Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Oscillation control in the underactuated "Ball and Beam" system

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Abstract: The problem of oscillatory motion construction and stabilization for the underactuated "Ball and Beam" system is considered. Virtual holonomic constraints approach is used. System's dynamics equations are derived, their transverse linearization is implemented, the controllability is proven, the stabilization algorithm is constructed. Obtained results are confirmed with computer simulation.

Keywords: Oscillation, automatic control, robotics, stabilization, underactuation, ball and beam.

1. INTRODUCTION

One may define an underactuated system (see, for example, Spong (1998)) as a controlled mechanical system that has more degrees of freedom than control inputs. One of the main reasons of the underactuation is the presence of elastic or nonholonomic constraints in the system. Dynamics of the underactuated systems are predominantly nonlinear which makes the analysis of such systems with classical control theory methods difficult or impossible. Moreover, not every trajectory in such systems can be physically realizable which leads to a problem of finding suitable solutions. The underactuated systems emerge in many fields of industry and robotics. Examples include industrial manipulators, walking robots, aircraft and watercraft. The widespread of such systems and a large number of unsolved problems gives rise to serious scientific interest and attracts many researchers, as seen in, for example, Lynch et al. (1998); Shiriaev et al. (2014).

This work considers a problem of motion stabilization in a "Ball and Beam" system, one of the most well-known examples of an underactuated mechanical system. It consists of a straight line segment ("beam") which can be rotated by a servo and a ball which lies on the beam and can roll left and right due to the beam's tilt. The beam can move the ball without any fixation and this movement is very unstable and cannot be controlled directly which makes the system underactuated and highly nonlinear.

First results in controlling "Ball and Beam" (for example, Wellstead et al. (1978)) date back to the late seventies. One of the first and most cited papers on the matter (Hauser et al. (1992)) considers the impossibility of system's equations linearization due to underactuation and proposes an approximate tracking approach. This work seems to have given rise to the interest around "Ball and Beam". In following years a number of papers were issued, covering different problems: from achieving global stability (see, for example, Barbu et al. (1997)) to considering variations of the system including adaptive and fuzzy control (see, for example, Eaton et al. (2000)). Most of the works consider stabilization of the ball in a certain point on the beam. This problem is a popular example in control theory and is well studied.

The goal of this work is to realize stable periodic rolling of the ball from one side to another and back. This problem is studied much less than the point stabilization but is much trickier. It was considered before in Gordillo et al. (2002), but the authors used greatly simplified motion equations that do not describe the real dynamics of the system. In this work the Lagrange's equations of the second kind without any simplifying assumptions are derived. Due to the impossibility of a usual linearisation procedure, a new method of virtual holonomic constraints described in Shiriaev et al. (2005) is applied to this system for the first time. This approach makes it possible to analyze the behaviour of the underactuated system without knowing the control input, find suitable feasible trajectories and simplify the equations of the system. The method has shown to give great results in stabilizing periodic motion of the underactuated systems (see, for example, Surov et al. (2015)).

The structure of the paper is as follows. First of all, the system's dynamics equations are derived. Then a virtual holonomic constraint is introduced and with its help suitable periodic trajectories are found and one of them is chosen as a desired motion of the system. After that via nonlinear transform the dynamics equations are rewritten in terms of special transverse coordinates and are linearised in a specific manner. Then the stabilizing feedback is computed and the stabilization process modelling is realised.

2. MOTION EQUATIONS

In order to obtain system's dynamics equations the suitable coordinate system is introduced. It is assumed that the beam is connected to a servo with a joint of length

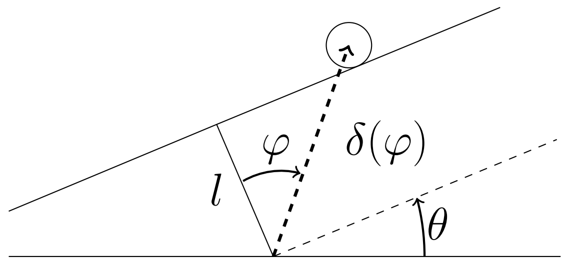

Fig. 1. The scheme of the "Ball and Beam" system.

l. Let the distance between the servo and the center of the ball be δ. Let the angle between the joint and the line whose length is δ be φ and the angle between the horizon and the line, parallel to the beam and coming out of the servo be θ (see Fig. 1).
Lagrange's equations of the second kind in matrix form are obtained in order to describe the system's dynamics and are written as follows:

$$M(q)\ddot{q} + C(q,\dot{q}) + G(q) = (u(t), 0)^{\mathrm{T}}, \tag{1}$$

where $u(t)$ is a control input, $q = (\theta, \varphi)^{\mathrm{T}}$. The explicit forms of the matrices are as follows (see, for example, Surov et al. (2015)):

$$M(q) = \begin{pmatrix} J_f + J_b + m\delta^2 & -J_b\mu - m\delta^2 \\ -J_b\mu - m\delta^2 & J_b\mu^2 + m\left(\delta^2 + \delta'^2\right) \end{pmatrix},$$

$$C(q,\dot{q}) = \begin{pmatrix} 2\dot{\theta}\dot{\varphi}m\delta\delta' - \dot{\varphi}^2\left(J_b\mu' + 2m\delta\delta'\right) \\ \dot{\varphi}^2\left(J_b\mu\mu' + m\delta'\left(\delta + \delta''\right)\right) - \dot{\theta}^2 m\delta\delta' \end{pmatrix},$$

$$G(q) = mg\begin{pmatrix} \delta\sin(\varphi - \theta) \\ \delta'\cos(\varphi - \theta) - \delta\sin(\varphi - \theta) \end{pmatrix}.$$

Here R is ball's radius, m is ball's mass, J_b is the ball's moment of inertia, J_f is the beam's moment of inertia, g is gravitational acceleration, $\mu(\varphi) = \sqrt{\delta^2 + \delta'^2}/R$.

3. VIRTUAL HOLONOMIC CONSTRAINTS

The next step is to obtain a trajectory of the desired periodic motion of the system. In accordance with the virtual holonomic constraints approach developed in Shiriaev et al. (2005) a function which expresses one variable through another is introduced: $\theta = \Theta(\varphi)$. After substituting new expressions for θ and it's derivatives into the second line of (1) that does not contain the input $u(t)$ the so-called $\alpha\beta\gamma$-equation is obtained:

$$\alpha(\varphi)\ddot{\varphi} \; + \; \beta(\varphi)\dot{\varphi}^2 + \gamma(\varphi) = 0, \tag{2}$$

where

$$\alpha(\varphi) \; = \; \left(-J_b\mu - m\delta^2\right)\Theta' + m\Upsilon\left(\delta^2 + \delta'^2\right),$$

$$\beta(\varphi) = \left(-J_b\mu - m\delta^2\right)\Theta'' + m\delta'\left(\Upsilon\left(\delta + \delta''\right) - \delta\Theta'^2\right),$$

$$\gamma(\varphi) = mg\left(\delta'\cos(\varphi - \Theta) - \delta\sin(\varphi - \Theta)\right),$$

$\Upsilon = J_b/mR^2 + 1$. This second order differential equation describes the dynamics of φ and has a number of useful properties described in Shiriaev et al. (2005). Following classical theorem (see, for example, Bautin N.N. and Leontovich E.A. (1990)) can provide one especially significant property.

Theorem (Lyapunov). *Consider a system of ordinary differential equations of first order and dimension 2: $\dot{x} = F(x)$, $x = (x_1, x_2)$. If it has an equilibrium point x_0, $F \in C^\infty$, linearised equation matrix has purely imaginary eigenvalues and there exists a nontrivial integral of the equation, then x_0 is a center.*

One can easily check that the linearised matrix of (2) has the eigenvalues $\lambda_{1,2} = \pm i\sqrt{(\gamma/\alpha)'_\varphi}$ and that the radicand is positive for the derived $\alpha(\varphi)$ and $\gamma(\varphi)$. The existence of a nontrivial integral will be discussed in section 4. Other conditions of the theorem are also obviously satisfied. It means that the trajectories of (2) are necessarily periodic. Thus one of them (denoted $\varphi_d(t)$) can be chosen to represent the desired motion of the system: $q_d = (\Theta(\varphi_d), \varphi_d)^{\mathrm{T}}$.

Now it is convenient to strictly formulate the motion stabilization problem in order to better understand the goal of the further work. Let $z = (q, \dot{q})^{\mathrm{T}}$ be the system's phase vector and $z_d = (q_d, \dot{q}_d)^{\mathrm{T}}$ be the desired trajectory vector. Then $\Gamma \; = \; \{z_d(t) \,|\, t \in [0; +\infty)\}$ is a curve in phase plane which sets the desired motion. The problem is to obtain the control law $u = U(z)$ that is independent of time t and that will satisfy the following condition: $\operatorname{dist}\{z(t), \Gamma\} \xrightarrow[t\to\infty]{} 0$, i.e. orbital asymptotic stability. Moreover, the problem of trajectory tracking in time is not considered, as the standard condition $|z_d(t) - z(t)| \xrightarrow[t\to\infty]{} 0$ may not be satisfied.

4. TRANSVERSE COORDINATES

A number of new special variables has to be introduced in order to simplify the solution of the stated problem. First of all, a variable that represents the initial coordinates' deviation from virtual constraints fulfillment is defined as follows: $y = \theta - \Theta(\varphi)$. This yields $q \; = \; (y \; + \; \Theta(\varphi), \varphi)^{\mathrm{T}}$. After the substitution of q and it's derivatives into (1) the first line becomes the following expression:

$$\ddot{y} = P(\varphi, y)u - R(\varphi, \dot{\varphi}, y, \dot{y}),$$

where $P = (1,0)\left(\Lambda M^{-1}\right)(1,0)^{\mathrm{T}}$, $\Lambda(\varphi) = \left(\begin{smallmatrix} 1 & -\Theta'(\varphi) \\ 0 & 1 \end{smallmatrix}\right)$,

$$R = (1,0)\left(\dot{\varphi}^2\Lambda\left(\Theta''(\varphi), 0\right)^{\mathrm{T}} + \Lambda M^{-1}\left(C + G\right)\right),$$

Let v be the new control: $v \; = \; P(\varphi, y)u \; - \; R(\varphi, \dot{\varphi}, y, \dot{y})$. Hence

$$\ddot{y} = v. \tag{3}$$

One can notice that P is invertible, which makes the transition from u to v and back justified.
The second line of (1) after the substitution of q becomes the nonhomogeneous $\alpha\beta\gamma$-equation:

$$\begin{aligned} &\alpha(\varphi)\ddot{\varphi} + \beta(\varphi)\dot{\varphi}^2 + \gamma(\varphi) \\ &\qquad = g_v(\varphi)v + g_{\dot{y}}(\varphi, \dot{\varphi}, \dot{y})\dot{y} + g_y(\varphi, y)y, \end{aligned} \tag{4}$$

where $g_v = -J_b\mu - m\delta^2$, $g_{\dot{y}} = m\delta\delta'(\dot{y} + 2\Theta'\dot{\varphi})$,

$$\begin{aligned} g_y = mg(&\delta'\sin(\varphi - \Theta - y/2) \\ &+ \delta\cos(\varphi - \Theta - y/2))\operatorname{sinc}(y/2). \end{aligned}$$

As shown in Shiriaev et al. (2005), via the coefficients of (2) and its solution $\varphi(t)$ with initial conditions $\varphi(0) = \varphi_0$, $\dot{\varphi}(0) = \dot{\varphi}_0$ one can define a function I as follows:

$$I(\varphi, \dot{\varphi}, \varphi_0, \dot{\varphi}_0) = \dot{\varphi}^2 - \psi(\varphi_0, \varphi)\dot{\varphi}_0^2 + \int\limits_{\varphi_0}^{\varphi} \frac{2\gamma(s)}{\alpha(s)}\psi(s, \varphi)\, ds,$$

where $\psi(\varphi_0, \varphi) \; = \; \exp\left\{-\int_{\varphi_0}^{\varphi} \frac{2\beta(s)}{\alpha(s)}\, ds\right\}$. This function is a nontrivial integral of (2), it is equal to zero on its trajectories (Shiriaev et al., 2005, p.4, theorem 1). Furthermore, for (4) the following relation holds (Shiriaev et al., 2005, p.4, theorem 2):

$$\dot{I} = \frac{2\dot{\varphi}}{\alpha}(g_v v + g_{\dot{y}}\dot{y} + g_y y - \beta I). \tag{5}$$

From now on the system composed of equations (3) and (5) will be considered. This system is not equivalent to (1) as the definition of I requires a certain trajectory of (2). The system describes the dynamics of transverse coordinates vector $x = (I, \dot{y}, y)^{\mathrm{T}}$, i.e. the vector that lies in the hyperplane that is orthogonal to the system's phase vector z, which is of dimension 4. Along the system's trajectories $y = \dot{y} = I = 0$. Moreover, I can be used to express the distance D between the trajectory $\varphi_d(t)$ and an arbitrary phase space point $(\varphi, \dot{\varphi})$ which is reached in a time moment t_D (see Shiriaev et al. (2010)):

$$I^2(\varphi, \dot{\varphi}, \varphi_d(t_D), \dot{\varphi}_d(t_D)) = 4\left(\dot{\varphi}_d^2(t_D) + \ddot{\varphi}_d^2(t_D)\right) D^2 + o(D^2),$$
$$D^2(\varphi, \dot{\varphi}) = \min_t \left\{ |\varphi - \varphi_d(t)|^2 + |\dot{\varphi} - \dot{\varphi}_d(t)|^2 \right\}.$$

Therefore these coordinates can be used to estimate distance to the desired trajectory and check the fulfillment of virtual constraints at any time moment. Thus it is sufficient to stabilize the system in transverse coordinates in order to solve the initial problem.

New equations can be rewritten in a following matrix form:

$$\dot{x} \;=\; \bar{A}(\varphi, \dot{\varphi}, y, \dot{y})x + \bar{B}(\varphi, \dot{\varphi})v, \qquad (6)$$

where

$$\bar{A} = \begin{pmatrix} k_1(t) & k_2(t) & k_3(t) \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; \bar{B} = \begin{pmatrix} \rho(t) \\ 1 \\ 0 \end{pmatrix},$$

$$k_1 = -\frac{2\dot{\varphi}\beta}{\alpha}, \; k_2 = \frac{2\dot{\varphi}g_{\dot{y}}}{\alpha}, \; k_3 = \frac{2\dot{\varphi}g_y}{\alpha}, \; \rho = \frac{2\dot{\varphi}g_v}{\alpha}.$$

5. LINEARIZATION

The next step in the stabilization problem solution is the linearisation of (6). But as it was mentioned in section 3, the system's trajectories may not be properly trackable in time, so the usual procedure of linearisation along some trajectory (for example, $\varphi_d(t)$) will not give the desired result. One has to exclude the time t from the equations on this step. In order to do that a new variable τ that represents a time moment, in which the aforementioned distance D is reached, is introduced as follows:

$$\tau(\varphi, \dot{\varphi}) = \arg\min_t \operatorname{dist}\left\{(\varphi, \dot{\varphi}), (\varphi_d(t), \dot{\varphi}_d(t))\right\}.$$

It can be shown that τ can be uniquely defined in some neighborhood of $\varphi_d(t)$ and that $\dot{\tau} = 1 + O(|x| + |v|)$, i.e. this new time variable is changing with almost the same speed as the old one. Furthermore, τ depends only on the current system state and does not depend on time t. That makes it possible to linearise (6) in a right way and to obtain a time-invariant control law.

After the substitution of the desired trajectory $\varphi_d(t)$ together with τ and the fulfillment of the virtual holonomic constraints ($y = \dot{y} = 0$) into (6) one can consider a new system of linear equations:

$$\frac{dx}{d\tau} = A(\tau)x + B(\tau)v, \qquad (7)$$

where

$$A(\tau) = \bar{A}(\varphi_d(\tau), \dot{\varphi}_d(\tau), 0, 0), \; B(\tau) = \bar{B}(\varphi_d(\tau), \dot{\varphi}_d(\tau)).$$

The matrices $A(\tau)$ and $B(\tau)$ will be periodic because of the periodicity of $\varphi_d(t)$ and their period will be the same as of the trajectory.

6. CONTROLLABILITY AND STABILIZABILITY

The following proposition is quite important for the further solution of the problem.

Proposition. *Pair of matrices $(A(\tau),\ B(\tau))$ defined in section 5 is controllable and stabilizable for every τ if for some τ_0 the following relation holds:*

$$\rho''(\tau_0) - \rho'(\tau_0)k_1(\tau_0) - \rho(\tau_0)k_1'(\tau_0) - k_2'(\tau_0)$$
$$\neq (\rho'(\tau_0) - \rho(\tau_0)k_1(\tau_0) - k_2(\tau_0))k_1(\tau_0) - k_3(\tau_0).$$

Proof. Let the vectors $K_0(\tau)$, $K_1(\tau)$ and $K_2(\tau)$ be defined as follows:

$$K_0(\tau) = B(\tau), \; K_j(\tau) = -A(\tau)K_{j-1}(\tau) + K_{j-1}'(\tau),$$

$j = 1, 2$. Then (Rugh, 1995, p. 145, theorem 9.4) the pair of matrices $(A(\tau),\ B(\tau))$ is controllable over period if the matrix made of columns K_0, K_1 and K_2 is nonsingular for some positive τ_0 that is smaller than the period. The direct calculations yield that the matrix in question has the following form:

$$\begin{pmatrix} \rho & \rho' - \rho k_1 - k_2 & \xi(\tau) \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

where $\xi = \rho'' - \rho' k_1 - \rho k_1' - k_2' - (\rho' - \rho k_1 - k_2)k_1 + k_3$. This matrix is nonsingular at the point τ_0 if and only if $\xi(\tau_0) \neq 0$. A half of the period can be chosen as τ_0, numerical computations show that the inequality holds in this point. The controllability of a periodic system over period implies the controllability for every real number which implies the system's stabilizability, as shown in Brunovsky (1969). The proof is complete.

7. STABILIZATION AND RETURN TO INITIAL COORDINATES

Stabilizability of (7) implies the existence of a stabilizing linear feedback: $v \;=\; K(\tau)x$. The suitable matrix $K(\tau)$ can be found by solving numerically the corresponding matrix Riccati differential equation with periodic coefficients, as shown in Gusev et al. (2016). After having found $K(\tau)$ one can return to nonlinear control in initial coordinates:

$$u = P^{-1}(\varphi, \theta - \Theta(\varphi))\Bigg[K(\tau)\begin{pmatrix} I(\varphi, \dot{\varphi}, \varphi_d, \dot{\varphi}_d) \\ \dot{\theta} - \Theta'(\varphi)\dot{\varphi} \\ \theta - \Theta(\varphi) \end{pmatrix}$$
$$+ R(\varphi, \dot{\varphi}, \theta - \Theta(\varphi), \dot{\theta} - \Theta'(\varphi)\dot{\varphi}\Bigg].$$

Such u provides orbital stability of the chosen trajectory φ_d of the initial Lagrange equations (1) (Shiriaev et al., 2010, p. 5, theorem 3).

8. MODELLING

The following parameter values were chosen for the modelling: $R = 0.015$ m; $l = 0.015$ m; ball's density: 1 kg/m^3; beam's length: 1 m. All the other parameters were calculated using corresponding formulae.

Denoting $h = l + R$ one can easily see from Fig. 1 that $\delta(\varphi) = h/\cos\varphi$. Therefore

$$\delta'(\varphi) = \frac{h\sin\varphi}{\cos^2\varphi}, \; \delta''(\varphi) = h\,\frac{1 + \sin^2\varphi}{\cos^3\varphi},$$
$$\mu(\varphi) = \frac{h}{R\cos^2\varphi}, \; \mu'(\varphi) = \frac{2h\sin\varphi}{R\cos^3\varphi}.$$

Thus matrix elements and vectors from (1) can be explicitly written as follows:

$$M_{11}(q) = J_f + J_b + \frac{mh^2}{\cos^2\varphi}, \; M_{22}(q) = \frac{m\Upsilon h^2}{\cos^4\varphi},$$
$$M_{12}(q) = M_{21}(q) = -\left(\frac{J_b}{R} + mh\right)\frac{h}{\cos^2\varphi},$$

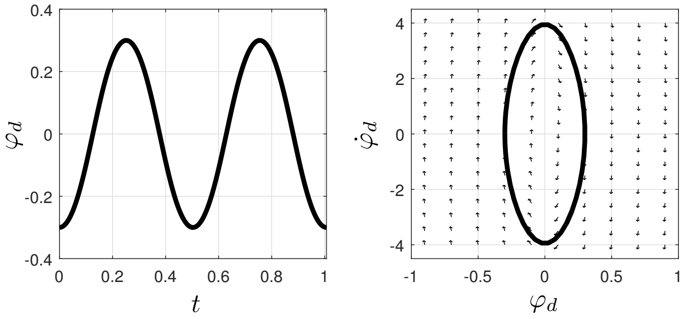

Fig. 2. Behaviour of the desired trajectory φ_d (left) and its phase portrait (right).

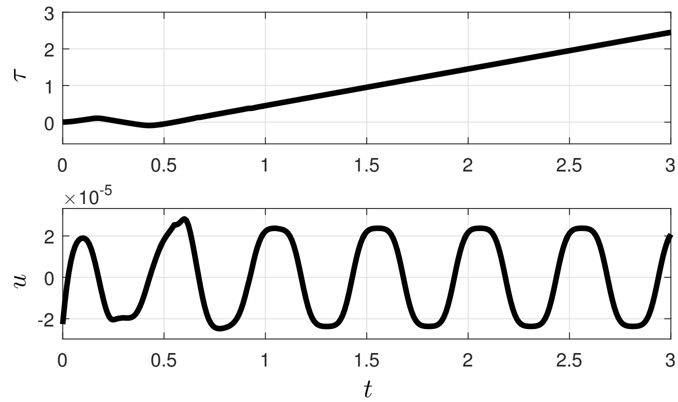

Fig. 3. Behaviour of τ (above) and u (below).

$$C(q,\dot{q}) = \begin{pmatrix} \dfrac{2h\sin\varphi}{\cos^3\varphi}\left(\dot{\theta}\dot{\varphi}mh - \dot{\varphi}^2\left(\dfrac{J_b}{R}+mh\right)\right) \\ \dfrac{mh^2\sin\varphi}{\cos^3\varphi}\left(\dfrac{2\Upsilon\dot{\varphi}^2}{\cos^2\varphi} - \dot{\theta}^2\right) \end{pmatrix},$$

$$G(q) = mgh\left(\frac{\sin(\varphi-\theta)}{\cos\varphi},\ \frac{\sin\theta}{\cos^2\varphi}\right)^{\mathrm{T}}.$$

One special feature of the virtual constraints method is that there is no particular way to derive such a constraint, one just has to come up with some suitable function. In this work the following virtual constraint was considered: $\Theta(\varphi) = k\varphi$. Therefore $\Theta'(\varphi) = k$, $\Theta''(\varphi) = 0$ and the coefficients of (2) are as follows:

$$\alpha(\varphi) = -\left(\frac{J_b}{R}+mh\right)\frac{hk}{\cos^2\varphi} + \frac{m\Upsilon h^2}{\cos^4\varphi},$$

$$\beta(\varphi) = \frac{mh^2\sin\varphi}{\cos^3\varphi}\left(\frac{2\Upsilon}{\cos^2\varphi} - k^2\right),\ \gamma(\varphi) = mgh\frac{\sin(k\varphi)}{\cos^2\varphi}.$$

Using these expressions and initial conditions $\varphi(0) = -0.3$, $\dot{\varphi}(0) = 10^{-6}$ the desired periodic trajectory $\varphi_d(t)$ was calculated numerically. Its period turned out to be approximately equal to 0.5032 s. Fig. 2 shows the behaviour of desired trajectory $\varphi_d(t)$ and its phase portrait.

The coefficients of the nonhomogeneous part of (4) with $y = \dot{y} = 0$ are as follows:

$$g_v(\varphi) = \frac{h}{\cos^2\varphi}\left(\frac{J_b}{R}+mh\right),\ g_{\dot{y}}(\varphi,\dot{\varphi},0) = \frac{2kmh^2\dot{\varphi}\sin\varphi}{\cos^3\varphi},$$

$$g_y(\varphi,0) = mgh\frac{\cos(k\varphi)}{\cos^2\varphi}.$$

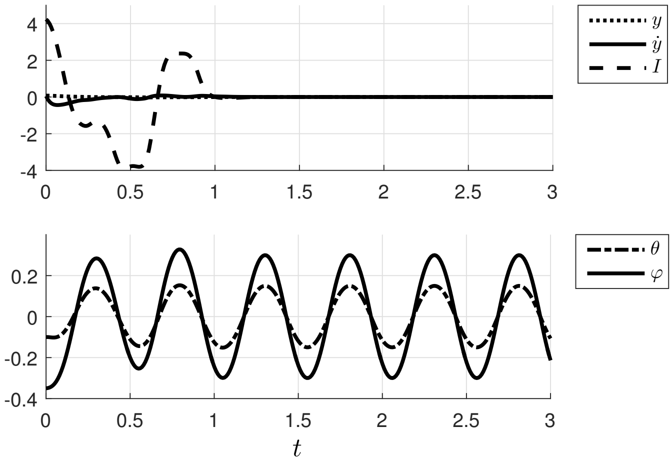

Fig. 4. Behaviour of transverse coordinates y, $\dot{y}$ and I (above) and angular variables θ and φ (below).

During the simulation the virtual constraint coefficient k was equal to 0.5. The variables I and τ and the stabilization matrix $K(\tau)$ were calculated numerically using formulae derived above. Figures 3 and 4 show the modelling results. Fig. 3 shows the behaviour of time substitute variable τ and the resulting control u. Fig. 4 shows the fading behaviour of the transverse coordinates y, $\dot{y}$ and I and the gradual stabilization of the initial angular coordinates θ and φ.

The simulation results allow us to conclude that the constructed controller provides stabilization of oscillations in the system under consideration and is robust with respect to disturbances associated with computational errors.

9. CONCLUSION

In this work the problem of oscillatory motion construction and stabilization in an underactuated "Ball and Beam" system was considered. The transverse linearization of the initial equations near the desired trajectory was carried out in accordance with the virtual constraints approach. The main contribution of the paper are: the finding of a suitable virtual holonomic constraint in order to provide the desired oscillations, synthesis of stabilizing control based on detailed non-simplified dynamics equations, the proof of stabilizability of periodic linear system, describing the transverse dynamics, and the orbital stability of desired trajectory of non-linear system, the validation of results using the computer simulation.

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