Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

On Homogeneous Lyapunov Function Theorem for Evolution Equations ⋆

Andrey Polyakov *,**

* *Inria Lille, Univ. Lille, CNRS, UMR 9189 - CRIStAL, (F-59000 Lille, France) (andrey.polyakov@inria.fr)*
** *ITMO University, Saint-Petersburg, 197101, Russia*

Abstract: The existence of homogeneous Lyapunov function for a stable homogeneous ordinary differential equation (ODE) is proven by V. Zubov in 1958 and refined by L. Rosier in 1992. The present paper proposes an extension of this result to evolution equations in Banach spaces.

Keywords: stability of distributed parameter systems; semigroup and operator theory; homogeneous systems.

1. INTRODUCTION

Homogeneity is a dilation symmetry, which can be discovered for functions and vector fields in $\mathbb{R}^n$ (Zubov (1958), Hermes (1986), Bernuau et al. (2014)) as well as for functionals and operators in Banach and Hilbert spaces (see e.g. Folland (1975), Polyakov et al. (2016)). Homogeneity simplify stability and robustness analysis of control systems (Zubov (1958), Hermes (1986), Kawski (1991), Ryan (1995), Andrieu et al. (2008), Polyakov (2018)) as well as non-linear controllers/observers design (Kawski (1990), Coron and Praly (1991), Andrieu et al. (2008), Polyakov et al. (2018), Lopez-Ramirez et al. (2018)).

The method of Lyapunov function is one of the main tools for stability analysis and stabilization of both finite Bacciotti and Rosier (2001) and infinite dimensional nonlinear systems Karafyllis and Krstic (2018), Prieur and Mazenc (2012). Characterization (necessary and sufficient conditions) of stability of evolution systems is an important problem in this context (see Mironchenko and Wirth (2019), Jacob et al. (2018)). It is known since Zubov (1958) and Rosier (1992) that any stable homogeneous ODE admit a homogeneous Lyapunov function. This paper presents a similar result for evolution equations in Banach spaces.

Notation. $\mathbb{R}_+ = [0, +\infty)$; $\mathbb{B}$ is a real Banach; $\mathbb{H}$ is a real Hilbert space; $\mathbf{0}$ is the zero element of $\mathbb{B}$; $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ denotes the space of linear bounded operators $\mathbb{B}_1 \to \mathbb{B}_2$; S is the unit sphere in $\mathbb{B}$; $B(r) \subset \mathbb{B}$ denotes the open ball in $\mathbb{B}$ centered at $\mathbf{0}$ of the radius $r > 0$; L^2 denotes the Lebesgue space of quadratically integrable functions; C_c^∞ is a set of infinitely smooth functions with a compact support; H^p is a Sobolev space; $\mathcal{K}$ is a set of strictly increasing functions $\sigma : [0, +\infty) \to [0, +\infty)$ such that $\sigma(0) = 0$; $\sigma \in \mathcal{K}^\infty$ if $\sigma \in \mathcal{K}$, $\sigma(s) \to +\infty$ as $s \to +\infty$; $B(r)$ is a ball of the radius $r > 0$ centered at $\mathbf{0}$.

⋆ The author acknowledges the support of ANR DIGISLID project (ANR-18-CE40-0008), the Grant 08-08 of the Government of Russian Federation and the Ministry of Science and Higher Education of Russian Federation, passport of goszadanie no. 2019-0898.

2. SYSTEM DESCRIPTION AND BASIC ASSUMPTIONS

Let us consider the nonlinear system

$$\dot{x} = Ax + f(x), \quad t > 0, \tag{1}$$

where $t_0 > 0$ is an initial instant of time, $x(t) \in \mathbb{B}$ is the system state, $A : \mathcal{D}(A) \subset \mathbb{B} \to \mathbb{B}$ is a linear (possibly) unbounded closed densely defined operator which generates a strongly continuous semi-group Φ of linear bounded operators on $\mathbb{B}$ (see e.g. (Pazy, 1983, Chapter 2) for more details) and $f : \mathbb{B} \to \mathbb{B}$ is a non-linear mapping defined on whole $\mathbb{B}$ (i.e. $\|f(x)\| < +\infty$ for any $x \in \mathbb{B}$) such that $f(\mathbf{0}) = \mathbf{0}$. The latter means that the evolution equation always has the zero solution.

Definition 1. A continuous function $x : [0, T) \to \mathbb{B}$ is said to be a *mild solution* of (1) if

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s) f(x(s)) ds, \quad \forall t \in [0, T).$$

If this mild solution satisfies (1) for (almost) all $t \in (0, T)$ then x is called classical (strong) solution of (1).

The integral in the above definition is understood in the sense of Bochner (see e.g. Driver (2003), page 187).

Assumption 1. For any $x_0 \in \mathbb{B}$ a mild solution x with $x(0) = x_0$ exists and is defined on a time interval $[0, \bar{t})$, where $\bar{t} = +\infty$ or $\bar{t} < +\infty : \lim_{t \to \bar{t}} \|x(t)\| = +\infty$. The time instant $\bar{t}$ may depend on x_0 and/or on a concrete solution x if solutions are not unique.

We do not ask explicitly any regularity of f, but it is hidden in Assumption 1. Indeed, the existence of mild solutions can be proven only under some restrictions on f (see, for example, Pazy (1983); Engel and Nagel (2000)). Notice that for $\mathbb{B} = \mathbb{R}^n$ the system (1) becomes a nonlinear ODE and any mild solution (if it exists) becomes the strong one.

3. STABILITY OF EVOLUTION SYSTEMS

3.1 Stability definitions

Let x_{x_0} denote a solution of (1) with the initial condition $x(0) = x_0$. Let us recall some definitions of stability.

Definition 2. (Lyapunov stability). The origin of the system (1) is said to be globally uniformly Lyapunov stable if there exists $\varepsilon \in \mathcal{K}^\infty$ such that $\|x_{x_0}(t)\| \leq \varepsilon(\|x_0\|), t > 0$ for any solution x_{x_0} of (1) and any $x_0 \in \mathbb{B}$.

Let us recall the well-known result about Lyapunov stable ODEs(see e.g. Bhat and Bernstein (2000)), which also holds for evolution equation Polyakov (2020).

Proposition 1. If the origin of the system (1) is Lyapunov stable then $x_{\mathbf{0}}(t) \equiv \mathbf{0}$ is the unique solution of the system (1) with the initial condition $x(0) = \mathbf{0}$.

Definition 3. (Uniform asymptotic stability). The origin of the system (1) is said to be globally uniformly asymptotically stable if it is globally uniformly Lyapunov stable and $\forall \varepsilon > 0$, $\forall R > \varepsilon$, there exists $\tilde{T} = \tilde{T}(R, \varepsilon) > 0$ such that the inclusion $x_0 \in B(R)$ imply $x_{x_0}(t) \in B(\varepsilon)$ for $t > \tilde{T}$.

Definition 4. (Finite-time Stability). The origin of a dynamical system is said to be globally uniformly finite-time stable if it is globally uniformly Lyapunov stable and there exists a locally bounded functional $T : \mathbb{B} \to [0, +\infty)$ such that for any $x_0 \in \mathbb{B}$ we have $x_{x_0}(t) = 0, \forall t \geq T(x_0)$.

The functional T estimates a setting time of the system. Obviously, this functional is not unique. If T satisfies Definition 4 then $T + C$ with $C > 0$ also estimates the settling time. The minimal over all functionals satisfying Definition 4 is called the *settling-time function.*

Definition 5. (Practically fixed-time stability). The origin of a dynamical system is said to be globally practically fixed-time stable if it is globally uniformly Lyapunov stable and

$$\forall r > 0, \exists T_r > 0 : \quad \|x_{x_0}(t)\| \leq r, \quad \forall t \geq T_r, \forall x_0 \in \mathbb{B}.$$

Fixed-time stability can be discovered even for linear evolution systems (i.e. with $f \equiv \mathbf{0}$). For example, solutions of wave equations with the so-called transparent boundary conditions vanish in a fixed time independently of initial conditions (see e.g. Perrollaz and Rosier (2014) and Majda (1975)).

3.2 Generalized derivatives

Let $\mathcal{I}$ be one of the following sets: $[a, b]$, (a, b), $[a, b)$ or $(a, b]$, where $a, b \in [-\infty, +\infty]$, $a < b$. Let $\mathbb{K}$ be the set of sequences of real non-zero numbers converging to zero:

$$\{h_n\} \in \mathbb{K} \quad \Leftrightarrow \quad h_n \to 0,\ h_n \neq 0.$$

Definition 6. (page 207, Natanson (1955)). A number

$$D_{\{h_n\}}\varphi(t) = \lim_{n \to +\infty} \frac{\varphi(t+h_n) - \varphi(t)}{h_n}, \quad \{h_n\} \in \mathbb{K} : t + h_n \in \mathcal{I}$$

is called derivative number of the function $\varphi : \mathcal{I} \to \mathbb{R}$ at the point $t \in \mathcal{I}$, if the above limit exists (both finite and infinite limits are admissible).

The set of all derivative numbers of the function φ at the point $t \in \mathcal{I}$ is called contingent derivative:

$$D_{\mathbb{K}}\varphi(t) = \bigcup_{\{h_n\} \in \mathbb{K} : \exists D_{\{h_n\}}\varphi(t)} \{D_{\{h_n\}}\varphi(t)\} \subset [-\infty, +\infty].$$

Obviously, if a function φ is differentiable at a point $t \in \mathcal{I}$ then $D_{\mathbb{K}}\varphi(t) = \dot{\varphi}(t)$.

Lemma 1. (Natanson (1955), page 208). The set $D_{\mathbb{K}}\varphi(t) \subseteq [-\infty, +\infty]$ is nonempty for any $\varphi : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ and $t \in \mathcal{I}$.

If $\varphi : \mathcal{I} \to \mathbb{R}$ then the notation $D_{\mathbb{K}}\varphi(t) < 0$ means $y < 0$, $\forall y \in D_{\mathbb{K}}\varphi(t)$. In the similar component-wise sense we define$|D_{\mathbb{K}}\varphi(t)| < L$ with $L \in \mathbb{R}_+$, and $qD_{\mathbb{K}}\phi(t)$ with $q \in \mathbb{R}$. Definition 6 implies *if* $D_{\mathbb{K}}\phi_1(t) \leq c_1$ *and* $D_{\mathbb{K}}\phi_2(t) \leq c_2$ *for all* $t \in \mathcal{I}$, *where* $c_1, c_2 \in \mathbb{R}$, *then* $D_{\mathbb{K}}(\phi_1 + \phi_2)(t) \leq c_1 + c_2$.

Recall that a function $\varphi : \mathcal{I} \to \mathbb{R}$ is called *decreasing* on $\mathcal{I}$ if $\forall t_1, t_2 \in \mathcal{I} : t_1 \leq t_2 \ \Rightarrow\ \varphi(t_1) \geq \varphi(t_2)$.

Lemma 2. The function $\varphi : \mathcal{I} \in \mathbb{R} \to \mathbb{R}$ is decreasing on $\mathcal{I}$ if and only if $D_{\mathbb{K}}\varphi(t) \leq 0$ holds for all $t \in \mathcal{I}$.

Sufficiency is the straightforward consequence of Definition 6 (see also Natanson (1955), page 266, for more details). *Necessity* immediately follows from the definition of the decreasing function.

3.3 Positive Definiteness and Generalized Properness

Recall that a functional $W : \Omega \subset \mathbb{B} \to \mathbb{R}$ is said to be *positive definite* if $W(0) = 0$ and $W(x) > 0$ for $x \in \Omega \backslash \{\mathbf{0}\}$.

Lyapunov function candidates in $\mathbb{R}^n$ are usually positive definite and proper (see e.g. Clarke et al. (1998)). Recall that a mapping $f : \mathbb{R}^n \to \mathbb{R}$ is proper if the inverse image of any compact set is a compact set. In the general case, closedness and boundedness is not sufficient for compactness in Banach spaces, and the properness in the classical sense may be too strong condition for Lyapunov function candidate. For “generalized” proper functions given by Definition 7 an inverse image of any compact set belongs to a closed bounded set (which may be not compact in the general case). Below the word ”generalized” is omitted for shortness.

Definition 7. A positive definite functional $V : \mathbb{B} \to [0, +\infty)$ is said to be *locally (resp. globally) proper* if there exist $\underline{V}, \overline{V} \in \mathcal{K}$(resp. $\mathcal{K}^\infty$): $\underline{V}(\|x\|) \leq V(x) \leq \overline{V}(\|x\|)$ for $x \in \mathbb{B}$.

Notice that the above definition asks for the continuity of V only at $\mathbf{0}$. If $\mathbb{B} = \mathbb{R}^n$ and V is continuous on $\mathbb{R}^n$ then the globally proper (in the ”generalized” sense) functional V becomes proper in the classical sense.

The locally or globally proper positive definite functionals are conventional Lyapunov function candidates for stability analysis of dynamical systems (see Bacciotti and Rosier (2001), Mironchenko and Wirth (2019), Polyakov (2020)). To analyze a decay of such discontinuous function along trajectories of a system, the generalized derivatives considered above can be utilized.

3.4 Characterization of stability via Lyapunov functionals

A mild solution satisfies the evolution inclusion (1) in the mild sense (see Definition 1). Theoretically, its time derivative may nowhere exist even when the solution is locally Lipschitz continuous. Generalized derivatives (see e.g. Natanson (1955)) can be utilized in this case for Lyapunov analysis.

Lyapunov Function Method is founded on the so-called energetic approach to stability analysis. It considers a positive definite function as a possible energetic characteristic (energy) of a dynamical system and studies evolution of this function in time. If such a function is decreasing along any system trajectory then the system has some stability property.

Theorem 1. The origin of the system (1) is globally uniformly Lyapunov stable if and only if there exists a globally proper positive definite functional $V : \mathbb{B} \to \mathbb{R}$ such that the inequality

$$D_{\mathbb{K}} V(x_{x_0}(t)) \leq 0, \qquad \forall t > 0 \tag{2}$$

holds for any mild solution of (1) as long as $x_{x_0}(t) \in \mathbb{B}\backslash\{\mathbf{0}\}$.

For finite-dimensional time-invariant systems the latter theorem is proven in (Zubov, 1964, Theorem 12) or (Bacciotti and Rosier, 2001, Theorem 2.5). The case of a time-varying evolution inclusion

$$\dot{x} \in F(t, x), \quad F : \mathbb{R} \times \mathcal{D} \rightrightarrows \mathbb{B}, \quad \mathbf{0} \in \mathcal{D} \subset \mathbb{B} \tag{3}$$

is studied in (Deimling, 1992, Proposition 14.1), where the converse Lyapunov theorem is proven for a time-dependent Lyapunov functional $(t, x) \to V(t, x)$. If the system (3) is time-invariant then V becomes time-invariant as in Theorem 1.

Remark 1. **In Polyakov (2019), Theorem 1 was claimed to be true for the time-varying system (3). However, its proof contains an error in the necessity part. It works only for time-invariant system (3) even if $\mathbb{B} = \mathcal{D} = \mathbb{R}^n$. The author would like to thank an anonymous reviewer of the present paper who pointed out the error. Notice that this error has impacted some other Lyapunov theorems announced in Polyakov (2019) and utilized in Polyakov (2020). The necessity parts in the mentioned theorems are true only for the time-invariant case $F(t, x) = F(x)$. This fact does not impact the results of Polyakov (2020) (and the results of the present paper as well) since the converse Lyapunov theorems are utilized there only for autonomous systems.**

The finite-time stability combines two properties: Lyapunov stability and finite-time convergence. If the settling-time function T is continuous at the origin then the settling-time can be estimated using V (see Bhat and Bernstein (2000) for the corresponding analysis in the finite-dimensional case).

Corollary 1. Polyakov (2019) The origin of the autonomous system (1) is globally uniformly finite-time stable with a continuous at the origin settling-time function if and only if there exists a globally proper positive definite functional $V : \mathbb{B} \to [0, +\infty)$ such that the inequalities

$$D_{\mathbb{K}} V(x_{x_0}(t)) \leq -1, \qquad t > 0, \tag{4}$$

hold for any mild solution x_{x_0} of (1) as long as $x_{x_0}(t) \in \mathbb{B}\backslash\{\mathbf{0}\}$ and the settling-time T admits the estimate

$$T(x_0) \leq V(x_0), \quad \forall x_0 \in \mathbb{B}.$$

Below we use the latter corollary in order to prove existence of Lyapunov function for homogeneous systems.

4. HOMOGENEOUS SYSTEMS

4.1 Dilations in Banach Spaces

Recall that $\mathcal{L}(\mathbb{B}, \mathbb{B})$ denotes the space of linear bounded operators $\mathbb{B} \to \mathbb{B}$, where $\mathbb{B}$ is a real Banach space with a norm $\|\cdot\|$ and $I \in \mathcal{L}$ denotes the identity operator.

Definition 8. A mapping $\mathfrak{d} : \mathbb{R} \to \mathcal{L}(\mathbb{B}, \mathbb{B})$ is said to be *a group of linear dilations* (or simply *dilation*) in $\mathbb{B}$ if
1) *(Group property)* $\mathfrak{d}(0) = I$, $\mathfrak{d}(t+s) = \mathfrak{d}(t)\mathfrak{d}(s)$, $t, s \in \mathbb{R}$;
2) *(Limit property)* $\lim\limits_{s\to-\infty} \|\mathfrak{d}(s)z\| = 0$, $\lim\limits_{s\to+\infty} \|\mathfrak{d}(s)z\| = \infty$ uniformly on the unit sphere $S = \{z \in \mathbb{B} : \|z\| = 1\}$.

Obviously, $\mathfrak{d}$ is the group of linear bounded invertible operators and $\mathfrak{d}(-s) = (\mathfrak{d}(s))^{-1}$, $\forall s \in \mathbb{R}$. Let us introduce the notations

$$\|\mathfrak{d}(s)\| = \sup_{u\neq 0} \frac{\|\mathfrak{d}(s)u\|}{\|u\|} \quad \text{and} \quad \lfloor \mathfrak{d}(s) \rfloor = \inf_{u\neq 0} \frac{\|\mathfrak{d}(s)u\|}{\|u\|}.$$

The limit property specifies groups being dilations in $\mathbb{B}$. In particular, it implies (see Polyakov et al. (2016), Polyakov et al. (2018)): 1) $\|\mathfrak{d}(s)\| \to 0$ as $s \to -\infty$ and $\lfloor \mathfrak{d}(s) \rfloor \to +\infty$ as $s \to +\infty$; 2) $\lfloor \mathfrak{d}(s) \rfloor < 1$ for $s < 0$ and $\|\mathfrak{d}(s)\| > 1$ for $s > 0$.

Definition 9. A dilation $\mathfrak{d}$ is 1) *strongly continuous* if $\mathfrak{d}(\cdot)z : \mathbb{R} \to \mathbb{B}$ is continuous for any $z \in \mathbb{B}$; 2) *uniformly continuous* if $\mathfrak{d}(\cdot) : \mathbb{R} \to \mathcal{L}(\mathbb{B}, \mathbb{B})$ is continuous.

Examples of linear dilations in $\mathbb{R}^n$ are
1) *Uniform dilation* (L. Euler, 18th century): $\mathfrak{d}(s) = e^s I$, where I is the identity matrix $\mathbb{R}^n$(or identity operator in $\mathbb{B}$).
2) *Weighted dilation* (Zubov 1958 Zubov (1958)): $\mathfrak{d}(s) = \text{diag}\{e^{r_1 s}, ..., e^{r_n s}\} \in \mathbb{R}^n$, where $r_i > 0$, $i = 1, 2, ..., n$.
3) *Linear dilation* (Fischer and Ruzhansky (2016), Polyakov (2018)):

$$\mathfrak{d}(s) = e^{sG_{\mathfrak{d}}} = \sum_{i=0}^{\infty} \frac{s^i G_{\mathfrak{d}}^i}{i!}, \tag{5}$$

where $G_{\mathfrak{d}} \in \mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix.

Nonlinear dilations in $\mathbb{R}^n$ are studied in Khomenuk (1961), Kawski (1991), Rosier (1993).

Examples of linear dilations in function spaces are
1) For $\mathbb{B} = C([0, p], \mathbb{R})$ a uniformly continuous dilation in $\mathbb{B}$ can be defined as $(\mathfrak{d}(s)z)(y) = e^{s-0.5sy/p} z(y)$, where $s \in \mathbb{R}$, $y \in [0, p]$.
2) For $\mathbb{B} = L^2(\mathbb{R}, \mathbb{R})$ a strongly continuous dilation in $\mathbb{B}$ can be defined as $(\mathfrak{d}(s)z)(y) = e^{\alpha s} z(e^{\beta s} y)$, where $s \in \mathbb{R}$, $y \in \mathbb{R}$, $\alpha > \beta/2$.

For more details about dilations in function space (see (Polyakov, 2020, Chapter 6)).

Definition 10. A dilation $\mathfrak{d}$ in $\mathbb{B}$ is *monotone* if $\mathfrak{d}(s)$ is a strong contraction for any $s < 0$, i.e. $\|\mathfrak{d}(s)u\| < 1, \forall s < 0$, $\forall u \in S$. A dilation $\mathfrak{d}$ is *strictly monotone* in $\mathbb{B}$ if $\exists \beta > 0$:

$$\|\mathfrak{d}(s)\| \leq e^{\beta s}, \quad \forall s \leq 0. \tag{6}$$

Properties of monotone and strictly monotone dilations are studied in Polyakov et al. (2018), Polyakov (2018), where, in particular, it is shown that all dilations in $\mathbb{R}^n$ are strictly monotone under a proper selection of the weighted Euclidean norm in $\mathbb{R}^n$.

Definition 11. A nonempty set $\mathcal{D} \subseteq \mathbb{B}$ is said to be $\mathfrak{d}$-*homogeneous cone* $\mathbb{B}$ if $\mathfrak{d}(s)z \in \mathcal{D}, \forall z \in \mathcal{D}, \forall s \in \mathbb{R}$.

Homogeneous cone is a reasonable notion for a set satisfying the latter definition, since any point $z \in \mathcal{D}$ belongs to $\mathcal{D}$ together with the *homogeneous curve* $\Gamma_{\mathfrak{d}}(z) = \{y \in \mathbb{B} : y = \mathfrak{d}(s)z, s \in \mathbb{R}\}$. In particular, if $\mathfrak{d}$ is the uniform dilation (i.e. $\mathfrak{d}(s) = e^s I, s \in \mathbb{R}$), then the homogeneous curve becomes a ray and $\mathcal{D}$ becomes a positive cone in $\mathbb{B}$. Homogeneous cones in $\mathbb{B}$ are domains of homogeneous operators in $\mathbb{B}$ (see Polyakov (2020)).

4.2 The canonical homogeneous norm

The dilation in $\mathbb{B}$ introduces a new norm topology in $\mathbb{B}$ using the so-called canonical homogeneous norm.

Definition 12. Polyakov et al. (2018) The functional $\|\cdot\|_{\mathfrak{d}} : \mathbb{B}\backslash\{\mathbf{0}\} \to \mathbb{R}_+$:

$$\|u\|_{\mathfrak{d}} = e^{s_u}, \text{ where } s_u \in \mathbb{R} : \|\mathfrak{d}(-s_u)u\| = 1, \qquad (7)$$

is called the *canonical homogeneous norm* in the Banach space $\mathbb{B}$, where $\mathfrak{d}$ is a monotone dilation on $\mathbb{B}$.

Obviously, $\|\mathfrak{d}(s)u\|_{\mathfrak{d}} = e^s\|u\|_{\mathfrak{d}}$ and $\|u\|_{\mathfrak{d}} = \|-u\|_{\mathfrak{d}}$ for $\forall u \in \mathbb{B}$ and $\forall s \in \mathbb{R}$. The homogeneous norm $\|\cdot\|_{\mathfrak{d}}$ is called canonical since it is induced by a canonical norm $\|\cdot\|$ in $\mathbb{B}$ and $\|u\|_{\mathfrak{d}} = 1 \Leftrightarrow \|u\| = 1$. Notice that $\|\cdot\|_{\mathfrak{d}} = \|\cdot\|$ provided that $\mathfrak{d}$ is the standard dilation: $\mathfrak{d}(s) = e^s I$, $s \in \mathbb{R}$.

If $\mathfrak{d}$ is a strictly monotone dilation then the functional $\|\cdot\|_{\mathfrak{d}}$ has the following properties Polyakov et al. (2018), Polyakov (2020):
1) $\|\cdot\|_{\mathfrak{d}}$ is positive, single-valued and $\|u\|_{\mathfrak{d}} \to \mathbf{0}$ as $\|u\| \to \mathbf{0}$;
2) there exist $\alpha > \beta > 0$, $M \geq 1 : M^{-1}\|u\|_{\mathfrak{d}}^{\alpha} \leq \|u\| \leq \|u\|_{\mathfrak{d}}^{\beta}$ if $u \in B(1)$ and $\|u\|_{\mathfrak{d}}^{\beta} \leq \|u\| \leq M\|u\|_{\mathfrak{d}}^{\alpha}$, if $u \in \mathbb{B}\backslash B(1)$;
3) $\|\cdot\|_{\mathfrak{d}}$ is *locally Lipschitz continuous* on $\mathbb{B}\backslash\{\mathbf{0}\}$.

Notice that the property 1) implies that the functional $\|\cdot\|_{\mathfrak{d}}$ can be prolonged to zero by continuity, i.e. $\|\mathbf{0}\|_{\mathfrak{d}} := 0$.

4.3 Homogeneous operators

Definition 13. Polyakov et al. (2016) An operator $f : \mathcal{D}(f) \subset \mathbb{X} \to \mathbb{X}$ (a functional $h : \mathcal{D}(h) \subset \mathbb{X} \to \mathbb{R}$) is said to be $\mathfrak{d}$-homogeneous of a degree $\nu \in \mathbb{R}$ if the domain $\mathcal{D}(f)$ (resp. $\mathcal{D}(h)$) is a $\mathfrak{d}$-homogeneous cone and

$$\begin{array}{ll} e^{\nu s}\mathfrak{d}(s)f(u) = f(\mathfrak{d}(s)u), & \forall s \in \mathbb{R}, \quad \forall u \in \mathcal{D}(f), \\ (\text{resp. } h(\mathfrak{d}(s)u) = e^{\nu s}h(u), & \forall s \in \mathbb{R}, \quad \forall u \in \mathcal{D}(h)) \end{array} \qquad (8)$$

where $\mathbb{X}$ is a vector space and $\mathfrak{d}$ is a group of linear invertible operators in $\mathbb{X}$.

All linear and many nonlinear evolution models of mathematical physics are $\mathfrak{d}$-homogeneous Polyakov et al. (2016), Polyakov et al. (2018), Polyakov (2020). The identity (8) can be understood in the week sense as shown below.

On homogeneity of the Laplace operator. Let us consider the Laplace operator $\Delta : \mathcal{D}(\Delta) \subset L^2 \to L^2$,
$\mathcal{D}(\Delta) = \left\{u \in L^2 : \exists f \in L^2 \int u \cdot \Delta\phi = \int f \cdot \phi, \forall \phi \in C_c^{\infty}\right\}$.
Let us show that the operator Δ is $\mathfrak{d}$-homogeneous of degree 2β provided that the dilation $\mathfrak{d}$ is given by $(\mathfrak{d}(s)u)(x) = e^{\alpha s}u(e^{\beta s}x)$, where $s \in \mathbb{R}$, $x \in \mathbb{R}^n$, $\alpha > n\beta/2$.

Since, $\mathfrak{d}$ is a group of linear invertible operators on C_c^{∞} and, consequently, $\mathfrak{d}(s)$ maps C_c^{∞} onto C_c^{∞}. Notice that if $\phi \in C_c^{\infty}$ then, obviously,

$$(\Delta \circ \mathfrak{d}(s))\phi)(x) = e^{(\alpha+2\beta)s}(\Delta\phi)(e^{\beta s}x) = e^{2\beta s}((\mathfrak{d}(s) \circ \Delta)\phi)(x).$$

In other words, the Laplace operator is $\mathfrak{d}$-homogeneous as an operator $C_c^{\infty} \to C_c^{\infty}$. Since C_c^{∞} is dense in L^2, H^1 and H^2 then, it is expectable that, Δ is $\mathfrak{d}$-homogeneous as an operator in $\mathcal{D}(\Delta) \subset L^2 \to L^2$. Let us prove this rigorously.

Let $u \in \mathcal{D}(\Delta)$ and $\Delta u = f \in L^2$ in the weak sense, i.e.

$$\int_{\mathbb{R}^n} u\Delta\phi = \int_{\mathbb{R}^n} f\phi, \quad \forall \phi \in C_c^{\infty}.$$

Since $\mathfrak{d}(s)f \in L^2$ then using the change-of-variable theorem in the Lebesgue integral we derive

$$e^{2\beta s}\int(\mathfrak{d}(s)f) \cdot \phi = e^{(\alpha+2\beta)s}\int f(e^{\beta s}x) \cdot \phi(x)dx =$$

$$e^{(\alpha+(2-n)\beta)s}\int f(x) \cdot \phi(e^{-\beta s}x)dx = e^{(2\alpha+(2-n)\beta)s}\int f \cdot \tilde{\phi} =$$

$$e^{(2\alpha+(2-n)\beta)s}\int u \cdot \Delta\tilde{\phi} = e^{(2\alpha+(2-n)\beta)s}\int u \cdot (\Delta \circ \mathfrak{d}(-s))\phi =$$

$$e^{(\alpha-n\beta)s}\int u(x) \cdot \Delta\phi(e^{-\beta s}x)dx = e^{\alpha s}\int u(e^{\beta s}x) \cdot \Delta\phi(x)dx.$$

Hence, $\mathfrak{d}(s)u \in \mathcal{D}(\Delta)$ and $(\Delta \circ \mathfrak{d}(s))u = e^{2\beta s}\mathfrak{d}(s)f = e^{2\beta s}(\mathfrak{d}(s) \circ \Delta)u$ in the weak sense for any $s \in \mathbb{R}, u \in \mathcal{D}(\Delta)$.

Below we consider $\mathfrak{d}$-homogeneous operators in a Banach space $\mathbb{B}$. *We say the evolution equation* (1) *is* $\mathfrak{d}$-*homogeneous of a degree* $\mu \in \mathbb{R}$ *if the operators* A *and* f *are* $\mathfrak{d}$-*homogeneous of the degree* μ.

4.4 Symmetry of solutions of homogeneous systems

First, let us show that a semigroup generated by a closed densely defined linear homogeneous operator in $\mathbb{B}$ is homogeneous as well.

Lemma 3. Let a linear closed densely defined operator $A : \mathcal{D}(A) \subset \mathbb{B} \to \mathbb{B}$ generate a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{B}$ and $\mathfrak{d}$ be a group of linear bounded invertible operators on $\mathbb{B}$. If the operator A is $\mathfrak{d}$-homogeneous of a degree $\mu \in \mathbb{R}$ then

$$\Phi(t)\mathfrak{d}(s) = \mathfrak{d}(s)\Phi(e^{\mu s}t), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}. \qquad (9)$$

Lemma 3 proves the symmetry of solutions of (1) for $f \equiv \mathbf{0}$:

$$x_{\mathfrak{d}(s)x_0}(t) = \mathfrak{d}(s)x_{x_0}(e^{\mu s}t), \quad s \in \mathbb{R}, t \geq 0. \qquad (10)$$

Below we show that this identity holds for solutions of a $\mathfrak{d}$-homogeneous evolution equation (1) even if $f \neq \mathbf{0}$.

Theorem 2. Let $\mathfrak{d}$ be a group of linear bounded invertible operators on $\mathbb{B}$. Let a linear closed densely defined operator $A : \mathcal{D}(A) \subset \mathbb{B} \to \mathbb{B}$ generate a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{B}$. Let A and f be $\mathfrak{d}$-homogeneous operators of a degree $\mu \in \mathbb{R}$.

If $x : [0, T) \to \mathbb{B}$ is a mild solution of the evolution equation (1) then the function $x^s : [0, e^{-\mu s}T) \to \mathbb{B}$ given by

$$x^s(t) := \mathfrak{d}(s)x(e^{\mu s}t), \quad t \in [0, e^{-\mu s}T)$$

is also a mild solution of the evolution equation (1), $\forall s \in \mathbb{R}$.

The latter theorem allows us to expand globally any local result provided that $\mathfrak{d}$ is a dilation in $\mathbb{B}$. For example, if the origin of (1) is locally stable then, from (10) and the limit property of $\mathfrak{d}$, we immediately derive a global stability.

4.5 Convergence rates of homogeneous systems

The following theorem extends the result of Nakamura et al. (2002) to evolution equations in Banach spaces.

Theorem 3. Let $\mathbf{\mathfrak{d}}$ be a dilation in $\mathbb{B}$. If an evolution system (1) is $\mathbf{\mathfrak{d}}$-homogeneous of a degree $\nu \in \mathbb{R}$ and its origin is locally uniformly asymptotically stable then 1) it is globally uniformly finite-time stable if $\nu < 0$ and there exists a $\mathbf{\mathfrak{d}}$-homogeneous settling-time functional $T : \mathbb{B} \to [0, +\infty)$ of the degree $-\nu$ that is locally bounded and continuous at $\mathbf{0}$; 2) it is globally uniformly exponentially stable if $\nu = 0$; 3) it is globally uniformly practically fixed-time stable if $\nu > 0$.

5. HOMOGENEOUS LYAPUNOV FUNCTION THEOREM

Now we are ready to prove homogeneous Lyapunov functions theorems for evolution equations.

Theorem 4. Let $\mathbf{\mathfrak{d}}$ be a dilation in $\mathbb{B}$. A $\mathbf{\mathfrak{d}}$-homogeneous evolution equation (1) is globally uniformly Lyapunov stable if and only if there exists a globally proper positive definite $\mathbf{\mathfrak{d}}$-homogeneous functional $V : \mathbb{B} \to \mathbb{R}$ of the degree 1 such that the inequality

$$D_{\mathbb{K}} V(x_{x_0}(t)) \leq 0, \quad \forall t \geq 0$$

holds for any solution x_{x_0} with $x_0 \in \mathbb{B}\backslash\{\mathbf{0}\}$ as long as $x_{x_0}(t) \neq \mathbf{0}$, where $D_{\mathbb{K}}$ denotes the contingent derivative.

Combining the results of Theorem 3 and Corollary 1 we derive the homogeneous Lyapunov function theorem in the form.

Theorem 5. Let $\mathbf{\mathfrak{d}}$ be a dilation in $\mathbb{B}$ and the evolution equation (1) be $\mathbf{\mathfrak{d}}$-homogeneous of a degree $\mu \in \mathbb{R}$.

The origin of (1) is uniformly asymptotically stable if and only if there exists a globally proper positive definite $\mathbf{\mathfrak{d}}$-homogeneous functional $V : \mathbb{B} \to \mathbb{R}$ of the degree 1 such that the inequality

$$D_{\mathbb{K}} V(x_{x_0}(t)) \leq -\|x_{x_0}(t)\|_{\mathbf{\mathfrak{d}}}^{\mu+1}, \ \forall t \geq 0 \qquad (11)$$

holds for any solution x_{x_0} with $x_0 \in \mathbb{B}\backslash\{\mathbf{0}\}$ as long as $x_{x_0}(t) \neq \mathbf{0}$.

Remark 2. It is worth stressing that all Lyapunov theorems presented in this paper are obtained under the very mild assumption about an existence of a mild solution of the system (1). Basically, the only semigroup property of solutions and their continuity are utilized in the proofs. Therefore, the presented results can be straightforwardly extended to other types of solutions of dynamical systems (see e.g. Mironchenko and Wirth (2019) and Krein and Khazan (1985)). We do not assume that the system (1) is well-posed in the Hadamard's sense. This is the main reason why we cannot prove even continuity of the Lyapunov functional V.

Remark 3. Notice that the contingent derivative $D_{\mathbb{K}}$ cannot be replaced with the strong (conventional) derivative in the general case. Indeed, it is well-known (see e.g. Tabor (2006) and references therein) that even globally Lipschitz functions $t \to x(t) \in \mathbb{B}$ may be nowhere differentiable provided that the Banach space $\mathbb{B}$ does not satisfy the so-called Radon-Nikodym property. The latter means that, even if V is a Frechét differentiable functional, the scalar-valued function $t \to V(x(t))$ may be nowhere differentiable in the strong sense.

Assumption 2. Let any mild solution x_{x_0} of (1) with $x(0) = x_0 \in \mathbb{B}$ be a uniform (on compact intervals of time) limit of strong solutions x_{x_i} of (1) with $x_i \in \mathcal{D}(A)$, $x_i \to x_0$ as $i \to +\infty$.

The following assumption sufficient condition of stability of homogeneous system is more useful in practice.

Corollary 2. Let $\mathbf{\mathfrak{d}}$ be a dilation in $\mathbb{B}$ and the evolution equation (1) be $\mathbf{\mathfrak{d}}$-homogeneous of a degree $\mu \in \mathbb{R}$ and satisfy Assumptions 1 and 2. The origin of (1) is uniformly asymptotically stable if there exists a functional $V : \mathbb{B} \to \mathbb{R}$ that is globally proper, positive definite, $\mathbf{\mathfrak{d}}$-homogeneous of the degree 1, Frechét differentiable on $\mathcal{D}(A)\backslash\{\mathbf{0}\}$ and satisfies the inequality

$$DV(x)(Ax + f(x)) \leq -\|x\|_{\mathbf{\mathfrak{d}}}^{\mu+1}, \ \forall x \in \mathcal{D}(A)\backslash\{\mathbf{0}\} \qquad (12)$$

where $DV(x) \in \mathcal{L}(\mathbb{B}, \mathbb{R})$ denote the Frechét derivative of V at the point $x \in \mathcal{D}(A)\backslash\{\mathbf{0}\}$.

This corollary is the trivial consequence of Corollary 5.4 and Theorem 8.8 from Polyakov (2020). Notice that Gateaux (directional) derivative can be utilized in the latter corollary as well. For more details about Gateaux and Frechéch derivatives we refer the reader, for example, to Driver (2003).

Notice that if $\mathbb{B} = \mathbb{H}$ is a Hilbert space (or a reflexive Banach space), A is generator of a strongly continuous semigroup and f is locally Lipschitz continuous on $\mathbb{H}\backslash\{\mathbf{0}\}$ then Assumptions 1 and 2 are fulfilled (see Pazy (1983) or Polyakov (2020), Chapter 5).

Example. Let us consider the following partial differential equation (PDE)

$$\tfrac{\partial x}{\partial t} = \Delta x - \gamma \|x\|^q x,$$

where $\gamma > 0$, $x : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$,

$$\Delta : H^2(\mathbb{R}^n, \mathbb{R}) \subset \mathbb{H} \to \mathbb{H}$$

is the Laplace operator on $\mathbb{H} = L^2(\mathbb{R}^n, \mathbb{R})$ and

$$\|x\| = \sqrt{\langle x, x\rangle}, \qquad \langle x, y\rangle = \int_{\mathbb{R}^n} x \cdot y \, dz.$$

The latter PDE admits the representation (1) with $\mathbb{B} = \mathbb{H}$, $A = \Delta$ and $f : \mathbb{H} \to \mathbb{H}$ given by

$$f(x) = -\gamma \|x\|^q x.$$

The considered system satisfies Assumptions 1 and 2.

In Section IV it is already shown that the Laplace operator Δ is $\mathbf{\mathfrak{d}}$-homogeneous of the degree 2β with the dilation

$$(\mathbf{\mathfrak{d}}(s)x)(z) = e^{\alpha s} x(e^{\beta s} z)$$

with $\alpha > n\beta/2$. Since $\|\mathbf{\mathfrak{d}}(s)x\| = e^{\alpha - n\beta/2}\|x\|$ then

$$\|x\|_{\mathbf{\mathfrak{d}}} = \|x\|^{\frac{1}{\alpha - n\beta/2}}$$

and f is $\mathbf{\mathfrak{d}}$-homogeneous of the degree $q(\alpha - n\beta/2)$. Therefore, for $2\beta = q(\alpha - n\beta/2)$ the system is $\mathbf{\mathfrak{d}}$-homogeneous of the degree 2β and the sign of the homogeneity degree is defined by the sign of q. Simple computations show that a homogeneous Lyapunov functional of the degree 1 for the considered PDE model is given by $V(x) = \|x\|_{\mathbf{\mathfrak{d}}}$. Indeed, for $x \in \mathcal{D}(A)\backslash\{\mathbf{0}\}$ we derive

$$\begin{aligned} D\|x\|_{\mathbf{\mathfrak{d}}}(\Delta x + f(x)) &= \tfrac{1}{\alpha - n\beta/2}\|x\|^{\frac{1}{\alpha - n\beta/2} - 2}\langle x, \Delta x + f(x)\rangle \\ &\leq \tfrac{1}{\alpha - n\beta/2}\|x\|^{\frac{1}{\alpha - n\beta/2} - 2}\langle x, f(x)\rangle \\ &= -\tfrac{\gamma}{\alpha - n\beta/2}\|x\|_{\mathbf{\mathfrak{d}}}^{1 + q(\alpha - n\beta/2)}, \end{aligned}$$

where the dissipative property $\langle x, \Delta x\rangle \leq 0$ of the Laplace operator is utilized on the indeterminate step. Notice that, the canonical homogeneous norm is a homogeneous Lyapunov function of many stable homogeneous system (see Polyakov (2020)).

REFERENCES

Andrieu, V., Praly, L., and Astolfi, A. (2008). Homogeneous Approximation, Recursive Observer Design, and Output Feedback. *SIAM Journal of Control and Optimization*, 47(4), 1814–1850.

Bacciotti, A. and Rosier, L. (2001). *Lyapunov Functions and Stability in Control Theory*. Springer.

Bernuau, E., Efimov, D., Perruquetti, W., and Polyakov, A. (2014). On homogeneity and its application in sliding mode control. *Journal of The Franklin Institute*, 351(4), 1866–1901.

Bhat, S. and Bernstein, D. (2000). Finite time stability of continuous autonomous systems. *SIAM J. Control Optim.*, 38(3), 751–766.

Clarke, F.H., Ledyaev, Y.S., and Stern, R. (1998). Asymptotic stability and smooth Lyapunov functions. *Journal of differential Equations*, 149, 69–114.

Coron, J.M. and Praly, L. (1991). Adding an integrator for the stabilization problem. *Systems & Control Letters*, 17(2), 89–104.

Deimling, K. (1992). *Multivariable Differential Equations*. Walter de Gruyter, Berlin, New York.

Driver, B. (2003). *Analysis Tools with Applications*. Springer.

Engel, K.J. and Nagel, R. (2000). *One-Parameter Semigroups for Linear Evolution Equations*. Springer Verlag: Berlin, Heidelberg, New York.

Fischer, V. and Ruzhansky, M. (2016). *Quantization on Nilpotent Lie Groups*. Springer.

Folland, G. (1975). Subelliptic estimates and function spaces on nilpotent lie groups. *Arkiv for Matematik*, 13(1-2), 161–207.

Hermes, H. (1986). Nilpotent approximations of control systems and distributions. *SIAM Journal of Control and Optimization*, 24(4), 731.

Jacob, B., Mironchenko, A., Partington, J., and Wirth, F. (2018). Remarks on input-to-state stability and non-coercive lyapunov functions. In *2018 IEEE Conference on Decision and Control (CDC)*, 4803–4808.

Karafyllis, I. and Krstic, M. (2018). *Input-to-State Stability for PDEs*. Springer.

Kawski, M. (1990). Homogeneous stabilizing feedback laws. *Control Theory and Advanced Technology*, 6(4), 497–516.

Kawski, M. (1991). Families of dilations and asymptotic stability. *Analysis of Controlled Dynamical Systems*, 285–294.

Khomenuk, V.V. (1961). On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Mathematica (in Russian).*, 3(22), 157–164.

Krein, S. and Khazan, M. (1985). Differential equations in a banach space. *Journal of Soviet Mathematics*, 30(3), 2154–2239.

Lopez-Ramirez, F., Polyakov, A., Efimov, D., and Perruquetti, W. (2018). Finite-time and fixed-time observer design: Implicit Lyapunov function approach. *Automatica*, 87(1), 52–60.

Majda, A. (1975). Disappearing solutions for the dissipative wave equation. *Indiana University Mathematics Journal*, 24(12), 1119–1133.

Mironchenko, A. and Wirth, F. (2019). Non-coercive lyapunov functions for infinite-dimensional systems. *Journal of Differential Equations*. doi: https://doi.org/10.1016/j.jde.2018.11.026.

Nakamura, H., Yamashita, Y., and Nishitani, H. (2002). Smooth Lyapunov functions for homogeneous differential inclusions. In *Proceedings of the 41st SICE Annual Conference*, 1974–1979.

Natanson, I. (1955). *Theory of functions of a real variable*. Frederick Ungar Publishing Co., New York.

Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer.

Perrollaz, V. and Rosier, L. (2014). Finite-time stabilization of 2x2 hyperbolic systems on tree-shaped networks. *SIAM Journal of Control and Optimization*, 52(1), 143–163.

Polyakov, A. (2018). Sliding mode control design using canonical homogeneous norm. *International Journal of Robust and Nonlinear Control*, 29(3), 682–701. doi: 10.1002/rnc.4058.

Polyakov, A. (2019). Characterization of finite/fixed-time stability of evolution inclusions. In *Conference on Decision and Control*.

Polyakov, A. (2020). *Generalized Homogeneity in Systems and Control*. Springer.

Polyakov, A., Coron, J.M., and Rosier, L. (2018). On homogeneous finite-time control for linear evolution equation in hilbert space. *IEEE Transactions on Automatic Control*, 63(9), 3143–3150.

Polyakov, A., Efimov, D., Fridman, E., and Perruquetti, W. (2016). On homogeneous distributed parameters equations. *IEEE Transactions on Automatic Control*, 61(11), 3657–3662.

Prieur, C. and Mazenc, F. (2012). Iss-lyapunov functions for time-varying hyperbolic systems of balance laws. *Mathematics of Control, Signals, and Systems*, 24(1), 111–134.

Rosier, L. (1992). Homogeneous Lyapunov function for homogeneous continuous vector field. *Systems & Control Letters*, 19, 467–473.

Rosier, L. (1993). Etude de quelques problèmes de stabilization. *PhD Thesis, Ecole Normale Superieure de Cachan (France)*.

Ryan, E. (1995). Universal stabilization of a class of nonlinear systems with homogeneous vector fields. *Systems & Control Letters*, 26, 177–184.

Tabor, J. (2006). Generalized differential inclusions in banach spaces. *Set-Valued Analysis*, 14, 121–148.

Zubov, V.I. (1964). *Methods of A.M. Lyapunov and Their Applications*. Noordhoff, Leiden (Translated from Russian: V.I. Zubov, Metody Lyapunova i ih primenenie, Leningrad: LGU, 1957).

Zubov, V. (1958). On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Mathematica (in Russian).*, 1, 80–88.