max_{z\in[0,l]} r(z)$, $q_{\max} = \max_{z\in[0,l]} q(z)$, $\rho = \frac{r_{\min}\pi^2}{l^2} - q_{\max}$. Now we can state our second technical lemma as follows.

Lemma 2. Let $\rho > 0$ and

$$B_m := \sup_{k\in\mathbb{Z}_+} \sup_{t\in[k\theta+mh, k\theta+(m+1)h]} \|B(\cdot,t)\|_{L^2_r[0,l]}.$$

Then for $t \in (k\theta{+}mh, k\theta{+}(m{+}1)h]$ the following estimate holds

$$\frac{dV_1(v(\cdot,t))}{dt} \leq -\frac{2\rho}{r_{\max}} V_1(v(\cdot,t)) +$$
$$2B_m\|P^{-1/2}(t)\|p^{-1/2}(t)\sqrt{p(t)V_1(v(\cdot,t))V_2(t,x(t))}.$$

Proof. The time derivative of the function $V_1(v(\cdot,t))$ along solutions of the auxiliary equation (14) is

$$\frac{dV_1(v(\cdot,t))}{dt} = 2\int_0^l r(z)v(z,t)v_t(z,t)\,dz$$
$$= 2\int_0^l r(z)v(z,t)\mathcal{A}v(z,t)\,dz + 2\int_0^l r(z)v(z,t)B(z,t)\,dz x(t)$$

The first integral in last line can be estimated as follows

$$\int_0^l r(z)v(z,t)\mathcal{A}v(z,t)\,dz = \int_0^l v(z,t)\frac{\partial}{\partial z}\Big(r(z)\frac{\partial v}{\partial z}(z,t)\Big)\,dz$$
$$+ \int_0^l q(z)v^2(z,t)\,dz$$
$$= -\int_0^l r(z)v_z^2(z,t)\,dz + \int_0^l q(z)v^2(z,t)\,dz$$
$$\leq -r_{\min}\int_0^l v_z^2(z,t)\,dz + q_{\max}\int_0^l v^2(z,t)\,dz.$$

With help of the Friedrichs inequality

$$\int_0^l v^2(z,t)\,dz \leq \frac{l^2}{\pi^2}\int_0^l v_z^2(z,t)\,dz$$

we get

$$\int_0^l r(z)v(z,t)\mathcal{A}v(z,t)\,dz$$
$$\leq \Big(-\frac{r_{\min}\pi^2}{l^2} + q_{\max}\Big)\int_0^l v^2(z,t)\,dz.$$

From the obvious estimation

$$V_1(v(\cdot,t)) = \int_0^l r(z)v^2(z,t)\,dz \leq r_{\max}\int_0^l v^2(z,t)\,dz$$

we get the inequality

$$\frac{dV_1(v(\cdot,t))}{dt}$$
$$\leq -\frac{2\rho}{r_{\max}}V_1(v(\cdot,t)) + 2\int_0^l r(z)v(z,t)B(z,t)\,dz x(t).$$

And finally by the Cauchy inequality we arrive at the desired estimate

$$\frac{dV_1(v(\cdot,t))}{dt} \le -\frac{2\rho}{r_{\max}} V_1(v(\cdot,t))$$
$$+2\|v(\cdot,t)\|_{L_r^2[0,l]} B_m \|P^{-1/2}(t)\| \sqrt{V_2(t,x(t))}$$
$$\le -\frac{2\rho}{r_{\max}} V_1(v(\cdot,t))$$
$$+2B_m\|P^{-1/2}(t)\| p^{-1/2}(t)\sqrt{p(t)V_1(v(\cdot,t))V_2(t,x(t))},$$

which proves the lemma.

And the last lemma that we need for the estimation of the time derivative of the function V_2 is as follows.

Lemma 3. Let

$$D_m = \sup_{k\in\mathbb{Z}_+} \sup_{t\in[k\theta+mh,k\theta+(m+1)h]} \Big\|\frac{D(\cdot,t)}{r(\cdot)}\Big\|_{L_r^2[0,l]},$$
$$\eta_m = b_m \sum_{k=1}^{\infty} \frac{(2c_m h)^{k-1}}{(k+1)!}.$$

Then for $t \in (k\theta + mh, k\theta + (m+1)h]$ the following inequality holds

$$\frac{dV_2(t,x(t))}{dt} \le 2e^{2c_m h}\sqrt{\frac{\lambda_{\max}(P_m)}{\lambda_{\min}(P_m)}} h\eta_m V_2(t,x)$$
$$+2p^{-1/2}(t) D_m \|P^{1/2}(t)\| \sqrt{p(t)V_1(v(\cdot,t))V_2(t,x(t))}$$
$$+2\sqrt{l r_{\max}} d_\infty D_m \|P^{1/2}(t)\| \sqrt{V_2(t,x(t))}.$$

Proof. Let us first consider the expression $\dot P(t)+C^{\mathrm T}(t)P(t)+P(t)C(t)$. To calculate the derivative $\dot P(t)$ we first derive an expression for the derivative $\frac{d}{dt}e^{-\widehat C_m(t)}$. By the chain rule we have

$$\frac{d}{dt}e^{-\widehat C_m(t)} = -\Big(\frac{d\widehat C_m(t)}{dt}\frac{\partial}{\partial X}\Big)e^X\Big|_{X=-\widehat C_m(t)}$$
$$= -\Big(C(t)\frac{\partial}{\partial X}\Big)e^X\Big|_{X=-\widehat C_m(t)}.$$

Then by the Hausdorff equality we get

$$\frac{d}{dt}e^{-\widehat C_m(t)} = -e^{-\widehat C_m(t)}\Big(C(t) + \sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{C(t),\widehat C_m^k(t)\}\Big).$$

Hence we can write

$$\frac{d}{dt}e^{-\widehat C_m^{\mathrm T}(t)} = -\Big(C(t) + \sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{C(t),\widehat C_m^k(t)\}\Big)^{\mathrm T} e^{-\widehat C_m^{\mathrm T}(t)}$$

Finally we obtain

$$\dot P(t) + C^{\mathrm T}(t)P(t) + P(t)C(t)$$
$$= -\Big(\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{C(t),\widehat C_m^k(t)\}\Big)^{\mathrm T} P(t)$$
$$-P(t)\Big(\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{C(t),\widehat C_m^k(t)\}\Big)$$

Taking into account that for all $t \in (k\theta+mh, k\theta+(m+1)h]$ we have

$$\|\{C(t),\widehat C_m^k(t)\}\| \le b_m(2c_m)^{k-1}(t-k\theta-mh)^k,$$
$$\Big\|\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{C(t),\widehat C_m^k(t)\}\Big\|$$
$$\le b_m\sum_{k=1}^{\infty}\frac{(2c_m)^{k-1}(t-k\theta-mh)^k}{(k+1)!} \le \eta_m h,$$

we can estimate

$$x^{\mathrm T}(\dot P(t) + C^{\mathrm T}(t)P(t) + P(t)C(t))x$$
$$\le 2\|P^{-1/2}(t)\|\|P^{1/2}(t)\| h\eta_m x^{\mathrm T}P(t)x(t).$$

Then from the inequalities

$$\|P^{1/2}(t)\| = \lambda_{\max}^{1/2}(P(t)) = \|P(t)\|^{1/2}$$
$$= \|e^{-\widehat C_m^{\mathrm T}(t)}P_m e^{-\widehat C_m(t)}\|^{1/2} \le e^{c_m(t-k\theta-mh)}\|P_m\|^{1/2},$$
$$\lambda_{\min}(P(t)) = \min_{\|x\|=1} x^{\mathrm T}P(t)x$$
$$= \min_{\|x\|=1}(e^{-\widehat C_m(t)}x)^{\mathrm T}P_m e^{-\widehat C_m(t)}x$$
$$\ge \lambda_{\min}(P_m)\min_{\|x\|=1}\|e^{-\widehat C_m(t)}x\|^2 \tag{18}$$
$$\ge \min_{\|x\|=1}\lambda_{\min}(P_m)e^{-2c_m(t-k\theta-mh)}\|x\|^2$$
$$= \lambda_{\min}(P_m)e^{-2c_m(t-k\theta-mh)}$$

and

$$\|P^{-1/2}(t)\| = \lambda_{\max}^{1/2}(P^{-1}(t)) = \lambda_{\min}^{-1/2}(P(t))$$
$$\le \lambda_{\min}^{-1/2}(P_m)e^{c_m(t-k\theta-mh)},$$

it follows that

$$x^{\mathrm T}(\dot P(t) + C^{\mathrm T}(t)P(t) + P(t)C(t))x$$
$$\le 2e^{2c_m h}\sqrt{\frac{\lambda_{\max}(P_m)}{\lambda_{\min}(P_m)}} h\eta_m V_2(t,x).$$

for all $t \in (k\theta + mh, k\theta + (m+1)h]$.

Now let us estimate the time derivative of $V_2(t,x(t))$ along solutions of the system (10) for $t \in [k\theta+mh, k\theta+(m+1)h]$. First we calculate that

$$\frac{dV_2(t,x(t))}{dt} = x^{\mathrm T}(t)(\dot P(t) + C^{\mathrm T}(t)P(t) + P(t)C(t))x(t)$$
$$+2\int_0^l v(z,t)D^{\mathrm T}(t,z)\,dz P(t)x(t)$$
$$+2\int_0^l w(z,t)D^{\mathrm T}(t,z)\,dz P(t)x(t).$$

Then by the Cauchy inequality we get that for $t \in (k\theta + mh, k\theta + (m+1)h]$

$$\int_0^l v(z,t)D^{\mathrm T}(t,z)\,dz P(t)x(t)$$
$$\le \|v(\cdot,t)\|_{L_r^2[0,l]} D_m\|P^{1/2}(t)\|\sqrt{V_2(t,x(t))}$$
$$\le p^{-1/2}(t)D_m\|P^{1/2}(t)\|\sqrt{p(t)V_1(v(\cdot,t))V_2(t,x(t))}$$

and hence from the inequality (9),

$$\int_0^l w(z,t)D^{\mathrm T}(t,z)\,dz P(t)x(t)$$
$$\le \sqrt{l r_{\max}} d_\infty D_m\|P^{1/2}(t)\|\sqrt{V_2(t,x(t))}$$
$$\le \sqrt{l r_{\max}} d_\infty D_m\|P^{1/2}(t)\|\sqrt{V(t,v(\cdot,t),x(t))}$$

the statement of the lemma follows.

5. MAIN RESULTS

Let us introduce the following matrices $\Gamma_m = (\gamma_{ij}^{(m)})_{i,j=1,2}$, $m = 0,\ldots,N-1$

$$\gamma_{11}^{(m)}=0,\quad \gamma_{22}^{(m)}=2e^{2c_mh}\sqrt{\frac{\lambda_{\max}(P_m)}{\lambda_{\min}(P_m)}}h\eta_m,$$
$$\gamma_{12}=\gamma_{21}=e^{c_mh}(B_m\lambda_{\min}^{-1/2}(P_m)e^{-\frac{\rho}{2r_{\max}}(N-2m-2)h}+e^{\frac{\rho}{2r_{\max}}(N-2m)h}D_m\lambda_{\max}^{1/2}(P_m)).$$
and constants
$$\delta_m=\sqrt{lr_{\max}}D_me^{c_mh}\lambda_{\max}^{1/2}(P_m),$$
$$Q_m=e^{\frac{h}{2}\sum\limits_{j=0}^{m}\lambda_{\max}(\Gamma_j)},\quad \zeta_m=\frac{2\delta_m(e^{\frac{h\lambda_{\max}(\Gamma_m)}{2}}-1)}{\lambda_{\max}(\Gamma_m)},$$
$$m=0,\dots,N-1.$$
$$q_0=\max\{e^{-\frac{2\rho\theta}{r_{\max}}},\lambda_{\max}(P_0P_N^{-1})\},\qquad \alpha=\sqrt{q_0}Q_{N-1},$$
$$\beta=\sqrt{q_0}Q_{N-1}\sum_{l=0}^{N-1}Q_l^{-1}\zeta_l,\quad \lambda^*=\max_{m=\overline{0,N-1}}\lambda_{\max}(\Gamma_m),$$
$$\zeta^*=\max_{m=\overline{0,N-1}}\zeta_m,\quad K=\frac{\alpha\beta e^{\lambda^*h/2}}{\sqrt{q_0}(1-\alpha)}+\frac{\beta e^{\lambda^*h/2}}{\sqrt{q_0}}+\zeta^*,$$
$$\lambda_*=\min_{m=\overline{0,N-1}}\lambda_{\min}^{-1/2}(P_m)e^{-c_mh}.$$
Then we can state our first main result as follows.

Theorem 1. Let P_0 be positive definite and satisfy the linear matrix inequality (17). If $\alpha\in(0,1)$ and $\rho=\frac{r_{\min}\pi^2}{l^2}-q_{\max}>0$, then the linear system (3)-(5) satisfies the ISS property and, in particular, the following estimations for its solution are valid
$$\|x(t)\|\le\frac{e^{\lambda^*h/2}}{\lambda_*\sqrt{q}}e^{\frac{\ln\alpha}{\theta}t}\sqrt{e^{-\frac{\rho}{r_{\max}}\theta}\|\varphi_0\|^2_{L^2_r[0,l]}+\lambda_{\max}(P_0)\|x_0\|^2}+\frac{K}{\lambda_*}d_\infty.$$
$$\|u(\cdot,t)\|_{L^2_r[0,l]}\le\frac{e^{\lambda^*h/2}}{\sqrt{q}}e^{\frac{\ln\alpha}{\theta}t}\sqrt{\|\varphi_0\|^2_{L^2_r[0,l]}+e^{\frac{\rho}{2r_{\max}}\theta}\lambda_{\max}(P_0)\|x_0\|^2}+\Big(\sqrt{lr_{\max}}+Ke^{\frac{\rho}{2r_{\max}}\theta}\Big)d_\infty.$$

Proof. For the calculation of the time derivative of the Lyapunov function $\dot V(t,v(\cdot,t),x(t))$ along solutions of the system (9) the Lemmas 2 and 3 can be applied, which imply that the following inequality holds
$$\dot V(t,v(\cdot,t),x(t))\le(\dot p(t)-\frac{2\rho}{r_{\max}}p(t))V_1(v(\cdot,t))+2e^{c_mh}(B_m\lambda_{\min}^{-1/2}(P_m)p^{1/2}(t)+p^{-1/2}(t)D_m\lambda_{\max}^{1/2}(P_m))\sqrt{p(t)V_1(v(\cdot,t))V_2(t,x(t))}+2e^{2c_mh}\sqrt{\frac{\lambda_{\max}(P_m)}{\lambda_{\min}(P_m)}}h\eta_mV_2(t,x)+2\sqrt{lr_{\max}}D_m\|P^{1/2}(t)\|\sqrt{V(t,v(\cdot,t),x(t))}d_\infty$$
for all $t\in(k\theta+mh,k\theta+(m+1)h]$. Let us choose the function $p(t)$ so that
$$\dot p(t)-\frac{2\rho}{r_{\max}}p(t)=0,$$
$$p(k\theta+mh+0)=p_m.$$
Then $p(t)=e^{\frac{2\rho}{r_{\max}}(t-mh-k\theta)}p_m$ for $t\in(k\theta+mh,k\theta+(m+1)h]$, where $p_m=p_0e^{\frac{2\rho}{r_{\max}}mh}$. The constant p_0 we choose as follows: $p_0=e^{-\frac{\rho}{r_{\max}}Nh}=e^{-\frac{\rho}{r_{\max}}\theta}$. Then
$$\dot V(t,v(\cdot,t),x(t))\le2\gamma_{12}^{(m)}\sqrt{p(t)V_1(v(\cdot,t))V_2(t,x(t))}+\gamma_{22}^{(m)}V_2(t,x)+2\delta_md_\infty\sqrt{V(t,v(\cdot,t),x(t))}$$
$$\le\lambda_{\max}(\Gamma_m)V(t,v(\cdot,t),x(t))+2\delta_md_\infty\sqrt{V(t,v(\cdot,t),x(t))}.$$
Hence our Lyapunov function satisfies the following differential inequality for all $t\in(k\theta+mh,k\theta+(m+1)h]$
$$\dot V(t,v(\cdot,t),x(t))\le\lambda_{\max}(\Gamma_m)V(t,v(\cdot,t),x(t))+2\delta_md_\infty\sqrt{V(t,v(\cdot,t),x(t))}.$$
Hence by the comparison principle we get the following estimate for $t\in(k\theta+mh,k\theta+(m+1)h]$
$$V(t,v(\cdot,t),x(t))\le u(t,mh+k\theta,V_{mk}),\tag{19}$$
where
$$V_{mk}=V(k\theta+mh+0,v(\cdot,k\theta+mh+0),x(k\theta+mh+0))=V(k\theta+mh-0,v(\cdot,k\theta+mh-0),x(k\theta+mh-0)),$$
for $m=1,\dots,N-1$,
$$V_{0k}=V(k\theta+0,v(\cdot,k\theta+0),x(k\theta+0)),$$
$$V_{Nk}=V((k+1)\theta-0,v(\cdot,(k+1)\theta-0),x((k+1)\theta-0)),$$
and $u(t,k\theta+mh,u_0)$, $t\in(k\theta+mh,k\theta+(m+1)h]$ is the solution to the initial value problem
$$\dot u(t)=\lambda_{\max}(\Gamma_m)u(t)+2\delta_md_\infty\sqrt{u(t)},$$
$$u(k\theta+mh+0)=u_0.$$
Solving this initial value problem we get
$$\sqrt{V_{m+1,k}}\le e^{\frac{h\lambda_{\max}(\Gamma_m)}{2}}\sqrt{V_{mk}}+\frac{2\delta_m(e^{\frac{h\lambda_{\max}(\Gamma_m)}{2}}-1)}{\lambda_{\max}(\Gamma_m)}d_\infty,\quad m=0,\dots,N-1\tag{20}$$
And in particular
$$\sqrt{V_{Nk}}\le Q_{N-1}\sqrt{V_{0k}}+Q_{N-1}\sum_{l=0}^{N-1}Q_l^{-1}\zeta_ld_\infty.$$
Or equivalently
$$\sqrt{V((k+1)\theta,v(\cdot,(k+1)\theta),x((k+1)\theta))}\le Q_{N-1}\sqrt{V(k\theta+0,v(\cdot,k\theta+0),x(k\theta+0))}+Q_{N-1}\sum_{k=0}^{N-1}Q_l^{-1}\zeta_ld_\infty.$$
Since functions $p(t)$ are $P(t)$ are θ-periodic we can write
$$V((k+1)\theta+0,v(\cdot,(k+1)\theta+0),x((k+1)\theta+0))$$
$$=p((k+1)\theta+0)V_1(v(\cdot,(k+1)\theta+0))+x^{\mathrm T}((k+1)\theta+0)P((k+1)\theta+0)x((k+1)\theta+0)$$
$$=p(0+0)V_1(v(\cdot,(k+1)\theta))+x^{\mathrm T}((k+1)\theta)P(0+0)x((k+1)\theta)$$
$$=e^{-\frac{2\rho\theta}{r_{\max}}}p(\theta)V_1(v(\cdot,(k+1)\theta))+(P_N^{1/2}x((k+1)\theta))^{\mathrm T}P_N^{-1/2}P_0P_N^{-1/2}P_N^{1/2}x((k+1)\theta)$$
$$\le e^{-\frac{2\rho\theta}{r_{\max}}}p(\theta)V_1(v(\cdot,(k+1)\theta))+\lambda_{\max}(P_0P_N^{-1})V_2((k+1)\theta,x((k+1)\theta))$$
$$\le q_0(p(\theta)V_1(v(\cdot,(k+1)\theta))+V_2((k+1)\theta,x((k+1)\theta)))$$
$$=q_0V((k+1)\theta,v(\cdot,(k+1)\theta),x((k+1)\theta)).$$
From the last estimate it follows that
$$\sqrt{V((k+1)\theta+0,v(\cdot,(k+1)\theta+0),x((k+1)\theta+0))}\le\sqrt{q_0}Q_{N-1}\sqrt{V(k\theta+0,v(\cdot,k\theta+0),x(k\theta+0))}+\sqrt{q_0}Q_{N-1}\sum_{l=0}^{N-1}Q_l^{-1}\zeta_ld_\infty.$$

and

$$\sqrt{V(k\theta+0,v(\cdot,k\theta+0),x(k\theta+0))} \leq \alpha^k\sqrt{V(0+0,v(\cdot,0+0),x(0+0))}+\frac{\beta}{1-\alpha}d_\infty. \quad (21)$$

Let us denote $k_0 = \left[\frac{t}{\theta}\right]$ and $m_0 = \left[\frac{t-[t/\theta]k_0}{h}\right]$, then $t = k_0\theta + m_0 h + t_1$, $t_1 \in [0,h)$.

From the inequality (20) we get

$$\sqrt{V_{m_0,k_0}} \leq Q_{N-1}\sqrt{V_{0k_0}} + Q_{N-1}\sum_{l=0}^{N-1} Q_l^{-1}\zeta_l d_\infty = \frac{1}{\sqrt{q_0}}(\alpha\sqrt{V_{0,k_0}}+\beta d_\infty),$$

and from the inequality (19) we get

$$\sqrt{V(t,v(\cdot,t),x(t))} \leq e^{\lambda_{\max}(\Gamma_{m_0})h/2}\sqrt{V_{m_0,k_0}}+\zeta_{m_0}d_\infty$$

Hence

$$\sqrt{V(t,v(\cdot,t),x(t))} \leq e^{\lambda^* h/2}\sqrt{V_{m_0,k_0}}+\zeta^* d_\infty \leq \frac{\alpha e^{\lambda^* h/2}}{\sqrt{q_0}}\sqrt{V_{0,k_0}}+\Big(\frac{\beta e^{\lambda^* h/2}}{\sqrt{q_0}}+\zeta^*\Big)d_\infty.$$

Then from (21) we obtain

$$\begin{aligned}&\sqrt{V(t,v(\cdot,t),x(t))}\\ &\leq \frac{\alpha e^{\lambda^* h/2}}{\sqrt{q_0}}(\alpha^{k_0}\sqrt{V(0+0,v(\cdot,0+0),x(0+0))}\\ &\quad+\frac{\beta}{1-\alpha}d_\infty)+\Big(\frac{\beta e^{\lambda^* h/2}}{\sqrt{q_0}}+\zeta^*\Big)d_\infty\\ &=\frac{\alpha e^{\lambda^* h/2}}{\sqrt{q_0}}e^{k_0\ln\alpha}\sqrt{V(0+0,v(\cdot,0+0),x(0+0))}\\ &\quad+(\frac{\alpha\beta e^{\lambda^* h/2}}{\sqrt{q_0}(1-\alpha)}+\frac{\beta e^{\lambda^* h/2}}{\sqrt{q_0}}+\zeta^*)d_\infty\\ &\leq \frac{e^{\lambda^* h/2}}{\sqrt{q_0}}e^{\frac{\ln\alpha}{\theta}t}\sqrt{V(0+0,v(\cdot,0+0),x(0+0))}+Kd_\infty.\end{aligned}$$

Hence,

$$p^{1/2}(t)\|v(\cdot,t)\|_{L_r^2[0,l]} \leq \frac{e^{\lambda^* h/2}}{\sqrt{q_0}}e^{\frac{\ln\alpha}{\theta}t}\sqrt{p_0\|\varphi_0\|^2_{L_r^2[0,l]}+\lambda_{\max}(P_0)\|x_0\|^2}+Kd_\infty.$$

Since $p(t) \geq p_0$ for all $t \geq 0$, then

$$\|v(\cdot,t)\|_{L_r^2[0,l]} \leq \frac{e^{\lambda^* h/2}}{\sqrt{q_0}}e^{\frac{\ln\alpha}{\theta}t}\sqrt{\|\varphi_0\|^2_{L_r^2[0,l]}+e^{\frac{\rho}{r_{\max}}\theta}\lambda_{\max}(P_0)\|x_0\|^2} +Ke^{\frac{\rho}{2r_{\max}}\theta}d_\infty,$$

and as well

$$\lambda_{\min}^{1/2}(P(t))\|x(t)\| \leq \frac{e^{\lambda^* h/2}}{\sqrt{q_0}}e^{\frac{\ln\alpha}{\theta}t}\sqrt{p_0\|\varphi_0\|^2_{L_r^2[0,l]}+\lambda_{\max}(P_0)\|x_0\|^2}+Kd_\infty.$$

Now from (18) we obtain the estimate

$$\|x(t)\| \leq \frac{e^{\lambda^* h/2}}{\lambda_*\sqrt{q_0}}e^{\frac{\ln\alpha}{\theta}t} \times\sqrt{e^{-\frac{\rho}{r_{\max}}\theta}\|\varphi_0\|^2_{L_r^2[0,l]}+\lambda_{\max}(P_0)\|x_0\|^2}+\frac{K}{\lambda_*}d_\infty$$

and the statement of the theorem follows immediately from Lemma 1.

6. CONCLUSION

We have established ISS properties for classical solutions for a feedback connection of an ODE with a parabolic PDE and provided explicit ISS estimates for the solutions. Connections of the considered class of systems was not discussed in the ISS framework to the best of authors knowledge. It should be mentioned that the ISS property can be rather restrictive in some applications. The study of such couplings can be performed similarly under such weaker properties as integral ISS or local ISS.

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